

Quasi normal modes and P-V criticality for scalar perturbations in a class of dRGT massive gravity around black holes

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Abstract We investigate black holes in a class of dRGT massive gravity for their quasi normal modes (QNMs) for neutral and charged ones using Improved Asymptotic Iteration Method and their thermodynamic behavior. The QNMs are studied for different values of the massive parameter m_g for both neutral and charged dRGT black holes under a massless scalar perturbation. As m_g increases, the magnitude of the quasi normal frequencies are found to be increasing. The results are also compared with the Schwarzschild de Sitter case. P-V criticality of the aforesaid black holes under massless scalar perturbation in the de Sitter space are also studied in this paper. It is found that the thermodynamic behavior of a neutral black hole shows no physically feasible phase transition while a charged black hole shows a definite phase transition.

Keywords Quasi normal modes · dRGT massive gravity · P-V criticality

1 Introduction

The existence of black holes is an outcome of Einstein's General Theory of Relativity (GTR). The question then is how to realize their existence and one natural way to identify them is to try to perturb and know their responses to the perturbation. Regge and Wheeler [1] started way back in 1950s studying perturbations of black-hole space times and later, serious studies were initiated by Zerilli [2]. It was Vishveshwara [3] who first noticed the existence of quasinormal modes (QNMs) by studying the scat-

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tering of gravitational waves by Schwarzschild black holes. Later, scattering of scalar, electromagnetic and Fermi fields by different black-hole spacetimes have been studied by many [4–6] and references cited therein. In the frame work of general relativity, QNMs arise as perturbations of black hole spacetimes. QNMs are the solutions to perturbation equations and they are distinguished from ordinary normal modes because they decay at certain rates, having complex frequencies. The remarkable property of the black hole QNMs (“ring down” of black holes) is that their frequencies are uniquely determined by the mass, angular momentum and charge(if any) of black holes. Black holes can be detected by observing the QNMs through gravitational waves. When a star collapses to form a black hole or when two black holes collide or a black hole and a star collide, Gravitational Waves (GWs) are emitted. The result of these processes is a black hole with higher mass that absorbs the GWs [8]. Hence the emitted GWs decay quickly. The decay of oscillations are characterized by complex frequencies.

The Quasi normal modes were first introduced by Vishveshwara [9,10]. Later, perturbation calculations have been done by many to get QNM oscillations [11–13]. To study the black hole QNMs, the solution of the perturbed field equation are separated for the radial and angular parts, whose radial part is the so called Regge-Wheeler equation. But this technique is time consuming and complicated that makes it difficult to survey QNMs for a wide range of parameter values. A semi analytic method has then been explored [14] that has its own limitations of accuracy. Later, the Continued Fraction Method (CFM) was proposed by Leaver. This method is a hybrid of analytic and numerical and can calculate QNM frequencies by making use of analytic infinite series representation of solution [15]. Another method is WKB approximation which is very commonly employed and a powerful one too. However all these methods have their own limitations. In recent years a new approach has been introduced to study black hole QNMs called Asymptotic Iteration Method (AIM) which is previously used to solve eigenvalue problems [16]. This method has been shown to be efficient and accurate for calculating QNMs of black holes [17,18].

The studies of Hawking and Bekenstein made in 1970s [19,20] helped us to view that black holes are thermal objects possessing temperature and entropy and that laws of black hole dynamics are analogous to the laws of classical thermodynamics. An immediate consequence of these studies is that they bring together quantum theory, gravity and thermodynamics and one can hope for a quantum theory of quantum gravity. Various methods [21,22] have been developed to study the thermodynamics of black holes. An important fact is that certain black holes make a transition from a stable phase to an unstable phase and some are thermodynamically unstable [23]. If the thermodynamic variables, pressure and volume, are identified, then an equation of state corresponding to the black hole can be found out and the critical points can be determined. The P-V isotherms then show their thermodynamic behavior.

GTR helped us to have a model for our universe and the universe can be considered as a dynamical system and most of the cosmological and astronomical observations could find meaningful explanations under GTR. But there are some fundamental issues like quantization of gravity, the initial stages of the evolution of the Universe under Big-Bang theory and also certain astronomical observations like dark matter and the late

time accelerating expansion of the universe which lacked proper explanations under GTR [24,25]. Hence attempts are being made for an alternative theory of gravitation.

From the perspective of the modern particle physics, GTR can be thought of as the unique theory of a massless spin 2 particle called graviton [26–28]. If the assumption behind the uniqueness theorem is broken, it can lead to alternative theories of gravity. Theories concerning the breaking of Lorentz invariance and spin have been explored in depth. Representing gravity as a manifestation of a higher order spin, thereby maintaining the Lorentz invariance and spin has also been explored largely [29]. Yet another possibility that has been recently explored is the so called 'Massive Gravity' (MG) theory [30–32]. In this model gravity is considered to be propagated by a massive spin 2 particle. The theory gets complicated especially when the massive spin 2 field interacts with matter. In that case, the theory goes completely non-linear and consequently non renormalizable. A non self interacting massive graviton model was first suggested by Fierz and Pauli [33] which is now called as 'linear massive gravity'. However this model suffers from a pathology [34] thereby ruling out the theory on the basis of solar system tests. Later, Vainshtein [35] proposed that the linear massive gravity model can be recovered to GTR through 'Vainshtein Mechanism' at small scales by including non linear terms in the hypothetical massive gravity theory. But the Vainshtein mechanism is later found to suffer from the so called 'Boulware-Deser' (BD) ghost [36]. Recently it is shown by de Rham, Gabadadze and Tolley in their series of works [37–39] that the BD ghost can be avoided for a sub class of massive potentials. This is called dRGT massive gravity which includes one dynamical and one fixed metric. This also holds true for its bi gravity extension [30,34,40].

