Gen Relativ Gravit (2016) 48:44
DOI 10.1007/s10714-015-2009-y

GOLDEN OLDIE

Republication of: Contributions to the theory of gravitational radiation fields. Exact solutions of the field equations of the general theory of relativity V

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Published online: 18 March 2016

An editorial note to this paper and a biography can be found in this issue preceding this Golden Oldie and online via doi:10.1007/s10714-015-2008-z.

Original paper: Wolfgang Kundt and Manfred Trümper, Beiträge zur Theorie der Gravitations-Strahlungsfelder. Strenge Lösungen der Feldgleichungen der Allgemeinen Relativitätstheorie V, *Akademie der Wissenschaften und der Literatur, Abhandlungen der Mathematisch-naturwissenschaftliche Klasse* Nr 12 (1962). Reprinted with the kind permission of the Academy of Sciences and Literature, Mainz, and of the authors. © Akademie der Wissenschaften und der Literatur, Mainz 1963. Translated by the authors.

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ACADEMY OF SCIENCES AND LITERATURE

PROCEEDINGS OF THE
CLASS OF MATHEMATICS AND NATURAL SCIENCES
VOLUME 1962 · NO. 12

Contributions to the theory of gravitational radiation fields

by

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Hamburg

Exact solutions of the field equations of the general theory of relativity V

by

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PRESS OF THE
ACADEMY OF SCIENCES AND LITERATURE IN MAINZ
IN COMMISSION AT FRANZ STEINER VERLAG GMBH ·
WIESBADEN

Presented by Mr. Jordan at the plenary session on 27. July 1962,
approved for printing on the same day, published on 11. February 1963.

PRINTED BY: L. C. WITTICH, DARMSTADT

Contents

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[1]Editor's note: Page (4) in the original is completely blank and is omitted here. Note from the translator: Some elements of the notation used in the original text were modernised. Namely, the symbols $_|$ and $_{||}$ for partial and covariant derivatives were replaced by the comma and the semi-colon, respectively, and the symbol $\equiv$ for "equal by definition" was replaced by $A := B$ when A is defined by B, and by $A =: B$ when B is defined by A.

Chapter 1. Fields with matter

1. Field equations and Bianchi identities

The Einstein field equations (without cosmological term) are[1]

$$R^c_a - \frac{1}{2}R\delta^c_a = -T^c_a, \qquad (1.1.1)$$

where T^c_a is the energy-momentum tensor of the sources of the field.

For the investigation of properties of exact solutions of the gravitational field equations it is useful to consider not only the field equations, but also the Bianchi identities

$$R_{ab[cd;e]} = 0. \qquad (1.1.2)$$

Since these equations, written in the habitual form (1.1.2), are quite intransparent – though there are five indices, the equations (1.1.2) represent only 20 independent equations for the first derivatives of the Riemann tensor – we want to put them into a form which is more handy for practical calculations. For this purpose we recall the well-known decomposition of the Riemann tensor[2] into its components that are irreducible with respect to the Lorentz group. It is

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{R}{12}g_{abcd}. \qquad (1.1.3)$$

Here C_{abcd} is the Weyl tensor, $E_{abcd} = -g_{abe[c}S^e_{d]}$, ($S^e_d := R^e_d - \frac{R}{4}\delta^e_d$) is algebraically equivalent to the Ricci tensor.

With respect to the duality map the parts of the Riemann curvature tensor have the symmetries

$$^*C^*_{abcd} = -C_{abcd}, \quad ^*E^*_{abcd} = E_{abcd}, \quad ^*g^*_{abcd} = -g_{abcd}. \qquad (1.1.4)$$

Writing the Bianchi identities in the equivalent form $^*R^{*abcd}{}_{;d} = 0$ and inserting (1.1.3,4) we get

$$C^{abcd}{}_{;d} = E^{abcd}{}_{;d} - \frac{1}{12}g^{abcd}R_{,d}. \qquad (1.1.5)$$

[1] We are using units such that the velocity of light c and the Einstein gravitational constant are equal to 1.

[2] See e.g. (**1**), (**2**).

This equation implies by contraction[1]

$$\left(R^c_a - \tfrac{1}{2}R\delta^c_a\right)_{;c} = 0. \tag{1.1.6}$$

Eliminating E^{abcd} from (1.1.5) and using (1.1.6) we finally get

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6}g^{c[a}R^{,b]}. \tag{1.1.7}$$

Replacing the Ricci tensor by the energy-momentum tensor by using (1.1.1) we can write instead of (1.1.7)

$$C^{abcd}{}_{;d} = -T^{c[a;b]} + \frac{1}{3}g^{c[a}T^{,b]}. \tag{1.1.8}$$

From the last equation one gets by contraction the "conservation" law, equivalent to (1.1.6)

$$T^c_{a;c} = 0. \tag{1.1.9}$$

From (1.1.7) another important relation can be deduced. Taking the divergence with respect to x^c and using the Ricci identity we get

$$C^{abcd}{}_{;dc} = R^{c[a;b]}{}_{;c} = R^{c[a}{}_{;c}{}^{;b]} + R^{m[a}{}_{c}{}^{b]}R^c_m + R^{mc}{}_{c}{}^{[b}R^{a]}_m.$$

Each of the three terms to the right vanishes and we get the identity[2]

$$C^{abcd}{}_{;cd} = 0. \tag{1.1.10}$$

Because of (1.1.5) this equation is equivalent to

$$E^{abcd}{}_{;cd} = 0, \tag{1.1.11}$$

and thus it also follows

$$R^{abcd}{}_{;cd} = 0. \tag{1.1.12}$$

If we consider the Weyl tensor as the "free part" of the gravitational field, we can describe the meaning of equations (1.1.8), (1.1.9), (1.1.10) in this way: (1.1.9) are the equations of motion for the sources, (1.1.10) are differential equations for the free part of the field, and (1.1.8) give the interaction between the sources and the free part of the field. Here the metric field g_{ab} is considered as a kind of auxiliary field in which the "physical" fields are moving. Of course, it must not be forgotten that there is a reaction by the sources on the field.

[1] This is the Bianchi identity [translator].

[2] The identities (1.1.10) to (1.1.12) can be obtained equally fast from the Ricci identity alone.

2. Fields with matter

By field with matter we mean here a gravitational field created by an ideal fluid. This is described phenomenologically by the energy-momentum tensor

$$T_{ab} = \mu u_a u_b + p h_{ab}. \tag{1.2.1}$$

Hereby u_a (with $u_c u^c = -1$) is the four-velocity of matter, μ is its density, and p the (isotropic) pressure. As usual, $h_{ab} = g_{ab} + u_a u_b$ is the tensor which projects onto the space orthogonal to u^a.

From equation (1.1.9) follow the basic equations of hydrodynamics (the dot on a quantity means application of the operator $u^c\nabla_c$)

$$\dot{\mu} + (\mu + p)\theta = 0, \tag{1.2.2}$$

$$h^c_a p_{,c} + (\mu + p)\dot{u}_a = 0. \tag{1.2.3}$$

The kinematics of the fluid is described by the *kinematical quantities*[1] ω_{ab} (vorticity tensor), σ_{ab} (shear tensor), and θ (expansion scalar) which are defined by the identity

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b \tag{1.2.4}$$

and

$$\omega_{(ab)} = 0, \quad \sigma_{[ab]} = 0, \quad \theta = u^c_{;c}.$$

With (1.2.2,3,4) the Bianchi identities yield

$$\begin{aligned} C^{abcd}{}_{;d} &= \mu^{;[a}u^{b]}u^c + \tfrac{1}{3}\mu_{,d}h^{d[a}g^{b]c} \\ &-(\mu + p)\left[\omega^{ab}u^c - u^{[a}\omega^{b]c} + u^{[a}\sigma^{b]c}\right]. \end{aligned} \tag{1.2.5}$$

These relations show how the free part of the gravitational field is related to the hydrodynamical and kinematical properties of the flow of matter. The equations are useful for the investigation of fields with matter where assumptions are made either about the Weyl tensor or the movement of the fluid. It is remarkable that the acceleration vector of matter $\dot{u}^a$ does not explicitly occur in the Bianchi identities (1.2.5).

Applying the identity (1.1.10) to (1.2.5) and using equs. (1.2.2), (1.2.3) and the results of (**5**), chapter 1, paragraph 3, we get after some calculation

[1]For a more complete foundation of these quantities see (**5**) or (**6**).

$$\mu_{,c} h^{c[a} \dot{u}^{b]} + \left(\tfrac{2}{3}\dot{\mu} - \dot{p}\right) \omega^{ab}$$
$$+(\mu + p)\left(2\sigma_c{}^{[a}\omega^{b]c} - 2u^{[a}\omega^{b]c}\dot{u}_c - \dot{\omega}^{ab}\right) = 0. \tag{1.2.6}$$

We can simplify this equation by taking the dual in the space orthogonal to u^a. To this end we introduce $\beta_{ab} := \eta_{abcd}\dot{u}^c u^d$, the (skew) tensor of "acceleration". We get

$$\tfrac{1}{4}\beta_{ac}\mu^{,c} + (\mu + p)\left(h^c_a \dot{\omega}_c - \sigma^c_a \omega_c\right) - \left(\tfrac{2}{3}\dot{\mu} - \dot{p}\right)\omega_a = 0. \tag{1.2.7}$$

Here $\omega_a = \frac{1}{2}\eta_{abcd}u^b\omega^{cd}$ is the vorticity vector. If we further define the flow index F by

$$\frac{\dot{p}}{\mu + p} = \frac{\dot{F}}{F}$$

and the auxiliary quantity l by $\dot{l}/l = \frac{1}{3}\theta$ we get

$$\frac{1}{Fl^2}\left[h^c_a (Fl^2\omega_c)^{\cdot} - \sigma^c_a Fl^2\omega_c\right] = -\tfrac{1}{4}\beta_{ac}\frac{\mu^{,c}}{\mu + p}. \tag{1.2.8}$$

This formula is analogous to the equation of propagation of ω^a along the flow lines given by J. EHLERS in (**5**), chapter 1, paragraph 4. It means that, for a thin vorticity tube with cross section δS, the vorticity strength $F\omega\delta S$ is constant in time if and only if the space projection of the gradient of the density is collinear with the acceleration vector.

