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RESEARCH ARTICLE

Solutions for uniform acceleration in general relativity

Tzvi Scarr[1] · **Yaakov Friedman**[1,2]

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Abstract We explore two methods for obtaining solutions for uniformly accelerated motion in general curved spacetime. We provide an example in Schwarzschild spacetime.

Keywords Uniform acceleration · Schwarzschild spacetime · Frenet frame · Parallel transport

1 Introduction

Uniformly accelerated systems have been studied in [1] and, more recently, in [2,3]. In this paper, we present two methods for finding solutions for uniformly accelerated motion in a general curved spacetime. This extends the results of [4], where explicit solutions are computed for flat spacetime only.

We represent arbitrary curved spacetime by a time-orientable four-dimensional semi-Riemannian manifold M endowed with a locally smooth metric $g_{\mu\nu}$ of signature $(+,-,-,-)$. A *worldline* is a smooth future-pointing timelike curve $\gamma : I \to M$, where I is an interval of $\mathbb{R}$ containing 0. In a local coordinate system x^μ, we write

$$\gamma(s) = x^\mu(s), \qquad (1)$$

✉ Yaakov Friedman
friedman@jct.ac.il

1 Department of Mathematics, Jerusalem College of Technology, P.O.B. 16031, 91160 Jerusalem, Israel

2 Department of Physics, Jerusalem College of Technology, P.O.B. 16031, 91160 Jerusalem, Israel

where the parameter s is the arclength along the curve, that is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2}$$

At each point $\gamma(s)$, the *four-velocity* $u^\mu(s)$, defined by

$$u^\mu(s) = \frac{dx^\mu(s)}{ds}, \tag{3}$$

has unit length:

$$u^2 = g_{\mu\nu} u^\mu u^\nu = 1. \tag{4}$$

The goal of this paper is to find solutions $u(s)$ for uniformly accelerated motion in curved spacetime. In flat spacetime ($g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$), this goal has already been achieved. It is shown in [4,5] that, in flat spacetime, $\gamma(s)$ represents uniformly accelerated motion if and only if there is a constant, rank (0, 2) antisymmetric tensor $A_{\mu\nu}$ such that

$$\frac{du^\mu(s)}{ds} = A^\mu_\nu u^\nu(s). \tag{5}$$

Equation (5) can be extended to accommodate an orthonormal basis $\Lambda(s)$ of the tangent space at $\gamma(s)$, and we have

$$\frac{d\Lambda(s)}{ds} = A\Lambda(s). \tag{6}$$

The solution to (6) is

$$\Lambda(s) = \exp(As)\Lambda(0). \tag{7}$$

Explicit solutions to Eq. (6) are given in [6].

The plan of the paper is as follows. In Sect. 2, we construct a system of first-order ordinary differential equations for uniformly accelerated motion. We show that this system extends the geodesic equation. We provide an example in Sect. 3. Here we consider motion in the radial direction in Schwarzschild spacetime. We solve this system numerically and show that there are no bounded orbits. In Sect. 4, we explore an alternative method for finding solutions for uniformly accelerated motion. This method may also be used to find solutions for parallel transport. Directions for further research appear in Sect. 5. Throughout the paper, we use units in which $c = 1$.

2 Equations for uniform acceleration

In this section, we construct a first-order system of ordinary differential equations for uniform acceleration. The covariant derivative of a vector Z along a curve $\gamma(s)$ is defined by (see [7, 3.13]):

$$\frac{DZ^\mu}{ds} = \frac{dZ^\mu}{ds} + \Gamma^\mu_{\sigma\rho} Z^\sigma u^\rho, \tag{8}$$

where

$$\Gamma^{\mu}_{\sigma\rho} = \frac{1}{2} g^{\beta\mu} \left(g_{\beta\sigma,\rho} + g_{\beta\rho,\sigma} - g_{\sigma\rho,\beta} \right). \tag{9}$$