This paper deals with the study of quasinormal modes coming out of massless scalar perturbations of a class of dRGT massive gravity around both neutral and charged black holes. We use the Improved Asymptotic Iteration Method (AIM) to calculate the QNMs. The P-V criticality condition of such black holes are also verified in the de Sitter space. Section 2 deals with a review of the Asymptotic Iteration Method. In Sect. 3, the quasinormal modes of neutral and charged black holes coming under a class of dRGT massive gravity, proposed by Ghosh et al. [41], are found out. Section 4 deals with the P-V criticality in the extended phase space of black holes described in Sect. 3. Section 5 concludes the paper.

2 Review of Asymptotic Iteration Method

Asymptotic Iteration Method (AIM) was proposed initially for finding solutions of the second order differential equations of the form [42],

$$Y''(x) - \lambda_0(x)Y'(x) - s_0(x)Y(x) = 0, \quad (1)$$

where $\lambda_0(x)$ and $s_0(x)$ are coefficients of the differential equation and are well defined functions and sufficiently differentiable. By differentiating (1) with respect to x ,

$$Y'''(x) - \lambda_1(x)Y'(x) - s_1(x)Y(x) = 0, \quad (2)$$

where the new coefficients are $\lambda_1(x) = \lambda'_0 + \lambda_0^2 + s_0$ and $s_1(x) = s'_0 + s_0\lambda_0$. Differentiating (1) twice with respect to x leads to,

$$Y''''(x) - \lambda_2(x)Y'(x) - s_2(x)Y(x) = 0, \tag{3}$$

where the new coefficients are $\lambda_2(x) = \lambda'_1 + \lambda_1\lambda_0 + s_1$ and $s_2(x) = s'_1 + s_0\lambda_1$. This process is continued to get the n^{th} derivative of (1) with respect to x as,

$$Y^{(n)}(x) - \lambda_{n-2}(x)Y'(x) - s_{n-2}(x)Y(x) = 0, \tag{4}$$

where the new coefficients are related to the older ones through the following expressions,

$$\lambda_n(x) = \lambda'_{n-1} + \lambda_{n-1}\lambda_0 + s_{n-1}, \tag{5}$$

$$s_n(x) = s'_{n-1} + s_0\lambda_{n-1}, \tag{6}$$

where $n = 1, 2, 3, \dots$

The ratio of $(n+2)^{th}$ derivative and $(n+1)^{th}$ derivative can be obtained from (4) as,

$$\begin{aligned} \frac{Y^{(n+2)}(x)}{Y^{(n+1)}(x)} &= \frac{d}{dx}(\ln Y^{n+1}) \\ &= \frac{\lambda_n[Y'(x) + \frac{s_n}{\lambda_n}Y(x)]}{\lambda_{n-1}[Y'(x) + \frac{s_{n-1}}{\lambda_{n-1}}Y(x)]} \end{aligned}$$

By introducing the asymptotic concept that for sufficiently large values of n ,

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} \equiv \alpha, \tag{7}$$

where α is a constant, we get,

$$\frac{d}{dx}(\ln Y^{n+1}) = \frac{\lambda_n}{\lambda_{n-1}},$$

from which a general expression for $Y(x)$ can be found out [7]. From (7) we can write,

$$\lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)s_n(x) = 0. \tag{8}$$

The roots of this equation are used to obtain the eigenvalues of (1). The energy eigenvalues will be contained in the coefficients. To get the eigenvalues, each derivative of λ and s are found out and expressed in terms of the previous iteration. Then by applying the quantization condition given by (8), a general expression for the eigenvalue can be arrived at. Cifti et al. [43] first noted that this procedure has a difficulty in that, the process of taking the derivative of s and λ terms of the previous iteration at each step can consume time and also affect the numerical precision of calculations. To overcome

this difficulty, an improved version of AIM has been proposed that bypasses the need to take derivative at each iteration. This is shown to improve both accuracy and speed of the method. For that, λ_n and s_n are expanded in a Taylor series around the point at which AIM is performed, x' ,

$$\lambda_n(x') = \sum_{i=0}^{\infty} c_n^i (x - x')^i, \tag{9}$$

$$s_n(x') = \sum_{i=0}^{\infty} d_n^i (x - x')^i, \tag{10}$$

where c_n^i and d_n^i are the i^{th} Taylor coefficients of $\lambda_n(x')$ and $s_n(x')$ respectively. Substitution of Eqs. (9) and (10) in (5) and (6) lead to the recursion relation for the coefficients as,

$$c_n^i = (i + 1)c_{n-1}^{i+1} + d_{n-1}^i + \sum_{k=0}^i c_0^k c_{n-1}^{i-k}, \tag{11}$$