If the matter obeys an equation of state $\mu = \mu(p)$ the right hand side of (1.2.8) vanishes because of (1.2.3). In this case our formula reduces to the one given by EHLERS.

Equation (1.2.6) shows that special assumptions about the kinematics of the flow of matter impose a restriction of the admissible equation of state. E.g. for an ideal fluid with vanishing rotation the spatial projection of the density gradient is collinear with the acceleration vector because of (1.2.6) and (1.2.3) implies that only equations of state of the form $\mu = \mu(p, t)$ are admitted[1] (t is the universal time coordinate defined by $\dot{t}u_a = -t_{,a}$).

3. Type null fields with matter

The question whether gravitational fields of the null type can serve as models for the interaction between gravitational radiation and its sources that are physically interesting and useful for the theory of

[1]Cf. (**7**), p.38.

gravitational radiation can be decided only after one has succeeded in constructing a corresponding solution of the Einstein field equations.

Gravitational vacuum fields of type null have to be considered as the far fields of spatially bounded sources of radiation. If one has no bounded sources, but matter distributed continuously in space, it is conceivable that it gives rise to a null field. This view is supported by the fact that in the case of special movement of matter and special distribution of mass there are solutions of the field equations even with vanishing Weyl tensor (the Friedmann cosmological models).

We now want to investigate first what can be said about the movement of matter in a gravitational null field.

Let k^a denote the light-like eigendirection which is (up to a factor) uniquely determined by the Weyl tensor.

This tensor then has the form

$$C_{abcd} = 4C(V_{ab}V_{cd} - \overset{*}{V}_{ab}\overset{*}{V}_{cd}) \tag{1.3.1}$$

with

$$V_{ab} := k_{[a}\xi_{b]}, \quad k_c\xi^c = 0, \quad \xi_c\xi^c = 1.$$

ξ_a is a spacelike vector which is determined by the field up to null rotations.

We want to use the vectors k_a, u_a, and ξ_a to construct a pseudo-orthonormal tetrad. It will be convenient to impose the condition

$$k_a u^a = -1 \tag{1.3.2}$$

by using the time rotations ($k_a \to Ak_a$)). By a suitable null rotation ($\xi_a \to \xi_a + rk_a$) we then arrange

$$\xi_a u^a = 0. \tag{1.3.3}$$

We then define a null vector m_a by

$$m_a := \tfrac{1}{2}k_a - u_a \tag{1.3.4}$$

and, finally, we choose the space-like unit vector η_a which is orthogonal to k_a, m_a, and ξ_a. Then we have the orthogonality relations

$$k\cdot k = k\cdot\xi = k\cdot\eta = m\cdot m = m\cdot\xi = m\cdot\eta = \xi\cdot\eta = 0,$$

$$k\cdot m = \xi\cdot\xi = \eta\cdot\eta = 1.$$

In the following we will use the abbreviation $t_a := 1/\sqrt{2}(\xi_a + i\eta_a)$.

The tetrad is uniquely determined by the above conventions.

In order to get information on the movement of matter we shall exploit two sets of equations which follow from $C^{abcd}k_d = 0$:

$$\begin{aligned} C^{abcd}{}_{;d}k_c + C^{abcd}k_{c;d} &= 0 \\ C^{abcd}{}_{;d}k_b + C^{abcd}k_{b;d} &= 0. \end{aligned} \tag{1.3.5}$$

The terms to the left will be rewritten by use of the Bianchi identities (1.2.5) while to the right we replace $k_{a;b}$ by its decomposition with respect to the tetrad.[1] We are introducing the abbreviations

$$\gamma := k_{a;b}\bar{t}^a k^b$$

and

$$\sigma_{opt.} := k_{a;b}\bar{t}^a\bar{t}^b,$$

and note that γ vanishes if and only if k_a is geodesic. We further introduce the space-like vector

$$s^a := h^a_c k^c$$

which is the direction of the gravitational "ray" in the local rest frame of the fluid. Then, from (1.3.5) we get

$$(\mu + p)\omega^{ac} = 4\,C(\gamma s^{[a}\bar{t}^{c]} + \bar{\gamma} s^{[a}t^{c]}), \tag{1.3.6}$$

$$(\mu + p)\sigma^{ac} = \mu_{,b}s^b(s^a s^c - \tfrac{1}{3}h^{ac}), \tag{1.3.7}$$

$$h^a_c\mu^{,c} - \mu_{,c}s^c s^a = 6\,C(\gamma\bar{t}^a + \bar{\gamma}t^a), \tag{1.3.8}$$

$$\mu_{,c}s^c = 6\,C\sigma_{opt.}. \tag{1.3.9}$$

From (1.3.6) we derive an expression for the vorticity scalar ω

$$(\mu + p)\omega = \sqrt{8}\,C|\gamma|, \tag{1.3.10}$$

while (1.3.7) and (1.3.9) yield

$$(\mu + p)\sigma = 2\sqrt{3}\,C\sigma_{opt.}. \tag{1.3.11}$$

The last two equations contain the well-known result that in a vacuum field ($\mu = 0 = p$) of type null the vector k_a belongs to a null congruence which is geodesic and shear free.

The properties of the motion of matter which can be read directly from the equations (1.3.6) to (1.3.11) are summed up by

Theorem 1.3.1: In a matter field of type null (with $C_{abcd} \neq 0$) the flow of matter has the following properties:

[1]We use the identity $k_{a;b} = \delta^c_a\delta^d_b k_{c;d}$ with $\delta^a_b = k^a m_b + m^a k_b + t^a\bar{t}_b + \bar{t}^a t_b$ [translator].

I. The shear ellipsoid is a rotational ellipsoid whose longitudinal axis coincides with the spatial direction of the null ray.

II. The vorticity vector is an eigenvector of the shear tensor belonging to the degenerate eigenvalue.

III. The flow is irrotational if and only if the null ray is geodesic. For this to happen it is necessary and sufficient that the spatial direction of the ray coincides with the spatial projection of the density gradient.

IV. The shear vanishes if and only if the distortion of the rays (relative to the "observer" u_a) vanishes. This is the case if and only if the spatial direction of the ray is tangent to the hypersurfaces $\mu = \text{const}$.

With our results we can describe the case of conformally flat matter fields without further calculation. From (1.3.6,9) we infer: If the Weyl tensor of the field vanishes, i.e. if $C = 0$, the flow of matter is free of shear and rotation; the flowlines orthogonally intersect the hypersurfaces $\mu = \text{const}$. Using orthogonal coordinates with respect to these hypersurfaces we have $\mu = \mu(t)$. If the matter satisfies an equation of state $\mu = \mu(p, t)$ the flowlines are geodesics because of (1.2.3). From these facts it follows with use of the field equations that we have a Friedmann model[1].

4. Type null fields with irrotational and shearfree flow of matter

To approach the question of the existence of type null matter fields we start with a flow of particular kinematical simplicity. We assume $\omega_{ac} = 0$ and $\sigma_{ac} = 0$. According to Theorem 1.3.1 this means that the congruence of rays of the null field is geodesic and shear-free. Then, according to (1.3.8,9), we have $h^a_c \mu^{,c} = 0$. Therefore, the flow of matter satisfying an equation of state $\mu = \mu(p, t)$ is geodesic. The field equations then yield the Friedmann models, i.e. the Weyl tensor necessarily vanishes. We get **Theorem 1.4.1:** Matter fields of type null with irrotational and shear-free flow of matter satisfying an equation of state $\mu = \mu(p, t)$ are conformally flat.

[1]Cf. (**8**), (**9**).

5. Type null fields with a non-rotating fluid

We now turn to the question whether or not null fields with non-rotating flow of matter can exist. Since the calculations to treat this problem are somewhat complicated we shall only give a sketchy account of them. We shall use the following information on the field:

a) The field equations, which are, by well-known auxiliary formulae,[1] reduced to equations for the spatial sections (orthogonal to u^μ). Here we are using a modified form of those auxiliary formulae, by introducing the kinematical quantities into them. – If we choose coordinates referred to the hypersurfaces that are orthogonal to u^μ, then we can write the metric in the form

$$Q = g_{\mu\nu}(x^a)dx^\mu dx^\nu - V(x^a)^2 dt^2 = q - V(x^a)^2 dt^2, \qquad (1.5.1)$$

$$u^a = V^{-1}\delta^a_0, \;\; u_a = -V\delta^0_a.$$

We shall denote the covariant derivative with respect to the space-metric q by a colon in front of the differentiation index; the partial derivative with respect to t shall be denoted by a prime, e.g. $\theta' := \frac{\partial\theta}{\partial t}$. The Ricci tensor of q is denoted $\overline{R}^\lambda_\nu$.