In [4], we showed how to construct the orthonormal *Frenet basis* $\{\lambda_{(0)}(s) = u(s), \lambda_{(1)}(s), \lambda_{(2)}(s), \lambda_{(3)}(s)\}$ of the tangent space $T_{\gamma(s)}M$ at the point $\gamma(s)$. The basis vectors $\lambda_{(\alpha)}(s)$ satisfy the *Frenet equations*

$$\begin{aligned}
\frac{D\lambda_{(0)}(s)}{ds} &= \kappa(s)\lambda_{(1)}(s) \\
\frac{D\lambda_{(1)}(s)}{ds} &= \kappa(s)\lambda_{(0)}(s) + \tau_1(s)\lambda_{(2)}(s) \\
\frac{D\lambda_{(2)}(s)}{ds} &= -\tau_1(s)\lambda_{(1)}(s) + \tau_2(s)\lambda_{(3)}(s) \\
\frac{D\lambda_{(3)}(s)}{ds} &= -\tau_2(s)\lambda_{(2)}(s),
\end{aligned}$$

where the scalar function $\kappa(s)$ is called the *curvature* of the curve γ, and $\tau_1(s)$ and $\tau_2(s)$ are known as the *first* and *second torsion*, respectively, of γ. The Frenet equations may be written compactly as

$$\boxed{\frac{D\lambda_{(\alpha)}(s)}{ds} = \lambda_{(\beta)}(s)A(s)^{(\beta)}_{(\alpha)},} \tag{10}$$

where

$$A(s) = A(s)^{(\beta)}_{(\alpha)} = \begin{pmatrix} 0 & \kappa(s) & 0 & 0 \\ \kappa(s) & 0 & -\tau_1(s) & 0 \\ 0 & \tau_1(s) & 0 & -\tau_2(s) \\ 0 & 0 & \tau_2(s) & 0 \end{pmatrix}. \tag{11}$$

Note that $A(s)$ is not a tensor, since it remains the same under a coordinate transformation. Thus, its two indices are coordinate-free, so we place them in parentheses. It is a 4×4 matrix of scalar functions which we call the *acceleration matrix*.

Recall from [4] that a worldline represents *uniformly accelerated motion* if the acceleration matrix $A(s)$ is constant along γ, that is, if $\frac{dA}{ds} = 0$. Let $\gamma(s)$ be a uniformly accelerated worldline, with acceleration matrix A. Using (8), we can write (10) as

$$\frac{d\lambda^{\mu}_{(\alpha)}(s)}{ds} + \Gamma^{\mu}_{\sigma\rho}\lambda^{\sigma}_{(\alpha)}(s)\lambda^{\rho}_{(0)}(s) = \lambda^{\mu}_{(\beta)}(s)A^{(\beta)}_{(\alpha)}. \tag{12}$$

The Christoffel symbols $\Gamma^{\mu}_{\sigma\rho}$ are smooth functions of the coordinates x^{μ}. Thus, in order to have a complete system of differential equations, we need equations for $\frac{dx^{\mu}}{ds}$. Using $\lambda_{(0)}(s) = u(s)$, we arrive at the following first-order system of ordinary differential equations:

$$\left\{ \begin{array}{l} \frac{d\lambda^{\mu}_{(\alpha)}(s)}{ds} = -\Gamma^{\mu}_{\sigma\rho}\lambda^{\sigma}_{(\alpha)}(s)\lambda^{\rho}_{(0)}(s) + \lambda^{\mu}_{(\beta)}(s)A^{(\beta)}_{(\alpha)} \\ \frac{dx^{\mu}(s)}{ds} = \lambda^{\mu}_{(0)}(s) \end{array} \right\}. \tag{13}$$

It is easily checked that this system satisfies the condition for existence and uniqueness of solutions. The solutions are exactly the uniformly accelerated motions.

Note that (12) is a generalization of the geodesic equation. Along a geodesic, there is zero acceleration, so $A = 0$. Thus, (12) becomes

$$\frac{d\lambda^{\mu}_{(\alpha)}(s)}{ds} + \Gamma^{\mu}_{\sigma\rho}\lambda^{\sigma}_{(\alpha)}(s)\lambda^{\rho}_{(0)}(s) = 0. \tag{14}$$

Setting $\alpha = 0$ and using $\lambda_{(0)}(s) = u(s)$, we obtain

$$\frac{du^{\mu}(s)}{ds} + \Gamma^{\mu}_{\sigma\rho}u^{\sigma}(s)u^{\rho}(s) = 0,$$

which is the geodesic equation.