$$d_n^i = (i + 1)d_{n-1}^{i+1} + \sum_{k=0}^i d_0^k c_{n-1}^{i-k}, \tag{12}$$

Applying (11) and (12) in (8), the quantization condition can be rewritten as,

$$d_n^0 c_{n-1}^0 - d_{n-1}^0 c_n^0 = 0. \tag{13}$$

This gives a set of recursion relations that do not require any derivatives. The coefficients given by (11) and (12) can be computed by starting at $n = 0$ and iterating up to $(n + 1)$ until the desired number of recursions are reached. The quantization condition given by (13) contains only $i = 0$ term. So, only the coefficients with $i < N - n$ where N is the maximum number of iterations to be performed needs to be determined. The perturbed radial wave equation of a black hole can be written in the form of a second order differential equation similar to (1) with the coefficients containing their quasi-normal frequencies. Hence the condition (13) can be employed to extract the QNMs of a black hole [17, 18]. This method is used in this paper to determine the QNMs of dRGT black hole.

3 Quasinormal modes of black holes in dRGT massive gravity

3.1 Neutral dRGT black hole

In the standard formalism of dRGT massive gravity theory, the Einstein-Hilbert action is given by Berezhiani et al. [44] and Babichev and Brito [45],

$$S = \int d^4x \sqrt{-g} \frac{1}{2\kappa^2} \left[R + m_g^2 U(g, \phi) \right], \tag{14}$$

where g is the metric tensor, R is the Ricci scalar, m_g represents the graviton mass and U is the effective potential for the graviton and is given by Kodama and Arraut [46],

$$U(g, \phi) = U_2 + \alpha_3 U_3 + \alpha_4 U_4, \quad (15)$$

where α_3 and α_4 are two free parameters. These parameters are redefined by introducing two new parameters α and β as,

$$\alpha_3 = \frac{\alpha - 1}{3}, \quad (16)$$

$$\alpha_4 = \frac{\beta}{4} + \frac{1 - \alpha}{12}. \quad (17)$$

Varying the action given by (14) with respect to the metric leads to the field equation,

$$G_{\mu\nu} = -m^2 X_{\mu\nu}, \quad (18)$$

where,

$$X_{\mu\nu} = \frac{\delta U}{\delta g_{\mu\nu}} - \frac{1}{2} U g^{\mu\nu}. \quad (19)$$

The constraints of this field Eq. (16) can be obtained by using the Bianchi identity,

$$\nabla^{\mu\nu} X_{\mu\nu} = 0. \quad (20)$$

A spherically symmetric metric has a form given by,

$$ds^2 = g_{tt}(r)dt^2 + 2g_{tr}(r)dt dr + g_{rr}(r)dr^2 + h(r)^2 d\Omega^2, \quad (21)$$

with $g_{tt}(r) = -\eta(r)$, $g_{rr} = \frac{1}{f(r)}$ and $h(r) = h_0 r$ where h_0 is a constant in terms of α and β [47–49]. The exact solution for this ansatz is complicated. It is simplified by choosing specific relations for the parameters. In this paper, we take $\alpha = -3\beta$. Since the fiducial metric acts like a Lagrangian multiplier to eliminate the BD ghost, to simplify the calculations, we choose the fiducial metric as, [50],

$$f_{\mu\nu} = (0, 0, c^2, c^2 \sin^2 \theta), \quad (22)$$

where c is a constant.

In this paper we consider only the diagonal branch of the physical metric for simplicity i.e., $g_{tr} = 0$. Then,

$$ds^2 = -\eta(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

By taking $\eta(r) = f(r)$ we get,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2$$

The non-zero components of the Einstein tensor are given by Ghosh et al. [41],

$$G_t^t = \frac{f'}{r} + \frac{f}{r^2} - \frac{1}{r^2}, \tag{23}$$

$$G_r^r = \frac{(rf' + f)}{r^2} - \frac{1}{r^2}, \tag{24}$$

$$G_\theta^\theta = G_\phi^\phi, \tag{25}$$

$$= f' \left(\frac{f'}{4f} + \frac{1}{2r} \right) + \left(\frac{f''}{2} + \frac{f'}{2r} - \frac{(f')^2}{4f} \right), \tag{26}$$

and the $X_{\mu\nu}$ tensor as,

$$X_t^t = \left(\frac{\alpha(3r - c)(r - c)}{r^2} + \frac{3\beta(r - c)^2}{r^2} + \frac{3r - 2c}{r} \right), \tag{27}$$

$$X_r^r = - \left(\frac{\alpha(3r - c)(r - c)}{r^2} + \frac{3\beta(r - c)^2}{r^2} + \frac{3r - 2c}{r} \right), \tag{28}$$

$$X_\theta^\theta = X_\phi^\phi, \tag{29}$$

$$= \frac{\alpha(2c - 3r)}{r} + \frac{3\beta(c - r)}{r} + \frac{c - 3r}{r}, \tag{30}$$

Solving (18) using these expressions for G_μ^ν and X_μ^ν gives the form of the metric as,

$$f(r) = 1 - \frac{2M}{r} + \frac{\Lambda}{3}r^2 + \gamma r + \zeta, \tag{31}$$

where,

$$\Lambda = 3m_g^2 (1 + \alpha + \beta), \tag{32}$$

$$\gamma = -cm_g^2 (1 + 2\alpha + 3\beta), \tag{33}$$

$$\zeta = c^2m_g^2 (\alpha + 3\beta). \tag{34}$$

The details of the above calculations are given by Ghosh et al. [41]. When $\gamma = \zeta = 0$, α and β will determine the nature of the solution. ie., if $(1 + \alpha + \beta) < 0$ we get a Schwarzschild-de Sitter type solution, if $(1 + \alpha + \beta) > 0$, we will get a Schwarzschild-anti de Sitter type solution and when $m_g \rightarrow 0$ we get a Schwarzschild black hole.