The field equations $R^c_a = -(\mu + p)u_a u^c + \frac{1}{2}(p - \mu)\delta^c_a$, equivalent to (1.2.1), then read

$$\overline{R}^\mu_\nu + \frac{1}{V}V^{:\mu}{}_\nu - \left(\frac{1}{V}(\sigma^\mu_\nu)' + \theta\sigma^\mu_\nu\right) - \frac{1}{3}\left(\frac{1}{V}\theta' + \theta\right)\delta^\mu_\nu = \tfrac{1}{2}(p - \mu)\delta^\mu_\nu\ , \qquad (1.5.2)$$

$$\left(\sigma^\lambda_\nu - \tfrac{2}{3}\theta\delta^\lambda_\nu\right)_{:\lambda} = 0, \qquad (1.5.3)$$

$$\frac{1}{V}(V^{:\lambda}{}_\lambda - \theta') - 2\sigma^2 - \tfrac{1}{3}\theta^2 = \tfrac{1}{2}(3p + \mu). \qquad (1.5.4)$$

[1] (**1**), p. 53 and following.

b) The hydrodynamic and kinematic relations (1.2.2,3,6) and (1.3.6,7,8,10) which simplify, because of $\omega = 0$ and the ensuing relation $\gamma = 0$:

$$\frac{1}{V}\mu' + (\mu + p)\theta = 0, \tag{1.5.5}$$

$$p_{,\lambda} + (\mu + p)\frac{V_{,\lambda}}{V} = 0, \tag{1.5.6}$$

$$\sigma^{\nu}_{\lambda} = \sqrt{3}\sigma(s^{\nu}s_{\lambda} - \tfrac{1}{3}\delta^{\nu}_{\lambda}) \tag{1.5.7}$$

with

$$\sqrt{3}(\mu + p)\sigma = \mu_{,\lambda}s^{\lambda}, \tag{1.5.8}$$

$$\mu_{,\lambda} = \sqrt{3}(\mu + p)\sigma s_{\lambda}, \tag{1.5.9}$$

$$\mu_{,[\lambda}V_{,\nu]} = 0. \tag{1.5.10}$$

c) The "null field conditions", $C^{ab}{}_{cd}k^{d} = 0$. With help of the equations $k^{d} = s^{d} + u^{d}$ and $C^{ab}{}_{cd} = R^{ab}{}_{cd} - E^{ab}{}_{cd} - \frac{1}{12}Rg^{ab}{}_{cd}$ and the auxiliary equations in (**1**), pp. 53 and following, the null field conditions can be reduced to equations for the spatial hypersurfaces. We are presenting the final equations:

$\underline{C^{\mu\nu}{}_{\rho d}k^{d} = 0:}$

$$\overline{R}^{[\mu}_{\rho}s^{\nu]} + \delta^{[\mu}_{\rho}\overline{R}^{\nu]}_{\lambda} + \sigma^{[\mu:\nu]}_{\rho} + \tfrac{1}{3}\delta^{[\mu}_{\rho}\theta^{,\nu]} + \left(\tfrac{2}{3}\sigma^{2} - \tfrac{1}{3\sqrt{3}}\theta\sigma - \tfrac{1}{9}\theta^{2} + \tfrac{1}{3}\mu - \tfrac{1}{2}\overline{R}\right)\delta^{[\mu}_{\rho}s^{\nu]} = 0,$$

$\underline{C^{\mu\nu}{}_{0d}k^{d} = 0:}$

$$\sigma^{\rho}_{[\mu:\nu]}s_{\rho} + \tfrac{1}{3}s_{[\mu}\theta_{,\nu]} = 0,$$

$\underline{C^{0\nu}{}_{0d}k^{d} = 0:}$

$$\frac{1}{V}[V^{:\nu}{}_{\lambda} - (\sigma^{\nu}_{\lambda})']s^{\lambda} - \left(\frac{1}{3V}\theta' + \tfrac{4}{3\sqrt{3}}\theta\sigma + \tfrac{4}{3}\sigma^{2} + \tfrac{1}{9}\theta^{2} + \frac{\mu + 3p}{6}\right)s^{\nu} = 0,$$

$\underline{C^{0\nu}{}_{\rho d}k^{d} = 0:}$

$$2\left(\sigma^{\nu}_{[\rho:\lambda]} + \tfrac{1}{3}\delta^{\nu}_{[\rho}\theta_{,\lambda]}\right)s^{\lambda} = \frac{1}{V}\left(V^{:\nu}{}_{\rho} - (\sigma^{\nu}_{\rho})' - \tfrac{1}{3}\theta'\delta^{\nu}_{\rho}\right)$$

$$-\sigma^2\left(s^\nu s_\rho + \tfrac{1}{3}\delta^\nu_\rho\right) - \tfrac{2}{3}\theta\sigma^\nu_\rho - \left(\tfrac{1}{9}\theta^2 + \frac{\mu+3p}{6}\right)\delta^\nu_\rho.$$

d) The time derivative of σ (cf. (**5**), p.806)

$$\frac{1}{V}(\sigma^2)' = \frac{1}{V}V^{:\nu}{}_\lambda\sigma^\lambda_\nu - \tfrac{4}{3}\theta\sigma^2 - \tfrac{2}{\sqrt{3}}\sigma^3.$$

(The corresponding equation for θ is identical with the field equation $R^0_0 = \frac{1}{2}(\mu + 3p)$).

From these equations which, of course, are not independent of each other, the following equations can be extracted by substituting and combining.

$$\frac{1}{V}\left(V^{:\nu}{}_\lambda - \tfrac{1}{3}V^{:\mu}{}_\mu\delta^\nu_\lambda - (\sigma^\nu_\lambda)'\right)s^\lambda = \left(\tfrac{2}{3}\sigma^2 + \tfrac{4}{3\sqrt{3}}\theta\sigma\right)s^\nu, \tag{1.5.11}$$

$$\overline{R}^\nu_\lambda - \tfrac{1}{3}\overline{R}\delta^\nu_\lambda + \frac{1}{V}\left(V^{:\nu}{}_\lambda - \tfrac{1}{3}V^{:\mu}{}_\mu\delta^\nu_\lambda\right) = \frac{1}{V}(\sigma^\nu_\lambda)' + \theta\sigma^\nu_\lambda, \tag{1.5.12}$$

$$\overline{R} = -2\sigma^2 + \tfrac{2}{3}\theta^2 - 2\mu, \tag{1.5.13}$$

$$\theta_{,[\lambda}s_{\nu]} - \sqrt{3}\sigma_{,[\lambda}s_{\nu]} = 0, \tag{1.5.14}$$

$$\sqrt{3}\sigma s_{[\lambda:\nu]} + s_{[\lambda}\theta_{,\nu]} = 0, \tag{1.5.15}$$

$$\begin{aligned}\left[\frac{1}{V}V_{,\mu}s^\mu + \sqrt{3}\sigma\right]\Big[s^\nu{}_{:\lambda} - \tfrac{1}{2}s^\mu{}_{:\mu}(\delta^\nu_\lambda - s^\nu s_\lambda)&\\ + \tfrac{1}{\sqrt{3}\sigma}(\theta_{,\lambda} - \theta_{,\mu}s^\mu s_\lambda)s^\nu\Big] = 0.&\end{aligned} \tag{1.5.16}$$

Now the Weyl tensor becomes (because of its symmetry we need only to give the components listed below)

$$C^{0\nu}{}_{\mu\lambda} = -2\sqrt{3}\frac{\sigma}{V}\left(s^\nu{}_{:[\mu}s_{\lambda]} - \tfrac{1}{2}s^\rho{}_{:\rho}\delta^\nu_{[\mu}s_{\lambda]}\right), \tag{1.5.17}$$

$$\begin{aligned}C^{0\nu}{}_{0\lambda} = -\sqrt{3}\sigma\Big[s^\nu{}_{:\lambda} - \tfrac{1}{2}s^\mu{}_{:\mu}(\delta^\nu_\lambda - s^\nu s_\lambda)&\\ - \tfrac{1}{\sqrt{3}\sigma}(\theta^{,\nu} - \theta_{,\mu}s^\mu s^\nu)s_\lambda\Big].&\end{aligned} \tag{1.5.18}$$

We now want to consider the case where the irrotational flow is geodesic. Since now we can choose the Gaussian coordinates with respect to the hypersurfaces which are orthogonal to the worldlines of the flow, the condition is expressed by $V = 1$. Because of theorem 1.4.1 we have to assume that $\sigma \neq 0$ and then it follows from (1.5.16) that $s^{\nu}{}_{:\lambda} = \frac{1}{2} s^{\mu}{}_{:\mu}(\delta^{\nu}_{\lambda} - s^{\nu} s_{\lambda})$ and $s_{[\nu}\theta_{,\lambda]} = 0$. Now (1.5.17,18) imply that the Weyl tensor vanishes. We have proved

Theorem 1.5.1: Gravitational fields of type null with a non-rotating inertial flow of matter do not exist.