3 Schwarzschild metric

In this section, we provide an example of uniform acceleration in the *Schwarzschild metric*

$$ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2 d\Omega^2, \tag{15}$$

where r_s is the Schwarzschild radius and $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$. Here θ is the colatitude (= the angle from north) and φ is longitude. The known (see [8,9]) nonzero Christoffel symbols, computed from (9), are

$$\begin{aligned}
&\Gamma^{0}_{01} = \frac{r_s}{2r(r - r_s)}, \quad \Gamma^{1}_{00} = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right), \quad \Gamma^{1}_{11} = \frac{-r_s}{2r(r - r_s)}\\
&\Gamma^{1}_{22} = r_s - r, \quad \Gamma^{1}_{33} = (r_s - r)\sin^2\theta, \quad \Gamma^{2}_{12} = \frac{1}{r}\\
&\Gamma^{2}_{33} = -\sin\theta\cos\theta, \quad \Gamma^{3}_{13} = \frac{1}{r}, \quad \Gamma^{3}_{23} = \frac{\cos\theta}{\sin\theta}.
\end{aligned} \tag{16}$$

For our example, we consider motion in the (t, r) plane and set $\theta = \frac{\pi}{2}, \varphi = 0$. Then

$$\Gamma^{1}_{33} = r_s - r, \quad \Gamma^{2}_{33} = \Gamma^{3}_{23} = 0.$$

Let the acceleration matrix $A = \begin{pmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The upper equation of system (13), with $\alpha = 2$, $\mu = 0$ is

$$\frac{d\lambda^{2}_{(0)}}{ds} + \frac{2}{r}\lambda^{1}_{(0)}\lambda^{2}_{(0)} = \kappa\lambda^{2}_{(1)}.$$

Since $\theta = \frac{\pi}{2}$, we have $\lambda^2_{(0)} = \frac{d\theta}{ds} = 0$. Thus, $\lambda^2_{(1)} = 0$. Similarly, since $\varphi = 0$, we have $\lambda^3_{(0)} = \lambda^3_{(1)} = 0$.

Using the orthonormality conditions

$$\left(1 - \frac{r_s}{r}\right) \lambda^0_{(0)} \lambda^0_{(1)} - \left(1 - \frac{r_s}{r}\right)^{-1} \lambda^1_{(0)} \lambda^1_{(1)} = 0, \tag{17}$$

$$\left(1 - \frac{r_s}{r}\right) \left(\lambda^0_{(0)}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \left(\lambda^1_{(0)}\right)^2 = 1, \tag{18}$$

$$\left(1 - \frac{r_s}{r}\right) \left(\lambda^0_{(1)}\right)^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \left(\lambda^1_{(1)}\right)^2 = -1, \tag{19}$$

we can write $\lambda^0_{(0)}, \lambda^0_{(1)}, \lambda^1_{(1)}$ in terms of $\lambda^1_{(0)}$.

From (18), we get

$$\lambda^0_{(0)} = \sqrt{\left(\frac{r}{r - r_s}\right) \left(1 + \frac{r}{r - r_s} \left(\lambda^1_{(0)}\right)^2\right)}. \tag{20}$$

From (19), we get

$$\lambda^0_{(1)} = \sqrt{\left(\frac{r}{r - r_s}\right) \left(-1 + \frac{r}{r - r_s} \left(\lambda^1_{(1)}\right)^2\right)}. \tag{21}$$

Substituting (20) and (21) into (17), we get

$$\lambda^1_{(1)} = \sqrt{1 - \frac{r_s}{r} + \left(\lambda^1_{(0)}\right)^2}. \tag{22}$$