In this paper, we consider a static spherically symmetric space time with vanishing Energy momentum tensor and hence the field perturbations in such background are not coupled to the perturbations of the metric and therefore are equivalent to test field

in black hole background. Consider a massless scalar field that satisfies the Klein-Gordon equation in curved space-time,

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^a} g^{ab} \sqrt{-g} \frac{\partial}{\partial x^b} \Phi = 0 \tag{35}$$

$$i.e., \frac{1}{f(r)} \frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial}{\partial r} f(r) \frac{\partial \Phi}{\partial r} - \frac{\Delta_{\theta,\phi} \Phi}{r^2} = 0, \tag{36}$$

where,

$$\Delta_{\theta,\phi} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \tag{37}$$

In order to separate out the angular variables we choose the ansatz:

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=0}^l \frac{R(r)}{r} e^{-i\omega t} Y_{l,m}(\theta, \phi), \tag{38}$$

where ω gives the frequency of the oscillations corresponding to the black hole perturbation, $Y_{l,m}(\theta, \phi)$ are the spherical harmonics and,

$$\Delta_{\theta,\phi} Y_{l,m}(\theta, \phi) = -l(l+1)Y_{l,m}(\theta, \phi). \tag{39}$$

Substituting (38) in (36) and using (31) and (39) we get the radial wave equation,

$$\frac{d^2 R}{dr^2} + \frac{f'(r)}{f(r)} \frac{dR}{dr} + \left[\frac{\omega^2}{f(r)^2} - \frac{(\frac{2M}{r^3} + \frac{\gamma}{r} + \frac{2\Lambda}{3} + \frac{l(l+1)}{r^2})}{f(r)} \right] R = 0. \tag{40}$$

By using tortoise coordinate $x = \int \frac{dr}{f(r)}$, the above equation can be brought into the standard form [51],

$$\frac{d^2 R}{dx^2} + [\omega^2 - V(r)]R = 0, \tag{41}$$

where,

$$V(r) = f(r) \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right). \tag{42}$$

The SdS black hole has three singularities given by the roots of $f(r) = 0$, which are the event horizon, r_1 , the cosmological horizon, r_2 and at $r_3 = (-r_1 + r_2)$. The QNMs are defined as solutions of the above equation with boundary conditions: $R(x) \rightarrow e^{i\omega x}$ as $x \rightarrow \infty$ and $R(x) \rightarrow e^{-i\omega x}$ as $x \rightarrow -\infty$ for an $e^{-i\omega t}$ time dependence that corresponds to ingoing waves at the horizon and out going waves at infinity. The surface gravity κ_i at these singular points are defined as,

$$\kappa_i = \frac{1}{2} \frac{\partial f}{\partial r} \Big|_{r \rightarrow r_i}. \tag{43}$$

In the present study we are using improved AIM for finding the QNMs of the dRGT black hole and hence it is convenient to make a change of variable as $\xi = 1/r$ in (40) leading to,

$$\frac{d^2R}{d\xi^2} + \frac{p'}{p} \frac{dR}{d\xi} + \left[\frac{\omega^2}{p^2} - \frac{l(l+1) + (2M\xi + \gamma/\xi + \frac{2\Lambda}{3\xi^2})}{p} \right] R = 0, \tag{44}$$

where,

$$p = -2M\xi^3 + \xi^2(1 + \zeta) + \gamma\xi + \frac{\Lambda}{3}, \tag{45}$$

$$p' = -6M\xi^2 + \gamma + 2\xi(1 + \zeta). \tag{46}$$

In de Sitter space, the radial equation has got 3 singularities and these are represented as ξ_1 (Event horizon), ξ_2 (Cosmological horizon) and $\xi_3 = -\left(\frac{\xi_1\xi_2}{\xi_1 + \xi_2}\right)$ and hence we can write [17–52],

$$e^{i\omega\xi} = (\xi - \xi_1)^{\frac{i\omega}{2\kappa_1}} (\xi - \xi_2)^{\frac{i\omega}{2\kappa_2}} (\xi - \xi_3)^{\frac{i\omega}{2\kappa_3}}, \tag{47}$$

The idea is to scale out the divergent behavior at the cosmological horizon first and then rescale at the event horizon for a convergent solution. Now to scale out the divergent behavior at cosmological horizon, we take,

$$R(\xi) = e^{i\omega\xi} u(\xi). \tag{48}$$

The master equation given by (44) then takes the form,

$$pu'' + (p' - 2i\omega)u' - \left[l(l+1) + \left(2M\xi + \gamma/\xi + \frac{2\Lambda}{3\xi^2} \right) \right] u = 0. \tag{49}$$