If the matter is free of pressure, the absence of rotation is sufficient for the above statement to be true; the assumption of an inertial flow is necessary only for matter with pressure.

As the conclusion just drawn from (1.5.16) does not lead to anything of interest, the only way to satisfy this equation is by

$$\frac{1}{V} V_{,\lambda} s^{\lambda} + \sqrt{3}\sigma = 0.$$

From $V_{,[\lambda} s_{\nu]} = 0$[1] it follows $\frac{1}{V} V_{,\lambda} = -\sqrt{3}\sigma s_{\lambda}$ and (1.5.6) then reads

$$p_{,\lambda} = \sqrt{3}(\mu + p)\sigma s_{\lambda}.$$

Comparing this result with (1.5.9) we realize that the equation of state $\mu = \mu(p,t)$ would need to take the special form $\mu = p + A(t)$. Such an equation of state can be discarded for physical reasons since the pressure could not have the same order of magnitude as the rest energy density. Thus we see that an irrotational ideal fluid cannot give rise to a gravitational field of type null. Should a gravitational null field exist in the presence of matter, kinematically more complicated flows would have to be invoked.

[1] with equs. (1.5.9,10) [translator].

Chapter 2. The twist-free pure radiation fields

1. Introduction

In this chapter, those (coordinate-independent!) statements are to be compiled which we can make (so far) about pure radiation fields in finite domains. For their *definition*, we choose the existence of a bundle of rays (see section 2.4), which provides distortion-free maps in the small; this definition is formed in analogy to Maxwell's theory and turns out to be an extremely useful formal criterion. For Maxwellian and Jordanian vacuum fields we postulate in addition that the respective field quantities are adapted to the ray bundle; i.e. that the additional fields have the same rays as the metric field. (Only for this case, transparent statements emerge). In the third section we can show that the above definition is broadly equivalent with the definition proposed by Pirani, based on the algebraic shape of the conformal, or Weyl, tensor, which likewise imitates the Maxwellian theory and intuitively states that in the case of radiation, a privileged flow of intensity in a preferred lightlike direction is described by the conformal tensor; cf. (**3**), p. 184. This definition will also be strengthened by further statements, which can be made for expansion-free radiation fields.

Sections 2.5 and 2.6 give a rough survey of the formal methods, with which a number of (local) properties of the twist-free pure radiation fields can be proven in the last section.

As concerns the foundations, we refer the reader to our paper II, whose notations and results will be currently used. Further proofs suppressed here can be found in the publications (**17**), (**18**), as well as (**21**).

2. Geodetic null congruences

This section will refer to the most important formal and intrinsic properties of a congruence of geodetic null lines. We shall make use of a more extensive presentation in our earlier paper II, Chapter 1.

Let l_a be the geodetically gauged tangent vector of the considered null geodesics: $l^a l_a = 0 = \overset{l}{\nabla} l_a$, with $\overset{\xi}{\nabla} := \xi^a \nabla_a$ for an arbitrary vector ξ^a. Its gradient being decomposed w.r.t. a (complex) null tetrad adapted to the congruence[1] $\{t_a, l_a, m_a\}$:

$$g_{ab} = 2\{t_{(a}\bar{t}_{b)} + l_{(a}m_{b)}\}, \tag{2.1}$$

l_a, m_a real, t_a complex, $t_a\bar{t}^a = 1 = l_a m^a$, all other scalar products equal to zero; we get

$$l_{a;b} = 2\ Re\ \{z t_a \bar{t}_b + \sigma t_a t_b + \Omega t_a l_b + \zeta l_a t_b\} + \beta l_a l_b. \tag{2.2}$$

We call the expansion coefficients $\theta, \omega, (z := \theta + i\omega), \sigma, \Omega, \zeta, \beta$ occurring herein "optical scalars". Such a congruence shall be called a "*ray congruence*" iff its distortion $|\sigma|$ vanishes; it shall be called "twist-free" iff l_a is hypersurface-normal.

Twisted bundles of electromagnetic radiation can be generated experimentally by cables made of fiberglass; all other standard optical equipment works with twist-free ray bundles.

We further denote by *wave hypersurface* every (lightlike) hypersurface which intersects all rays perpendicularly, and by *wave surface* every spacelike 2-surface with the same property. Wave surfaces and wave hypersurfaces exist for a congruence iff it is twist-free: the vanishing of the twist ω is a necessary integrability condition for the existence of spacelike 2-surfaces orthogonal to l_a:

$$0 \stackrel{!}{=} l_a(\overset{t}{\nabla}\bar{t}^a - \overset{\bar{t}}{\nabla}t^a) = -2i\omega;$$

conversely, $\omega = 0$ is clearly sufficient. We remark in passing that with this notation, the spheres $(r, t) = \text{const}$ of Schwarzschild's vacuum fields are (frozen) wave surfaces.

Locally, each spacetime contains twist-free null congruences. They can be described by their *phase*, or *retarded time* u, defined by $u_{,a} := l_a$; (l_a is automatically normalised geodetic). u is unique up to arbitrary monotonic transformations:

$$\overline{u} = f(u), \quad f'(u) \neq 0. \tag{2.3}$$

We now prove

Theorem 2.1: For an expansion-free ($\theta = 0$) geodetic null congruence with $R_{ab}l^a l^b = 0$, being twist-free is equivalent with being distortion-free i.e. in particular: a twisted ray congruence must expand[2].

[1] Also called "Sachsbein".

[2] The word "expansion" is used for both signs of θ.

Proof: From the Ricci identity

$$2\xi_{a;[bc]} = -R^{d}{}_{abc}\xi_d, \tag{2.4}$$

follows, for $\xi_a = l_a$ (by transvection with $g^{ab}l^c$ and partial differentiation):

$$\dot{z} + z^2 + |\sigma|^2 = 1/2\ R_{cd}l^c l^d. \tag{2.5}$$

Under our assumptions, the real part of this equation reads:

$$\omega^2 = |\sigma|^2.$$

We finally mention the easy-to-grasp[1]

Theorem 2.2: The existence of a twist-free and expansion-free ray congruence is equivalent with the existence of a real solution u of the equation

$$\nabla_a^a e^{iu} = 0.$$

3. Field equations and ray congruence

In the present chapter, we want to consider electromagnetic gravitational fields with variable scalar of gravitation κ, which satisfy the *Jordan-Maxwell vacuum field equations.* They are obtained as Lagrange equations for the function

$$L = \kappa^{-1}R - \zeta\kappa^{-3}\kappa^{,c}\kappa_{,c} - \frac{1}{2}F^{ab}F_{ab} \tag{3.1}$$

under variation w.r.t. the 15 potentials κ, g_{ab}, and Φ_c (with $F_{ab} = 2\Phi_{[b,a]}$). One finds

$$\left\{\begin{aligned} &\text{(a) } R + \zeta(3\kappa^{-2}\kappa^{,c}\kappa_{,c} - 2\kappa^{-1}\kappa^{;c}{}_{c}) = 0 \\ &\text{(b) } G_{ab} + (2-\zeta)\kappa^{-2}\kappa_{,a}\kappa_{,b} - \kappa^{-1}\kappa_{;ab} + \\ &\qquad \left[\left(\tfrac{\zeta}{2} - 2\right)\kappa^{-2}\kappa^{,c}\kappa_{,c} + \kappa^{-1}\kappa^{;c}{}_{c}\right] g_{ab} = -\kappa E_{ab} \end{aligned}\right\} \tag{3.2}$$

$$\Phi_{[ab,c]} = 0 \tag{3.3}$$

with $G_{ab} := R_{ab} - \frac{R}{2}\, g_{ab}$, $\Phi_{ab} := F_{ab} + i\overset{*}{F}_{ab}$, $2E_{ab} = \Phi_{ac}\overline{\Phi}_b{}^{c}$. For $\zeta \neq 3/2$, the equations (3.2) can be written more simply:

[1] See (**10**).

$$\left\{\begin{array}{l}\text{(a) } (\kappa^{-1})^{;c}{}_{c} = 0 \\ \text{(b) } R_{ab} + \kappa E_{ab} = \left\{\begin{array}{lll}\lambda_{;ab} & \text{for} \quad \zeta = 1, & \lambda := -\ln\kappa \\ \lambda_{;ab}/(\zeta-1)\lambda & \text{for} \quad \zeta \neq 1, & \lambda := -\kappa^{\zeta-1}.\end{array}\right.\end{array}\right\} \tag{3.4}$$

Even simpler become Jordan's field equations under Schücking's conformal transformation: $\widetilde{g}_{ab} = \kappa^{-1} g_{ab}$, $\zeta \neq 3/2 \implies$

$$\widetilde{R}_{ab} + \kappa E_{ab} = \left(\zeta - \frac{3}{2}\right) \kappa^{-2} \kappa_{,a} \kappa_{,b}. \tag{3.5}$$

The 10 equations (3.5) are equivalent with the 11 equations (3.4) (which are not all independent, because of the Bianchi identity) and have a quite simple structure so that it appears appropriate to work with the conformal metric $\widetilde{g}_{ab}$ in all formal operations. The field equations (3.3) with (3.4) or (3.5) change for $\kappa = \text{const}$ into the *Einstein-Maxwell vacuum field equations*, which are thus absolutely included into our considerations.