Substituting (22) into (21) yields

$$\lambda^0_{(1)} = \frac{r}{r - r_s} \lambda^1_{(0)}. \tag{23}$$

The equation for $\lambda^1_{(0)}$ from the system (13) is

$$\frac{d\lambda^1_{(0)}}{ds} + \frac{r_s}{2r^2} \left(1 - \frac{r_s}{r}\right) \left(\lambda^0_{(0)}\right)^2 - \frac{r_s}{2r(r - r_s)} \left(\lambda^1_{(0)}\right)^2 = \kappa \lambda^1_{(1)}. \tag{24}$$

Substituting (20) and (22) into this equation, the system (13) reduces to

$$\left\{ \begin{array}{l} \frac{d\lambda^1_{(0)}}{ds} + \frac{r_s}{2r^2} - \kappa \sqrt{1 - \frac{r_s}{r} + \left(\lambda^1_{(0)}\right)^2} = 0 \\ \frac{dr}{ds} = \lambda^1_{(0)} \end{array} \right\}, \tag{25}$$

or, equivalently,

$$\frac{d^2r}{ds^2}+\frac{r_s}{2r^2}-\kappa\sqrt{1-\frac{r_s}{r}+\left(\frac{dr}{ds}\right)^2}=0. \tag{26}$$

In the particular case $\kappa=0$, then, writing $\dot{r}$ for $\frac{dr}{ds}$ and $\ddot{r}$ for $\frac{d^2r}{ds^2}$, Eq. (26) becomes

$$\ddot{r}+\frac{r_s}{2r^2}=0. \tag{27}$$

Multiplying by $2\dot{r}$, we obtain

$$2\dot{r}\ddot{r}+\frac{\dot{r}r_s}{r^2}=0. \tag{28}$$

Integrating, we obtain

$$\dot{r}^2-\frac{r_s}{r}=\text{ constant.} \tag{29}$$

Hence, the total energy is conserved, as expected along a geodesic.

We consider now the general case of Eq. (26) ($\kappa\neq 0$). Define

$$E=\dot{r}^2-\frac{r_s}{r}. \tag{30}$$

The quantity E is the *total dimensionless energy*. It is the total energy divided by the maximal kinetic energy $\frac{mc^2}{2}$. Then

$$\dot{E}=2\dot{r}\ddot{r}+\frac{\dot{r}r_s}{r^2}. \tag{31}$$

Dividing by $2\dot{r}$ and using (26), we obtain

$$\frac{\dot{E}}{2\dot{r}}=\kappa\sqrt{1+E}. \tag{32}$$

Separating variables and integrating, we have

$$\sqrt{1+E}=\kappa r+C, \tag{33}$$

where C is a constant of integration. Squaring and using (30), we obtain

$$\dot{r}^2=\frac{r(\kappa r+C)^2+r_s-r}{r}. \tag{34}$$

We now show that there are no *bounded orbits*. Define

$$f(r)=r(\kappa r+C)^2+r_s-r. \tag{35}$$

To have a bounded orbit, say between r_1 and r_2, with $0<r_1<r_2$, we must have $f(r_1)=f(r_2)=0$ and $f(r)>0$ for $r_1<r<r_2$. However, $f(r)$ is a cubic

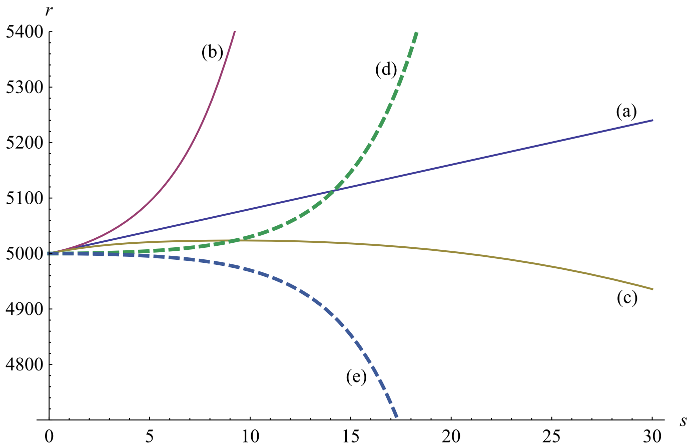

Fig. 1 Solutions for $r(s)$ starting at 5000 Schwarzschild radii, for (a) $\kappa = 0$, (b) $\kappa = 0.3$ compared to flat spacetime solution (d), (c) $\kappa = -0.3$ compared to flat spacetime solution (e)

polynomial, $f(0) > 0$ and $\lim_{r\to\infty} f(r) = +\infty$. This implies that f has at most two zeroes for $r > 0$ and between these two zeroes, $f(r) < 0$. Hence, there are no bounded orbits. See Fig. 1 for examples of solutions $r(s)$, compared to the corresponding solutions in flat spacetime.