The correct scaling condition of QNM at the event horizon implies,

$$u(\xi) = (\xi - \xi_1)^{-\frac{i\omega}{2\kappa_1}} \chi(\xi). \tag{50}$$

The master equation then can be viewed of the form as,

$$\chi'' = \lambda_0(\xi)\chi' + s_0(\xi)\chi, \tag{51}$$

where λ_0 and s_0 are the coefficients of the second order differential equation. It can be seen from (49) that the coefficient of u' includes the frequency ω . Therefore the quantization condition given by (13) can be used to find out the ω of (49) by iterating to some n maximum. For calculating the QNMs, we have used the MATHEMATICA NOTEBOOK given in the reference [53]. Initially the QNMs are calculated for the SdS by making $\gamma = \zeta = 0$ and the results are compared with reference [54,55] in

Table 1 Column 2 shows QNMs calculated for $\gamma = \zeta = 0$ for different values of Λ shown in column 1

Λ (for dRGT)	ω_{AIM}	Λ (for SdS)	ω_{WKB}
0	0.483644 – 0.0967588 i	0	0.48364 – 0.09677 i
–0.02	0.434585 – 0.0885944 i	0.02	0.43461 – 0.08858 i
–0.04	0.380784 – 0.0787610 i	0.04	0.38078 – 0.07876 i
–0.06	0.320021 – 0.0668449 i	0.06	0.32002 – 0.06685 i
–0.08	0.247470 – 0.0519043 i	0.08	0.24747 – 0.05197 i
–0.09	0.202960 – 0.0425584 i	0.09	0.20296 – 0.04256 i
–0.10	0.146610 – 0.0306869 i	0.10	0.14661 – 0.03069 i
–0.11	0.0461689 – 0.0063134 i	0.11	0.04617 – 0.00963 i

These are compared with the SdS case calculated in [54] shown in column 4. The results are found to agree quite well

Table 1. It can be seen that the results agree quite well with those found in the existing literature. We have executed 50 iterations while calculating the QNMs. We have taken $(1 + \alpha + \beta) < 0$ while calculating the QNMs so that the results of the calculations will correspond to that in de Sitter space.

Table 2 shows the quasi normal frequencies obtained through improved AIM method. The values of α and β are chosen so that Λ remains negative. We have chosen the values $M = c = 1$ in these calculations. The table shows the quasinormal modes calculated for $m_g = 0.8$ and $m_g = 1$ respectively for the same range of α and β values. It can be seen that for the same α and β , increasing the value of m_g increases the magnitude of the cosmological constant, which is obvious from (32). Also as m_g increases, the quasinormal frequencies are seen to be increasing in magnitude for both $l = 2$ and $l = 3$ modes. As for every m_g , both the real and imaginary parts of the quasinormal frequencies are seen to be continuously increasing in magnitude as Λ increases. Comparing these quasinormal frequencies with Table 1, it can be seen that the values of the quasinormal frequencies when m_g takes a finite value are higher in magnitude than when $m_g = 0$ which corresponds to a Schwarzschild case.

3.2 Charged dRGT black hole

Consider a charged black hole from the class of dRGT massive gravity with the metric,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \tag{52}$$

where [41],

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{\Lambda}{3}r^2 + \gamma r + \zeta, \tag{53}$$

Table 2 Quasinormal modes of black hole for massless scalar perturbations calculated by AIM (with 50 iterations) for a class of de- Sitter dRGT massive gravity for $l = 2$ and $l = 3$ modes

Λ	γ	ζ	$\omega(l = 2)$	$\omega(l = 3)$
$m_g = 0.8$				
-0.080	-0.80	1.9840	1.15155 - 0.348046 i	1.62914 - 0.341517 i
-0.088	-0.80	1.9904	1.15615 - 0.350418 i	1.63572 - 0.343749 i
-0.096	-0.80	1.9968	1.16081 - 0.352759 i	1.64237 - 0.346001 i
-0.104	-0.80	2.0032	1.16552 - 0.355121 i	1.64910 - 0.348271 i
-0.112	-0.80	2.0096	1.17030 - 0.357501 i	1.65590 - 0.350560 i
-0.120	-0.80	2.0160	1.17512 - 0.359902 i	1.66278 - 0.352868 i
-0.128	-0.80	2.0224	1.18001 - 0.362322 i	1.66974 - 0.355195 i
$m_g = 1.0$				
-0.100	-1.00	3.1000	2.81587 - 1.049800 i	3.90051 - 1.026860 i
-0.110	-1.00	3.1100	2.83013 - 1.057140 i	3.91984 - 1.033950 i
-0.120	-1.00	3.1200	2.84445 - 1.064510 i	3.93924 - 1.041070 i
-0.130	-1.00	3.1300	2.85881 - 1.071910 i	3.95870 - 1.048210 i
-0.140	-1.60	3.1400	2.87322 - 1.079340 i	3.97823 - 1.055380 i
-0.150	-1.75	3.1500	2.88768 - 1.086800 i	3.99781 - 1.062580 i
-0.160	-1.90	3.1600	2.90220 - 1.094280 i	4.10746 - 1.069800 i