Even though with our present state of insight we consider it at least early to try and define radiation fields also inside of matter, let us weaken the field equations (3.5) for just one theorem by complementing the electromagnetic energy-momentum tensor E_{ab} by a tensor T_{ab} describing possible matter, with the only constraint that it possess a timelike eigenvector and negative trace; these properties hold for every physically admissible energy-momentum tensor, see e.g. IV. With such an ansatz we claim, for $\kappa = \text{const}$:[1]

Theorem 3.1: The existence of a twist-free and expansion-free null congruence in a material world with $\kappa = \text{const}$ implies that 1. the congruence is a ray congruence ($\sigma = 0$), 2. a possible Maxwell field possesses the congruence as eigencongruence, and 3. the material content (described by T^a_b) necessarily vanishes.

Proof: Under our assumptions, (2.5) transforms to

$$\begin{aligned} 2|\sigma|^2 &= -\kappa(E_{ab} + T_{ab}) l^a l^b \\ &= -\kappa(l^a F_{ac} l^b F_b{}^c + l^a \overset{*}{F}_{ac} l^b \overset{*}{F}_b{}^c + T_{ab} l^a l^b); \end{aligned} \tag{3.6}$$

here the right-hand side is ≤ 0. For 1. $l^a F_{ca}$ and $l^a \overset{*}{F}_{ac}$ are real vectors perpendicular to l^a and hence spacelike or parallel to l^a. And 2. a symmetrical tensor T_{ab} with timelike eigenvector can be diagonalised simultaneously with the metric; expand l^a w.r.t. such a simultaneous

[1] Compare (**10**).

(orthonormal) eigentetrad of g_{ab} and T_{ab}, then follows $T_{ab}l^a l^b > 0$ from the trace condition $T^a_a < 0$. All statements can now be read off from (3.6).

The theorem just proven permits us to some extent to assume $T^a_b = 0$ in the future. Unfortunately, the case of a variable scalar of gravitation κ cannot be incorporated without additional assumptions into the above propositions; nevertheless, we want to go on occupying ourselves with it and prove

Theorem 3.2: In a Jordan-Maxwellian vacuum field (3.3), (3.5) with $\kappa_{,a} \neq 0$, $\zeta \neq 3/2$, the existence of an expansion-free ray congruence with $E_{ab}\kappa^{,a}\kappa^{,b} = 0$ follows from each of the subsequent three equations:
1) $\kappa^{,a}\kappa_{,a} = 0$, 2) $\kappa^{;a}{}_a = 0$, 3) $\widetilde{R} = 0$, as well as 4) from $R = 0$, if $\zeta \neq 0$.

Proof: First one concludes in each case, using (3.4a) and $\nabla^a_a f(\kappa) = f'\kappa^{;a}{}_a + f''\kappa^{,a}\kappa_{,a}$ that $\kappa^{,a}\kappa_{,a} = 0 = \kappa^{;a}{}_a$; i.e. κ is the phase of a twist-free and expansion-free null congruence. Now follows $R_{ab}\kappa^{,a}\kappa^{,b} \leq 0$ from (3.4b) because of $\kappa^{,a}\lambda_{;ab}$ and $E_{ab}\kappa^{,a}\kappa^{,b} \geq 0$, cf. proof of theorem 3.1. Insertion into (2.5) yields $\sigma = 0 = E_{ab}\kappa^{,a}\kappa^{,b}$ as claimed.

As a remarkable consequence from the theorem just proven results that there is no field with $\kappa^{,a}\kappa_{,a} = 0 \neq \kappa^{;a}{}_a$, for which κ would propagate at the speed of light along expanding rays. One should therefore expect that short-lasting spatial inhomogeneities of the κ field relax at a *speed* which *reaches the speed of light only at separations* from the source.

Of importance for our study of the connections between the different fields and the existence of a ray congruence is also the following theorem first proven in (**11**), (**12**) and then more systematically in II, theorems 3.2.1, 3.3.2:

Theorem 3.3: The congruence determined by the lightlike eigendirection of a null bivector obeying Maxwell's equations (3.3) is a ray congruence, and conversely, for every ray congruence there is a Maxwellian null field belonging to it.

A further deep insight yields the following theorem, whose proof one can find in (**13**), and which generalises the theorem of Goldberg and Sachs (**14**), which says that there exists a ray congruence in a vacuum field if and only if the Riemann tensor is of special type:

Theorem 3.4: The two following properties of a world domain:

(A) The conformal tensor is special

(B) There exists a ray congruence

are equivalent iff there is a complex lightlike bivector V_{ab} with:

$$(C)\quad \left\{\begin{array}{ll} V^{ea}V^{bc}C^{d}{}_{abc;d}=0, & V_{ab}=-i\overset{*}{V}_{ab},\quad V_{ab}k^{b}=0,\\ V^{bc}C^{d}{}_{abc;d}=0 & \text{for } C_{abcd} \text{ of type III,}\\ V^{ea}C^{d}{}_{abc;d}=0 & \text{for } C_{abcd} \text{ of type N.}\end{array}\right\}$$

Here k^b denotes a multiple (lightlike) eigendirection of C_{abcd} when (A) holds true, or the tangential vector of the congruence when (B) applies, respectively. The last two lines of (C) are only necessary for the conclusion (A) $\Longrightarrow$ (B).

In order to illuminate the range of applicability of this theorem, which establishes an important connection between two very different approaches to radiation theory, we prove

Lemma 3.5: The property (C) of theorem 3.4 follows from (B) if the Ricci tensor possesses the vector k_a as eigenvector, and if it satisfies the following two further (real) equations $R_{ab}t^at^b=0$. Consequently, (A) follows from (B) in particular for Einstein-Maxwell fields whose eigencongruence is a ray congruence.

Proof: Because of Bianchi's identity

$$C^{dabc}{}_{;d}=-P^{a[b;c]},\quad P^a_b=R^a_b-\frac{R}{6}\delta^a_b, \tag{3.7}$$

(C) from theorem 3.4 changes into[1].

$$V^{ea}V^{bc}R_{a[b;c]}=0. \tag{3.8}$$

As usual, we put $V_{ab}=2k_{[a}\bar{t}_{b]}$ and complete k_a, t_a by m_a to a null tetrad. Then by assumption, the Ricci tensor is the real part of a complex linear combination of terms of the form

$$k_ak_b,\quad k_{(a}t_{b)},\quad t_{(a}\bar{t}_{b)},\quad k_{(a}m_{b)} \tag{3.9}$$

and the only surviving terms after formation of (3.8), partial differentiation and use of the orthogonality relations (2.1) have the form

$$k_{a;b}\bar{t}^a\bar{t}^b \quad\text{and}\quad k_{a;b}\bar{t}^ak^b,$$

hence are proportional to σ and to a scalar γ which measures the deviation from geodesy. Consequently, when (B) holds, (3.8) holds likewise.

A deeper lying reverse of lemma 3.5 has been found by Trümper:

Theorem 3.6: When for an Einstein-Maxwell field, the conformal tensor is special and its multiple eigendirection is simultaneously an

[1]Only the first line of (C) must be proven: the others follow from theorem 3.4.

eigendirection of the Maxwell bivector, then this eigendirection describes a ray congruence; whereby in the proof, a special value of the ratio of the eigenvalues of $C^{ab}{}_{cd}C^{cd}{}_{ef}$ and E^a_b must be excluded.

Remark: The theorem holds also for Jordan-Maxwell fields when the gradient of κ spans the preferred congruence.

For the sake of transparency, we leave out in our proof the easy generalisation formulated in the remark. Then the Bianchi identity (3.7) reads in spinor notation, cf. (**3**) (2.9):

$$\Gamma_{ABCD}{}^{;D}{}_{\dot{E}} = E_{AB\dot{E}\dot{F};C}{}^{\dot{F}}, \tag{3.10}$$

where the field equations $G_{ab} = -E_{ab}$ were used. (As in II, Γ_{ABCD} denotes the totally symmetrical conformal spinor; spinor equivalents of tensors are denoted with the same core symbol). According to (3.2) we have:

$$2E_{AB\dot{E}\dot{F}} = \Phi_{AB}\overline{\Phi}_{\dot{E}\dot{F}}, \tag{3.11}$$

where $\Phi_{A\dot{C}B\dot{D}} := \Phi_{AB}\varepsilon_{\dot{C}\dot{D}}$; and Φ_{AB} satisfies, by assumption, the Maxwell equations (3.3):

$$\Phi_{AB}{}^{;B\dot{C}} = 0. \tag{3.12}$$

As the case of a null field has already been covered by theorem 3.3, we now restrict ourselves to non-null fields, and set:

$$\Phi_{AB} = -2\varphi\kappa_{(A}\mu_{B)} \tag{3.13}$$

with normalised κ_A, μ_A: $\kappa_A\mu^A = 1$; let κ_A be the multiple eigendirection of Γ_{ABCD}:

$$\kappa^A\kappa^B\Gamma_{ABCD} = -\gamma\kappa_C\kappa_D. \tag{3.14}$$

Application of $\kappa^C\nabla^D_{\dot{E}}$ to (3.14) yields

$$\kappa^A\kappa^B\kappa^C\Gamma_{ABCD}{}^{;D}{}_{\dot{E}} = 3\gamma\kappa^C\kappa^D\kappa_{C;D\dot{E}}. \tag{3.15}$$