4 Solutions via parallel transport

In this section, we explore an alternate method of obtaining solutions for uniform acceleration. For this, we need *parallel transport* and some additional properties of the covariant derivative.

Let $\gamma : I \to M$ be a worldline. For $s_1, s_2 \in I$, let $P_{s_1}^{s_2} : T_{\gamma(s_1)}M \to T_{\gamma(s_2)}M$ denote *parallel transport* from the tangent space at $\gamma(s_1)$ to the tangent space at $\gamma(s_2)$. Then, for $z \in T_{\gamma(s_1)}M$, we have

$$\frac{D}{ds}\left(P_{s_1}^{s}(z)\right) = 0. \tag{36}$$

We will also need the following properties of $\frac{D}{ds}$.

Theorem 1 ([7, 3.18])

(1) $\frac{D}{ds}(aZ_1 + bZ_2) = a\frac{DZ_1}{ds} + b\frac{DZ_2}{ds}$, *for* $a, b \in \mathbb{R}$
(2) $\frac{D}{ds}(fZ) = \frac{df}{ds}Z + f\frac{DZ}{ds}$, *for* $f \in \mathfrak{F}(I)$

Note that use of (1) and (2) implies that the Liebniz rule holds in the form

$$\frac{D}{ds}\left(\Sigma_{i=1}^{n} f_i Z_i\right) = \Sigma_{i=1}^{n}\left(\frac{df_i}{ds}Z_i + f_i\frac{DZ_i}{ds}\right). \tag{37}$$

Let $\gamma(s), s \in I$ be a uniformly accelerated worldline, with acceleration matrix A. We seek the orthonormal basis vectors $\lambda_{(\alpha)}(s)$ which solve Eq. (10). For each s, there is an additional basis $B(s)$ consisting of the initial basis vectors $\lambda_{(\alpha)}(0)$ parallel transported along γ from $s=0$ to s. To this end, we define

$$v_{(\alpha)}(s) = P_0^s \lambda_{(\alpha)}(0), \quad s \in I. \tag{38}$$

Note that $v_{(\alpha)}(0) = \lambda_{(\alpha)}(0)$. Since parallel transport is a linear isometry, the basis $B(s)$ is also orthonormal.

In this notation, the solution of Eq. (10) is

$$\boxed{\lambda_{(\alpha)}(s) = v_{(\beta)}(s)(\exp(As))^{(\beta)}_{(\alpha)}.} \tag{39}$$

We check now that this is, in fact, a solution of (10). First, by (37), we have

$$\frac{D\lambda_{(\alpha)}(s)}{ds} = \frac{D}{ds}(v_{(\beta)}(s))(\exp(As))^{(\beta)}_{(\alpha)} + v_{(\beta)}(s)\frac{d}{ds}(\exp(As))^{(\beta)}_{(\alpha)}. \tag{40}$$

By parallel transport (36), we have

$$\frac{D}{ds}(v_{(\beta)}(s)) = 0. \tag{41}$$

By the assumption of uniform acceleration $\left(\frac{dA}{ds} = 0\right)$, we have

$$\frac{d}{ds}\exp(As) = \exp(As)A. \tag{42}$$

Hence, using (41) and (42) in Eq. (40), we have

$$\frac{D\lambda_{(\alpha)}(s)}{ds} = v_{(\beta)}(s)(\exp(As))^{(\beta)}_{(\nu)}A^{(\nu)}_{(\alpha)} = \lambda_{(\nu)}(s)A^{(\nu)}_{(\alpha)},$$

and (10) holds.