The α and β values are kept same while QNMs are calculated by varying the m_g values

where Q corresponds to the charge. Proceeding as in Sect. 3.1, the wave equation is found as,

$$\frac{d^2 R}{d\xi^2} + \frac{p'}{p} \frac{dR}{d\xi} + \left[\frac{\omega^2}{p^2} - \frac{(2M\xi - 2Q^2\xi^4 + \gamma/\xi + \frac{2\Lambda}{3\xi^2})}{p} \right] R = 0, \tag{54}$$

where,

$$p = Q^2\xi^4 - 2M\xi^3 + \xi^2(1 + \zeta) + \gamma\xi + \frac{\Lambda}{3}, \tag{55}$$

$$p' = 4Q^2\xi^3 - 6M\xi^2 + \gamma + 2\xi(1 + \zeta). \tag{56}$$

Scaling out the divergent behavior at the event horizon leads to the master equation,

$$pu'' + (p' - 2i\omega)u' - \left[l(l + 1) + \left(2M\xi - 2Q^2\xi^2 + \gamma/\xi + \frac{2\Lambda}{3\xi^2} \right) \right] u = 0. \tag{57}$$

Again, the correct scaling condition of QNMs at the event horizon implies,

$$u(\xi) = (\xi - \xi_1)^{-\frac{i\omega}{2k_1}} \chi(\xi), \tag{58}$$

Table 3 The Quasinormal modes (after 50 iterations) for massless scalar perturbations of a charged black hole for the charge $Q = 0.5$ for the $l = 2$ and $l = 3$ modes

Λ	γ	ζ	$\omega(l = 2)$	$\omega(l = 3)$
<i>m</i> = 0.8				
-0.080	-0.80	1.9840	2.43544 - 0.523799 i	1.67618 - 0.168257 i
-0.088	-0.80	1.9904	2.43455 - 0.535763 i	1.67635 - 0.180489 i
-0.096	-0.80	1.9968	2.43252 - 0.547233 i	1.67351 - 0.195613 i
-0.104	-0.80	2.0032	2.42939 - 0.558215 i	1.67069 - 0.209057 i
-0.112	-0.80	2.0096	2.42523 - 0.568718 i	1.66693 - 0.222338 i
-0.120	-0.80	2.0160	2.42021 - 0.578624 i	1.66230 - 0.235427 i
-0.128	-0.80	2.0224	2.41399 - 0.588313 i	1.65677 - 0.248391 i
<i>m</i> = 1.0				
-0.10	-1.00	3.1000	0.304084 - 2.99974 i	0.9866449 - 4.93190 i
-0.11	-1.00	3.1100	0.342169 - 3.05263 i	1.0195500 - 5.01834 i
-0.12	-1.00	3.1200	0.378347 - 3.10442 i	1.0531600 - 5.10348 i
-0.13	-1.00	3.1300	0.413140 - 3.15511 i	1.0872800 - 5.18734 i
-0.14	-1.60	3.1400	0.446882 - 3.20472 i	1.1219000 - 5.26998 i
-0.15	-1.75	3.1500	0.479812 - 3.25326 i	1.1570200 - 5.35141 i
-0.16	-1.90	3.1600	0.512100 - 3.30072 i	1.1926600 - 5.43168 i

The α and β values are kept same while QNMs are calculated by varying the m_g values

where,

$$\begin{aligned} \kappa_1 &= \frac{1}{2} \frac{\partial f}{\partial r} \Big|_{r \rightarrow r_1}, \\ &= M\xi^2 - Q^2\xi^3 + \frac{\Lambda}{3}\xi + \frac{\gamma}{2}. \end{aligned} \tag{59}$$

The master equation is now in the form of (1) so that the quantization condition given by (13) can be employed to find out the QNMs.

Table 3 shows the quasinormal modes calculated using the improved AIM method for different values of α and β . We have chosen the values $M = c = 1$ and $Q = 0.5$ in these calculations. The QNMs are studied as in the previous section by varying the m_g value while keeping the values of α and β the same. It can be seen that as m_g increases, the real part of the quasi normal frequency decreases while the magnitude of the imaginary part increases. For each m_g the quasi normal frequency vary continuously. A black hole is stable only when the imaginary part in its Quasi normal spectrum is negative [56]. It is noted while calculating the Quasinormal modes that the roots of the frequency, ω give positive as well as negative imaginary frequencies. Here we are interested in the stable modes and therefore considered only the negative imaginary parts of ω . 50 iterations have been done for calculating the QNMs.

4 P-V Criticality of black holes

4.1 Black holes in dRGT massive gravity

In this section we look into the thermodynamic critical behavior of black holes described by the metric (31) in the extended phase space. We intend to check whether the black hole exhibits any phase transition by showing an inflection point in the $P - V$ indicator diagram. Here, the cosmological constant, Λ is treated as representing a negative pressure [57] as,

$$\Lambda = -8\pi P. \tag{60}$$

For $\gamma = \zeta = 0$ the metric given by (31) would lead to the case of a de Sitter space provided Λ is negative. Keeping this in mind we take,

$$\Lambda = 8\pi P, \tag{61}$$

where P is the pressure. The boundary of the black hole is described by the black hole horizon, r_h and is determined by the condition, $f(r)|_{r_h} = 0$. From this condition, the mass of the black hole can be expressed in terms of r_h as,

$$M = \frac{1}{6}r_h(3 + 3r_h\gamma + 3\zeta + r_h^2\Lambda), \tag{62}$$

and the black hole mass is considered to be the enthalpy of the system. The thermodynamic volume, V is given by Kstor et al. [58] and Dolan [59]

$$V = \frac{\partial M}{\partial P}. \tag{63}$$

Varying (62) partially with respect to the pressure P , we get

$$V = \frac{4}{3}\pi r_h^3. \tag{64}$$

The temperature of the black hole, described by the metric in (31), given by the Hawking temperature can be written as [60],

$$\begin{aligned} T &= \frac{1}{4\pi} f'(r_h), \\ &= \frac{1}{4\pi r_h} \left[\frac{2M}{r_h} + \gamma r_h + 2\frac{\Lambda}{3} r_h^2 \right]. \end{aligned} \tag{65}$$