Insertion of (3.10) with (3.11), (3.12) leads to

$$\frac{1}{2}\kappa^A\kappa^B\kappa^C\Phi_{AB;C}{}^{\dot{F}}\overline{\Phi}_{\dot{E}\dot{F}} = 3\gamma\kappa^C\kappa^D\kappa_{C;D\dot{E}}. \tag{3.16}$$

Application of the operator $\kappa^A\kappa^B\nabla^{\dot{F}}_C$ to (3.13) yields, on the other hand,

$$\kappa^A\kappa^B\Phi_{AB;C}{}^{\dot{F}} = 2\varphi\kappa^A\kappa_{A;C}{}^{\dot{F}}, \tag{3.17}$$

so that by insertion into (3.16) one obtains:

$$-2|\varphi|^2\overline{\kappa}_{(\dot{E}}\overline{\kappa}_{\dot{F})}\kappa^A\kappa^C\kappa_{A;C}{}^{\dot{F}} = 3\gamma\kappa^C\kappa^D\kappa_{C;D\dot{E}}. \tag{3.18}$$

Transvection of this equation with, respectively, $\overline{\kappa}^{\dot{E}}$, $\overline{\mu}^{\dot{E}}$ yields:

$$\left.\begin{cases} \text{(a) } (|\varphi|^2 + 3\gamma)\kappa^A\kappa^C\kappa^{\dot{E}}\kappa_{A;C\dot{E}} = 0 \\ \text{(b) } (|\varphi|^2 - 3\gamma)\kappa^A\kappa^C\overline{\mu}^{\dot{E}}\kappa_{A;C\dot{E}} = 0 \end{cases}\right\}, \tag{3.19}$$

so that for $|\varphi|^2 \neq 9\gamma^2$, the condition II (3.2.3) of geodesy and absence of distortion results:

$$\kappa^A\kappa^C\kappa_{A;C\dot{E}} = 0, \tag{3.20}$$

q.e.d..

4. Definition of pure radiation fields and classification

As already remarked in the introduction, the definition of gravitational waves requires an extension of the term *pure radiation* from other areas of physics, which is suggestive, but by no means "determined by correspondence". We choose the

Definition 4.1. An Einstein-Maxwell vacuum domain is called a *pure radiation field* when in it there is a ray congruence (= distortion-free geodetic null congruence), which is simultaneously an eigencongruence of the Maxwell bivector. (The case $R_{ab} = 0$ is considered a special case of this).

Theorems 3.4 to 3.6 have taught us that, cum grano salis, this definition is equivalent to the original one given by Pirani in (**15**). We prefer it for formal and intrinsic reasons.

Which fields are covered by our condition? For orientation we consider electromagnetic fields in flat spacetime. The well-known formula for retarded field strengths shows that an arbitrary pointlike distribution of charges and currents leads to "pure radiation fields" in our sense. Their common property is that the space projection of the (lightlike) eigendirection of the field strength points from the source to the point considered. In particular, static Coulomb fields are not excluded. We could exclude them because of the existence of a timelike symmetry group, but, for formal reasons, we consider this inappropriate; we keep them as "frozen waves". Simplest gravitational representatives of this kind are the spherically symmetric Schwarzschild metrics.

Open remains the question of what additional properties one should demand for a *pure Jordanian radiation field.* Here theorem 3.2 and the remark to theorem 3.6 teach us that, evidently, the following postulate appears appropriate:

Definition 4.2. A Jordan-Maxwell vacuum domain is called a *pure radiation field* when it possesses the properties of Definition 4.1 and the gradient of κ is lightlike.

We can immediately classify the thus defined radiation fields coarsely in terms of the invariant properties of their radiation bundles: the vanishing of ω characterises *twist-free* bundles, that of θ *expansion-free* ones; and inside the last mentioned class, $\Omega = 0$ is a characterising property for *parallel rays*, (i.e. for radiation whose direction appears spacelike and timelike constant to all observers). A finer classificatioin is possible, among others, by the type of the conformal tensor. In the following sections, we will strengthen and enlighten the given partition into classes by different properties.

5. Canonical coordinates of a twist-free ray congruence

When in some world domain, a twist-free null congruence with phase u is specified – such a congruence always exists locally – one can introduce two space-like coordinates x^A with $x^A{}_{,a}u^{,a} = 0$. Setting $(x^a) := (x^A, s, u)$ implies $g^{a4} = 0$ for $a \neq 3$, and because of $g^{4a}g_{ab} = \delta^4_b$, the metric fundamental form takes the form

$$G = g_{AB}dx^A dx^B + 2m_a dx^a du; \tag{5.1}$$

$(l_a := u_{,a})$. The wave hypersurfaces are described by $u = \text{const}$; they contain the wave surfaces $s = s(x^A)$. For the expansion θ one easily gets

$$\theta = \overset{l}{\nabla} \ln r \quad \text{with} \quad r := (det(g_{AB}))^{1/4}. \tag{5.2}$$

At this point, a first distinction of cases is required. For $\theta \neq 0$, (the "*Sachs parameter*") r yields the distance from the source judged by local measurements due to the rays' expansion; see e.g. II section 3.3, or (**17**) page 83; we specialise s to r. For $\theta = 0$, on the other hand, it is useful to specialise s as an affine parameter $(dx^a/ds \overset{!}{=} u^{,a})$ and to again call it r. We present the further possible specialisations of the coordinates under additional assumptions in the form of:[1]

[1] $\partial^A = p^{-2}\partial_A$.

Table 5.1

1) $\theta \neq 0$	$g_{AB} = r^2 h_{AB}, \quad \partial_r det(h_{AB}) = 0, \quad m_3 = \frac{1}{\theta r}$
$\sigma = 0$	$h_{AB} = p^2 \delta_{AB}, \quad \partial_r p = 0$
$R_{ab} l^a l^b = 0$	$m_3 = 1$
$R_{ab} l^a t^b = R_{ab} t^a t^b = 0$	$m_A = 0$
$R_{ab} t^a \bar{t}^b = 0$	$m_4 = r\partial_u \ln p + \frac{1}{2}\partial^A \partial_A \ln p - r^{-1} m(x, y, u)$
2) $\theta = 0$	$\partial_r det(h_{AB}) = 0, \quad m_3 = 1$
$\sigma = 0 (\Longleftrightarrow R_{ab} l^a l^b = 0)$	$g_{AB} = p^2 \delta_{AB}, \quad \partial_r p = 0$
$R_{ab} l^a t^b = 0$	$\partial_r{}^2 m_A = 0$
a) $\partial_r m_A \neq 0$,	
$R_{ab} l^a t^b = 0$	$\partial_r m_A = p^2 (m_{,1}{}^2 + m_{,2}{}^2)^{-1} m_{,A}, \quad \partial^A \partial_A m = -\frac{1}{2}$
$R_{ab} t^a \bar{t}^b = 0$	$(m^{,B} \partial_B - 1)(m^{,A} m_{,A})^{-1} = 2\partial^A \partial_A \ln p$
$R_{ab} l^a m^b = 0$	$\partial_r{}^2 m_4 = (4m^{,A} m_{,A})^{-1} - \partial^A \partial_A \ln p$
$III, N, 0$	$p = 1, \quad m_1 = -2\kappa^{-1} r + n_1(x^A, u), \quad m_2 = 0$
$N, 0$	$n_1 = 0$
b) $\partial_r m_A = 0$,	
$R_{ab} t^a \bar{t}^b = 0$	$p = 1, \quad m_2 = 0$
$R_{ab} l^a m^b = 0$	$\partial_r{}^2 m_4 = 0$
$N, 0$	$m_A = 0$

Please note: 1) $\partial_r m_A = 0$ is equivalent to the property $\Omega = 0$ of parallel rays.[1] 2) For $\sigma = 0$, the Gaussian curvature K of the wave surfaces is given by

$$K = \begin{Bmatrix} -r^{-2}\partial_A \partial_A \ln p & \text{for } \theta \neq 0 \\ -\partial^A \partial_A \ln p & \text{for } \theta = 0 \end{Bmatrix} \tag{5.3}$$

hence vanishes for $p = 1$. 3) The symbols II, D, III, N, 0 denote the types of the conformal tensor. 4) As soon as a ray congruence is uniquely marked as an eigencongruence to the conformal tensor – in theorem 3.4 we have given characterising conditions; the possible ambiguity for type D can be ignored for the present considerations – u is uniquely fixed up to monotonic transformations. One then sees easily that for $\sigma = 0$ precisely the following *gauge transformations* conserve the normal form of the metric (we put $x^1 + ix^2 = z$):

$$\text{I } \overline{u} = f(u), \;\; \overline{r} = r f'^{-1}(u), \quad \overline{z} = z \;\; \text{(change of phase)}$$

$$\text{II } \overline{z} = F(z; u), \;\; F \text{ analytic in } z, \quad \overline{r} = r, \;\; \overline{u} = u$$

[1] Author's note (2013): This equivalence holds for $\theta = 0$ only.

$$\text{(conformal transformations in wave surfaces)} \tag{5.4}$$

$$\text{III } \overline{r} = r + g(x,y,u), \;\; \overline{z} = z, \;\; \overline{u} = u \text{ for } \;\theta = 0$$

$$\text{(translation of the affine origin on the rays)}$$

To conserve $m_A = 0$ even requires $\Delta g = 0$, to conserve $p = 1$ requires instead $F = ze^{i\sigma(u)}$. With respect to more detailed discussions of possible further gauges and a simultaneous restriction of the gauge (semi)group we refer to (**18**) and (**17**) respectively, and content ourselves with a listing – in the last section – of the size of the space of functions remaining undetermined in the line element in spite of gauging for the case of Einstein's vacuum field equations $R_{ab} = 0$.