In particular, the four-velocity of a uniformly accelerated observer is

$$u(s) = \lambda_{(0)}(s) = v_{(\beta)}(s)(\exp(As))^{(\beta)}_{(0)}. \tag{43}$$

The solutions (39) gives the basis vectors $\lambda_{(\alpha)}$ in terms of the $B(s)$ basis vectors $v_{(\beta)}$. In order to compute the $v_{(\beta)}$, we now derive a system of differential equations which they satisfy.

By (8) and parallel transport, we have

$$\frac{Dv^{\mu}_{(\alpha)}(s)}{ds} = \frac{dv^{\mu}_{(\alpha)}(s)}{ds} + \Gamma^{\mu}_{\sigma\rho}v^{\sigma}_{(\alpha)}(s)u^{\rho}(s) = 0. \tag{44}$$

From (43), we have

$$\frac{dx^{\rho}(s)}{ds} = u^{\rho}(s) = v^{\rho}_{(\beta)}(s)(\exp(As))^{(\beta)}_{(0)}. \tag{45}$$

Substituting this last equation into (44), we obtain the following first-order system:

$$\left\{ \begin{array}{l} \frac{dv^{\mu}_{(\alpha)}(s)}{ds} = -\Gamma^{\mu}_{\sigma\rho} v^{\sigma}_{(\alpha)}(s) v^{\rho}_{(\beta)}(s)(\exp(As))^{(\beta)}_{(0)} \\ \frac{dx^{\rho}(s)}{ds} = v^{\rho}_{(\beta)}(s)(\exp(As))^{(\beta)}_{(0)} \end{array} \right\}. \tag{46}$$

It is easily checked that this system satisfies the condition for existence and uniqueness of solutions. The solutions $v_{(\beta)}$ are then substituted into (39) to yield a solution to (10).

5 Directions for further research

We plan to extend the example in Schwarzschild spacetime to include planetary motion. One may consider the gravitational pull of, say, Jupiter, on the Earth to be uniform acceleration. A solution of our equations would then predict the perturbation on the Earth's orbit caused by Jupiter. Alternatively, one could predict the perturbation of the Moon's orbit around the Earth caused by the Sun.

In [4,6], working in flat spacetime, we derived spacetime transformations, velocity transformations, and acceleration transformations from a uniformly accelerated system to an inertial frame. Given the results obtained in the current paper in *curved* spacetime, the next step is to derive spacetime, velocity, and acceleration transformations between uniformly accelerated systems in a general curved spacetime. We want to determine whether the spacetime transformations between uniformly accelerated systems form a *group*. If yes, we want to characterize this group, which will be an extension of the Lorentz group.

We also propose to compute the time dilation between clocks located at different positions in a uniformly accelerated system. We hypothesize that a system is uniformly accelerated if and only if all of the clocks in the system may be synchronized to each other.

References

1. Mashhoon, B., Muench, U.: Length measurement in accelerated systems. Ann. Phys. **11**, 532–547 (2002)
2. de la Fuente, D., Romero, A.: Uniformly accelerated motion in general relativity: completeness of inextensible trajectories. Gen. Relativ. Gravit. **47**, 33 (2015)
3. de la Fuente, D., Romero, A., Torres, P.: Unchanged direction motion in general relativity: the problems of prescribing acceleration and the extensibility of trajectories. J. Math. Phys. **56**, 112501 (2015)
4. Friedman, Y., Scarr, T.: Uniform acceleration in general relativity. Gen. Relativ. Gravit. **47**, 121 (2015)
5. Friedman, Y., Scarr, T.: Making the relativistic dynamics equation covariant: explicit solutions for motion under a constant force. Phys. Scr. **86**, 065008 (2012)
6. Friedman, Y., Scarr, T.: Spacetime transformations from a uniformly accelerated frame. Phys. Scr. **87**, 055004 (2013)
7. O'Neill, B.: Semi-Riemannian Geometry. Academic Press, London (1983)

8. Misner, C., Thorne, K., Wheeler, J.: Gravitation. Freeman, San Francisco (1973)
9. Grøn, Ø., Hervik, S.: Einstein's General Theory of Relativity with Modern Applications in Cosmology. Springer, New York (2007)