Substituting for M from (62) in the above equation and rearranging it we get an expression for the cosmological constant,

$$\Lambda = \frac{4\pi T - 2\gamma}{r_h} - \frac{(1 + \zeta)}{r_h^2}. \tag{66}$$

But from (61), the cosmological constant can be related to the pressure as $\Lambda = 8\pi P$. Therefore (66) can be written in terms of P as,

$$\begin{aligned} P &= \frac{\Lambda}{8\pi}, \\ &= \left(\frac{T}{2} - \frac{\gamma}{4\pi}\right) \frac{1}{r_h} - \left(\frac{1}{8\pi} + \frac{\zeta}{8\pi}\right) \frac{1}{r_h^2}, \end{aligned} \quad (67)$$

Or,

$$P = \frac{w_1}{r_h} + \frac{w_2}{r_h^2}, \quad (68)$$

where,

$$w_1 = \left(\frac{T}{2} - \frac{\gamma}{4\pi}\right), \quad (69)$$

$$w_2 = -\left(\frac{1}{8\pi} + \frac{\zeta}{8\pi}\right). \quad (70)$$

From (69), w_1 can be treated as a shifted temperature. From (64), thermodynamic volume V is a monotonic function of the horizon radius r_h . and hence r_h can be considered to be corresponding to V . Therefore, (68) can be treated as an equation of state describing the black hole. The critical point is then determined by the conditions,

$$\frac{\partial P}{\partial r_h} \Big|_{r_h=r_{hc}, T=T_c} = 0, \quad (71)$$

and

$$\frac{\partial^2 P}{\partial^2 r_h} \Big|_{r_h=r_{hc}, T=T_c} = 0. \quad (72)$$

Substituting for P from (68) in the above differential equation it is found that the conditions given by (71) and (72) are not simultaneously satisfied. The condition,

$$\frac{\partial P}{\partial r_h} \Big|_{r_h=r_{hc}, T=T_c} = 0, \quad (73)$$

gives the critical horizon as,

$$r_{hc} = -\frac{2w_2}{w_1}. \quad (74)$$

Evaluation of $\frac{\partial^2 P}{\partial^2 r_h} \Big|_{r_h=r_{hc}, T=T_c}$ gives a non zero value which can imply either a local maximum or a local minimum depending on whether the value is greater than or less than zero. The critical pressure is found out by substituting (74) in (68) which gives,

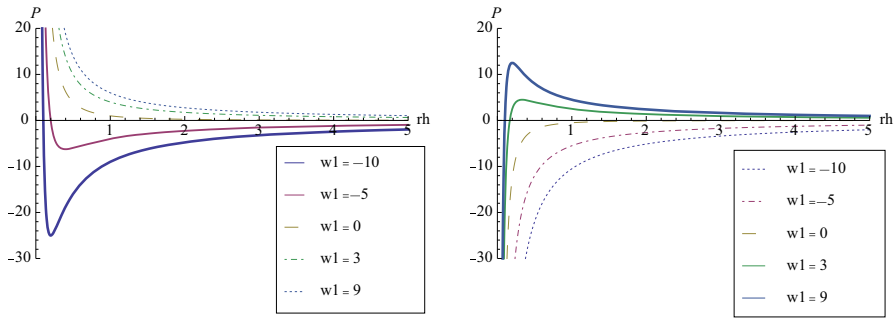


Fig. 1 In the first figure $p - r_h$ diagram is plotted for the value $w_2 = 1$ which shows critical behavior and the second figure shows plots for the value $w_2 = -0.5$ which shows an inflection

$$P_c = -\frac{w_1^2}{4w_2}. \tag{75}$$

This critical point corresponds to a physically feasible one if P_c is positive [61]. From (70) it can be seen that this happens only if w_2 is negative irrespective of the sign of w_1 . The relation between shifted temperature, w_1 , critical pressure, P_c and horizon radius r_h can be found out from (74) and 75 as,

$$\frac{P_c r_{hc}}{w_1} = \frac{1}{2}. \tag{76}$$

This ratio is called the ‘Compressibility Ratio’. The value of compressibility ratio for a Van der Waal’s gas is 0.375. Hence, the black hole system, with the Compressibility Ratio given by 76), can be thought of as behaving like a near Van der Waal’s system. The $P - r_h$ diagram plotted for different shifted temperature is shown in Fig. 1. In the first figure, the curves are plotted for $w_2 = 1$, the curves are seen to show critical behavior but it likely does not correspond to a physical one because, from (75), for the above said values of w_1 and w_2 the critical pressure P_c turns out to be negative for these curves. The second figure is plotted for $w_2 = -0.5$, they show inflection point but there is no phase transition.