As a concluding remark of our formal considerations, here a word concerning the simplest *invariants*: Starting from the gauge transformations (5.4), Robinson and Trautman have constructed in (**18**) a number of simple invariants for the case $\theta \neq 0 = \sigma = R_{ab}$. We mention the Gaussian curvature K of the wave surfaces calculated in (5.3) as well as the corresponding curvature $-r^{-3}(rm_4)_{r=0}$ of their orthogonal 2-surfaces. More easily reached is the goal for $\theta = 0$: here the gradient of the ray vector defines a vector p_a via

$$l_{a;b} = 2l_{(a}p_{b)}, \tag{5.5}$$

which is invariant up to an additive multiple of l_a,[1] from which one can form, among others, the following invariants:

$$p_a p^a = 2|\Omega|^2 = (4m^{,A}m_{,A})^{-1} \tag{5.6}$$

$$K = 4|\Omega|^2 - 2\overset{p}{\nabla} \ln|\Omega| = -\partial^A\partial_A \ln p \tag{5.7}$$

$$J := \partial_r{}^2 m_4 \quad \text{via}: \, 4p^{[a,b]}l_b = l^a J. \tag{5.8}$$

Of course, in each case one can form a basis of all invariants by forming scalars from the Riemann tensor and its covariant derivatives; for this, cf. (**23**).

Please note the occurrence of invariants in the canonical form of the metric, Table 5.1!

[1]Author's note (2013): This formulation corrects, in passing, an oversight in the original phrasing.

6. Solution of the field equations

For a complete solution of the vacuum field equations, one can start from the normal forms for the metric given in Table 5.1 and solve with them the equations $R_{ab}m^b = 0$. This step has been widely reduced, in (**18**) and (**17**), to the solution of differential equations of Poisson type. In this section we only want to discuss the additional difficulties, which arise when additional fields are present.

The case of a pure radiation field with a variable scalar of gravitation is quickly treated: according to (3.5), the conformal metric $\widetilde{g}_{ab}$ obeys the equation

$$\widetilde{R}_{ab} = l_a l_b, \tag{6.1}$$

if $(\zeta - 3/2)^{1/2} \ln \kappa$ is chosen as the (now invariant!) phase u of the ray congruence. Compared with the Einsteinian vacuum, here the only change (in canonical coordinates) is the addition of a constant to just one of the field equations. However, as stressed in section 2.3, these statements only hold for expansion-free radiation fields.

Also, the additional presence of an electromagnetic null field in a pure radiation field does not pose a serious obstacle to its formal evaluation. For this we first prove[1]

Theorem 6.1: A simple (real) bivector F_{ab} with

$$F_{a[b}F_{cd;e]} = 0 \tag{6.2}$$

is proportional to the skew product of two gradients:

$$F_{ab} = \lambda u_{[a} v_{b]}.$$

Proof: The bivector $\overset{*}{F}_{ab}$ dual to F_{ab} obeys the equation

$$\overset{*}{F}{}^{[ab}\overset{*}{F}{}^{c]d}{}_{;d} = 0, \tag{6.3}$$

and this in turn implies the integrability of the 2-surface elements spanned by $\overset{*}{F}{}^{ab}$: for it says that the Lie product $\overset{\eta}{\nabla}\xi^a - \overset{\xi}{\nabla}\eta^a$ of two vectors ξ^a, η^a spanning $\overset{*}{F}_{ab}$ is a linear combination of these vectors. Call u, v two local coordinates by whose being constant these surfaces can be described. Then the 2-surface elements orthogonal to them are spanned by $u_{,}{}^{a}$ and $v_{,}{}^{a}$, which yields the claim.

[1] Cf. (**19**), where a very similar theorem is proven; but also (**22**) page 82.

Among others, this theorem can be applied to every simple Maxwell bivector, in particular to Maxwell-null-bivectors. Here one can choose the lightlike eigenvector as one of the two gradients, if hypersurface normal, and one gets:

Theorem 6.2: A Maxwell null field F_{ab} with twist-free rays can be brought to the form

$$F_{ab} = \lambda u_{[,a} x_{,b]} \tag{6.4}$$

when $u_{,a}$ describes the ray congruence.[1]

For the sake of transparency, we now restrict ourselves to fields with constant scalar of gravity. Then the field equations read (with $\kappa = 1$):

$$\left\{\begin{matrix} \text{(a) } 2R_{ab} = -\Phi_{ac}\overline{\Phi}_b{}^c \\ \text{(b) } \Phi_{[ab;c]} = 0 \end{matrix}\right\} \tag{6.5}$$

Equ. (b) implies that Φ_{ab} is of the form

$$\Phi_{ab} = \lambda u_{[,a} z_{,b]}, \quad z := x^1 + ix^2, \quad \lambda \text{ complex}, \tag{6.6}$$

whereby (a), for $\theta = 0$ and by using Table 5.1, goes over into:

$$R_{ab} = -\frac{1}{4} p^{-2} |\lambda|^2 l_a l_b, \tag{6.7}$$

When one now sets[2]

$$\lambda = \sqrt{2\mu} p e^{i\varphi}, \quad \text{with} \quad 2R_{ab} = -\mu l_a l_b, \tag{6.8}$$

(6.5b) becomes equivalent with the equations

$$\overset{l}{\nabla}\varphi = 0, \quad \overset{t}{\nabla}\varphi = \frac{i}{2}\overset{t}{\nabla}\ln(\mu p^2), \tag{6.9}$$

with the only integrability condition:

$$\partial^A \partial_A \ln(\mu p^2) = 0 \tag{6.10}$$

(note: $\overset{t}{\nabla} \sim \partial_1 + i\partial_2$). (6.10) means that μp^2 is the magnitude of a function which is analytic in z. We have thus proven:

[1] Because of $F_{[ab;c]} = 0$, λ can even be absorbed in x.

[2] Even simpler gets the evaluation of (6.5b) when one works with the complex amplitude λ: (6.5b) just says that $\lambda = \lambda(z; u)$ is a u-dependent, analytic function in z.

Theorem 6.3: The field equations for an expansion-free Einstein-Maxwell null field are in canonical coordinates equivalent with

$$\left.\begin{cases} \text{(a) } 2R_{ab} = -\mu u_{,a} u_{,b} \\ \text{(b) } \mu = p^{-2}|F(z;u)|, \quad \text{F analytic in } z. \end{cases}\right\} \tag{6.11}$$

The Maxwellian bivector is then given by (6.6) with $\lambda = (2F)^{1/2}$.

Equation (6.11b) is the "geometrised" Einstein-Maxwell equation, cf. III, (**22**).

7. Properties of the different solution classes

We begin this section with a table of the *twist-free pure radiation fields*, restricted to Einstein's vacuum fields $R_{ab} = 0$:

Table 7.1 of the twist-free pure radiation fields in Einstein's vacuum:

<table>
<tr><th></th><th>Type</th><th>W.S.</th><th>B.C.</th><th colspan="2">Further division</th></tr>
<tr><td rowspan="5">$\theta \neq 0$</td><td>II, D</td><td>$\Delta K \neq 0$</td><td>$p(x,y)$</td><td rowspan="5" colspan="2"></td></tr>
<tr><td>II</td><td>$\Delta K = 0, K \neq \text{const}$</td><td>$K(x)$</td></tr>
<tr><td>D</td><td>$K = \text{const}$</td><td>K</td></tr>
<tr><td>III</td><td>$\Delta K = 0, K \neq \text{const}$</td><td>$K(x,u)$
$p(x,u)$</td></tr>
<tr><td>N</td><td>$K = \text{const}$</td><td>$p(x,u)$</td></tr>
<tr><td rowspan="5">$\theta = 0$</td><td>II, D</td><td>$K \neq 0$</td><td>$p(x,u)$
$m(x,u)$
$A(x,u)$</td><td>$\Omega \neq 0$</td><td rowspan="4"></td></tr>
<tr><td rowspan="2">III</td><td rowspan="2">$K = 0$
nongeodetic</td><td rowspan="2">$n(x,u)$
$A(x,u)$</td><td>$\Omega \neq 0$</td></tr>
<tr><td>$\Omega = 0$</td></tr>
<tr><td rowspan="2">$N, 0$</td><td rowspan="2">$K = 0$
geodetic</td><td rowspan="2">$A(x,u)$</td><td>$\Omega \neq 0$</td></tr>
<tr><td>$\Omega = 0$</td><td>plane waves</td></tr>
</table>

Herein the notations of Table 5.1 are used.

The *third column* lists the behaviour of the (invariant) Gaussian curvature K of the wave surfaces (= W.S.). One notices that *inner*

(K) and *outer* (θ, geodesy) *curvature* of the wave surfaces lead to a classification, which is almost equivalent to the one given by the first two columns. Δ denotes the Laplace operator w.r.t. the canonical coordinates x, y (x standing for x^1, y for x^2). The D fields with $\theta \neq 0$, $K = \text{const}$ are the static Schwarzschild-like fields, which were called the "DS-fields" in (**18**) and the "A-fields" in (**21**).