4.2 Charged dRGT Black Hole

Consider a charged black hole with the metric of the form (56). The Hawking Temperature for this metric can be found out as,

$$\begin{aligned} T &= \frac{1}{4\pi} f'(r_h), \\ &= \frac{1}{4\pi r_h} \left[\frac{2M}{r_h} - \frac{2Q^2}{r_h^2} + \gamma r_h + 2\frac{\Lambda}{3} r_h^2 \right]. \end{aligned} \tag{77}$$

From the above equation, the equation of state can obtained proceeding as described in Sect. 4.1. The mass, M of the black hole can be written in terms of the horizon radius r_h as,

$$M = \frac{1}{6r_h}(3Q^2 + 3r_h^2(1 + \zeta) + 3r_h^3\gamma + r_h^4\Lambda), \tag{78}$$

Substituting (78) in (77) we get,

$$\Lambda = \frac{4\pi T - 2\gamma}{r_h} - \frac{(1 + \zeta)}{r_h^2} + \frac{Q^2}{r_h^4}. \tag{79}$$

Writing this equation in terms of P ,

$$\begin{aligned} P &= \frac{\Lambda}{8\pi}, \\ &= \left(\frac{T}{2} - \frac{\gamma}{4\pi}\right) \frac{1}{r_h} - \left(\frac{1}{8\pi} + \frac{\zeta}{8\pi}\right) \frac{1}{r_h^2} + \frac{Q^2}{8\pi r_h^4}, \end{aligned} \tag{80}$$

Or,

$$P = \frac{w_1}{r_h} + \frac{w_2}{r_h^2} + \frac{w_3}{r_h^4}, \tag{81}$$

where,

$$w_1 = \left(\frac{T}{2} - \frac{\gamma}{4\pi}\right), \tag{82}$$

$$w_2 = -\left(\frac{1}{8\pi} + \frac{\zeta}{8\pi}\right), \tag{83}$$

$$w_3 = \frac{Q^2}{8\pi}. \tag{84}$$

(81) describes the equation of state. The critical point is then determined by the conditions,

$$\frac{\partial P}{\partial r_h} \Big|_{r_h=r_{hc}, T=T_c} = 0, \tag{85}$$

and

$$\frac{\partial^2 P}{\partial^2 r_h} \Big|_{r_h=r_{hc}, T=T_c} = 0. \tag{86}$$

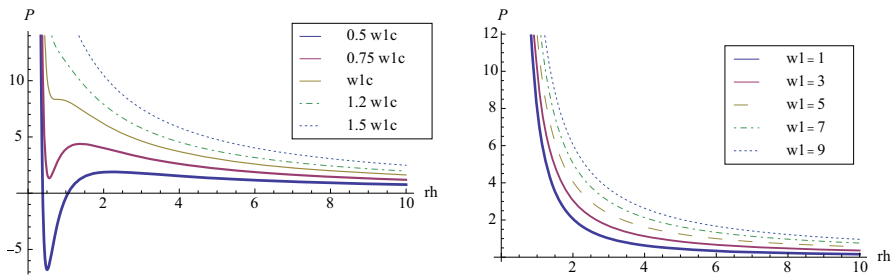


Fig. 2 In the first figure $p - r_h$ diagram is plotted for the vale $w_2 = -10$ and $w_3 = 1$ which shows a phase transition and the second figure shows plots for the value $w_2 = 6$ and $w_3 = 1$ which does not show any phase transition

Unlike in the Sect. 4.1, it is found that (85) and (86) are simultaneously satisfied which gives the solutions, for critical horizon as,

$$r_{hc} = \sqrt{-\frac{6w_4}{w_2}}, \tag{87}$$

and for the critical temperature as,

$$w_{1c} = \frac{2}{3} \sqrt{-\frac{2w_2^3}{w_3}}. \tag{88}$$

Using (81), (87) and (88), an expression for the critical pressure can be arrived at as,

$$P_c = \frac{w_2^2}{12w_3}. \tag{89}$$

The relation connecting shifted temperature w_{1c} , critical pressure, P_c and critical horizon radius r_{hc} are found as,

$$\frac{P_c r_{hc}}{w_{1c}} = \frac{3}{8}, \tag{90}$$

which is exactly the same as in the case for a Van der Waal’s system. The P-V diagram plotted for different shifted temperature is shown in Fig. 2. In the first figure, the curves are plotted for $w_2 = -10$ and $w_3 = 1$. The second figure is plotted for $w_2 = 6$ and $w_3 = 1$. The first figure shows an inflection point and a phase transition, but the second does not, as is obvious due to the sign change of w_2 .

5 Conclusion

In this paper, the quasinormal modes coming out of massless scalar perturbations in black hole space-time in a class of dRGT massive gravity, is studied. We have used the Improved Asymptotic Iteration Method (Improved AIM) to find out the QNMs in

the de Sitter space. We have done 50 iterations for calculating the QNMs. The Quasi normal modes are studied by varying the massive parameter, m_g . It is found that as m_g increases the magnitude of the quasi normal frequencies increase for neutral black hole. These QNMs are also higher in magnitude compared to the SdS case. It is also found that as γ and ζ tend to zero, the results converge to the SdS case. For a charged black hole, the real part of the quasi normal frequency decreases and the magnitude of imaginary part increases as m_g is increased.

The $P - V$ criticality in the extended phase space of the aforesaid black holes are also determined. The neutral black holes show a near Van der Waal behavior with the compressibility ratio of 0.5. But it does not show any physically feasible phase transition for the de Sitter space. The charged black hole on the other hand exactly shows a Van der Waal's behavior and clearly exhibits a phase transition.

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