The *fourth column* with the headline "*boundary conditions*" (= B.C.) lists the functions which can be arbitrarily prescribed, and cannot be further constrained by gauge transformations. Lower-dimensional data are suppressed. Left-out arguments in the functions mean that the corresponding coordinates are kept constant; and $(y, r, u) = \text{const}$ stands symbolically for a piece of boundary curve in a wave surface. $A(x, y, u)$ is the term in m_4 constant w.r.t. r, n the corresponding r-constant term in m_1 (for a suitable gauge).

In the *last column*, refinements of the given classification are listed: for expansion-free radiation, *parallelism* or *non-parallelism* of the rays ($\Omega = 0$ or $\neq 0$) motivate a splitting of classes of type III, N, or 0 into equally large partial classes. Here we have called the (electromagnetic) waves emitted by an ideal searchlight "parallel" if and only if the searchlight is not tilted. By "*plane-fronted*" we denote the waves with plane and geodetic wave surfaces; they are the expansion-free null fields, as has been shown in detail in (**29**) and (**17**). Inside the partial class of plane-fronted waves with parallel rays (= *pp-waves*), which can be characterised by, among others, the existence of a (necessarily lightlike) constant bivector, the *plane waves* can be defined by their property that the amplitudes (of relative accelerations of neighbouring test particles) in wave surfaces are constant. Other characterising properties of them are the existence of a 3-dimensional lightlike Abelian isometry group with 1-dimensional lightlike subgroup, or the existence of a 5-dimensional isometry group. Formally, they can be described by $m_{4,ABC} = 0$.

The partial classes described in the last section have been discovered very many times in the *literature*. For instance, the pp waves were already found in 1923 by Brinkmann (**24**), then in 1956 by I. Robinson (unpublished), and rediscovered in 1959 by (**25**) and (**26**). They are most thoroughly treated in (**21**); see also I. Among others, the partial class of plane waves is subject of the publications listed under (**27**). The more general class of expansion-free waves with parallel rays has been found in the papers listed under (**28**), and independently covered in (**29**).

(30)

We now present a series of further local properties of the twist-free pure radiation fields, and drop in particular the restriction to Einsteinian vacuum metrics. To begin with, let us remark that all statements of the Table, excepting only the column "boundary conditions", remain valid when the *field equations are weakened* to

$$2R_{ab} = -\mu l_a l_b. \tag{7.1}$$

The Jordanian radiation fields and electromagnetic null fields are then included, and, in the last line of the Table, there appear non-flat fields of type 0 ($C_{abcd} = 0$).

In the 6th section we have shown that every *Maxwellian null field* with *twist-free rays* can be written in the form

$$\Phi_{ab} = u_{,[a}\Phi_{,b]} \tag{7.2}$$

with analytic $\Phi(z; u)$. When one now asks for the restrictions which the Einstein-Maxwell equations impose in addition on a combined radiation field, one obtains from theorem 6.3 and the analogue for $\theta \neq 0$ described in (**18**):

Theorem 7.1: For every twist-free Maxwellian null field there are Einstein-Maxwell null fields (i.e. solutions of (6.5)). For a given complex amplitude $\lambda := \partial_z \Phi$ – the latter can always be reduced (locally) to 1 by gauges (5.4) I, II – there are no constraints imposed upon the type of the conformal tensor in the expansion-free case. In the expanding case, however, the conformal tensor must be of type II or D.

The theorem formulated right now teaches that in the expansion-free case, a rather arbitrary "mixture" is possible of electromagnetic and gravitational radiation. – In flat spacetime there are Maxwellian null fields with *spherical fronts*; they can be generated from pp-waves by reflection on a parabolic mirror, placed parallel to the ray direction. Also these fields have analogues in the Einstein-Maxwell theory: namely fields with constant inner and outer curvature of the wave surfaces. But the above theorem teaches that the conformal tensor cannot be of type III, N, or 0.

The large *similarity* between *electromagnetic* and *gravitational radiation* becomes clearer under confrontation of all expansion-free Maxwell waves of flat spacetime with the expansion-free pure gravitational waves. A comparison can be made on grounds of the invariant properties of the respective ray congruences. During this confrontation, because of $l_{a;[bc]} = 0$ in flat spacetime, only gravitational fields of type N qualify. Astonishingly, it could be shown in (**17**):

Theorem 7.2: The correspondence h:

$$Re(\Phi) \to \begin{Bmatrix} x^{-1}A & \text{for} & \Omega \neq 0 \\ A & \text{for} & \Omega = 0 \end{Bmatrix}$$

is a unique 1-to-1 map of all non-constant expansion-free ($\Leftrightarrow$ plane-fronted) Maxwellian null fields onto the plane-fronted gravitational waves; (Φ was introduced in (7.2); A is the term in m_4 constant w.r.t. r. For the rest see Table 5.1). h maps functions F onto each other which satisfy the equations $\overset{l}{\nabla} F = 0 = \Delta F$ (and only them).

Next we take interest in the possible *variability of the type* in a pure radiation field. Because the type of the conformal tensor in canonical coordinates is determined by functions which are independent of their distance from the source r, and which satisfy the Laplace equation in the wave surfaces, and because the local introduction of canonical coordinates does not mean a restriction, we conclude:

Theorem 7.3: In a twist-free pure radiation field, the type of the conformal tensor is constant in wave surfaces. As a function of the retarded time u, however, it can change arbitrarily (even for metrics of continuity class C^∞).

Hence there exist arbitrarily "multilayered sandwich waves".

Almost all pure radiation fields possess no *symmetries* at all. Of interest is the question for those matter-free spacetimes which permit a 1-dimensional lightlike isometry group. Here one finds, as expected, a partial set of the twist-free pure radiation fields; the result was already derived in (**17**), it reads:[1]

Theorem 7.4: The only metrics obeying (7.1) which allow a 1-dimensional group of lightlike isometries are 1) the pp waves ($\theta = 0$, type N, 0, $\Omega = 0$), and 2) a class of expansion-free II(D)-fields which in canonical coordinates are given by $p = x^{-1/4}$, $m_1 = -x^{-1}v + x^{-3/2}y\ n(u)$, $m_2 = 0$, $\partial_r m_4 = 0$. To the latter belongs the static, Schwarzschild-like vacuum field $B3$ from (**21**).

Up to now we have only studied the local properties of the pure radiation fields expecting that, domainwise, the latter form a good approximation to realistic radiation fields. But how does one have to continue them to *global* solutions? For this we can make two statements. The first statement was proven in (**21**):

[1] This result was found independently, and in coordinates adapted to the group, by Dautcourt, see (**30**).

Theorem 7.5: The plane waves are g-complete if m_4 in Table 5.1 belongs to continuity class C^3, and if R^4 is chosen as the canonical coordinate domain.

Here "*g-complete*" means that all geodesics have infinite length in both directions (for lightlike geodesics, an affine parameter should be taken as measure of length). Consequently, the plane waves are source-free world models, closed against action from beyond.

When one tries to extend Theorem 7.5, one already fails at the next simple class of waves: already the amplitudes (= components of the Riemann tensor) of the pp waves are non-constant potential functions w.r.t. the wave-surface coordinates, hence possess a *singularity* at finite or infinite separations. And one finds for the simplest representatives that these singularities are reached by timelike geodesics within finite proper time! By this one arrives at the impression that all non-planar waves play the role of interpolating fields in the interior or exterior of "*wave guides*"; exactly as in electrodynamics, the waves would possess singularities inside or outside of tubes or funnels if one tried to continue them to the full space. But how does a wave guide for gravitational waves look? It can certainly not consist of electromagnetic radiation; for the amplitudes of electromagnetic fields are likewise analytic functions in wave surfaces, hence vanish everywhere or almost nowhere. It could possibly consist of neutrinos, (i.e. matter with restmass zero). Instead, we assume in what follows that the *waveguide* consists *of matter*, with tensions which can be neglected in comparison with their energy density, at high approximation. Then Einstein's field equations read in the exterior domain:

$$R_{ab} = -\mu u_a u_b, \quad u^a u_a = -1, \quad \mu > 0, \tag{7.3}$$

and for every lightlike vector k_a we have

$$R_{ab} k^a k^b < 0. \tag{7.4}$$

In the exterior domain, therefore, at least inequality (7.4) must be satisfied for the ray direction l^a, through which the canonical coordinate system fails. Nevertheless, we can infer from (2.5) that inside the wave guide near its boundary[1], θ decreases approximately[2] linearly with the source distance r. From this behaviour of the expansion we can conclude at the behaviour of the rays in the rest space of an observer, and

[1]More precisely: for $2(z^2 + |\sigma|^2) \lesssim -R_{ab} l^a l^b$.

[2]For $l^a l^b \overset{l}{\nabla} R_{ab} \approx 0$, i.e. for an approximately homogeneous wave guide.

from here at the shape of the 2-surfaces of constant phase u: they are broken off at the boundary, and beyond, in the wave guide, they are curved concavely w.r.t. the wave channel. Unfortunately, the statements which we can make about the present continuation problem exhaust themselves with this.

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