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RESEARCH ARTICLE

Slowly rotating dilatonic black holes with exponential form of nonlinear electrodynamics

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Abstract The generalization of the four-dimensional Kerr–Newman black holes to include the nonlinear electrodynamics has been one of the famous problems in black hole physics. In this paper, we address the effects of the small rotation parameter on the exact black hole solutions of Einstein-dilaton gravity coupled to the exponential nonlinear electrodynamics. We find a new stationary black hole solutions of this theory, in the limit of small angular momentum, and in the presence of Liouville-type potential for the dilaton field and an arbitrary value of the dilaton coupling constant. We compute the angular momentum and the gyromagnetic ratio of these rotating dilaton black holes. Interestingly enough, we find that the nonlinearity of the electrodynamics do not affect the angular momentum and the gyromagnetic ratio of the spacetime, while in contrast, the dilaton field can modify the angular momentum as well as the gyromagnetic ratio

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of the rotating black holes. We find the gyromagnetic ratio as $g = 6/(3 - \alpha^2)$, where α is the coupling constant of the dilaton and the electrodynamic fields. For $\alpha = 0$, we arrive at $g = 2$, which is the gyromagnetic ratio of the Kerr–Newman black holes in four dimensions.

Keywords Slowly rotating black hole · Dilaton gravity · Nonlinear electrodynamics

1 Introduction

In the past decades there has been a growing interest for studying the rotating black hole solutions in the background of anti-de Sitter (AdS) spacetimes. The motivations for this studying mainly originates from the well-known correspondence between the gravitating fields in an AdS spacetime and the conformal field theory (CFT) living on the boundary of the AdS spacetime [1,2]. According to AdS/CFT correspondence, the rotating black holes in AdS space are dual to certain CFTs in a rotating space [3], while charged ones are dual to CFTs with chemical potential [4–6]. However, constructing rotating black holes coupled to the matter field are not a trivial task at all. For example, while the charged rotating black holes in four dimensions have been found several years ago, the counterpart of the Kerr–Newman solution in higher dimensions, that is the charged generalization of the Myers–Perry solution [7] in $(n + 1)$-dimensional Einstein–Maxwell gravity, still remains to be found analytically. The most general higher dimensional uncharged rotating black holes in AdS spaces with all rotation parameters have been found [8,9]. Rotating black holes for the Maxwell field minimally coupled to Einstein gravity in higher dimensions do not exist in a closed form, and one has to rely on perturbative or numerical methods to construct them in the background of asymptotically flat [10–12] and AdS [13,14] spacetimes. There has also been a lot of interests in constructing the analogous charged rotating solutions in the framework of gauged supergravity in various dimensions [15–19]. The studies were also applied to other gravity theories. In this regards, a class of charged slowly rotating black hole solutions in Gauss–Bonnet gravity [20,21], third order Lovelock gravity [22], Chern–Simons gravity [23,24], and Horava–Lifshitz gravity [25,26] has been constructed. In addition, a class of slowly rotating black hole solution of the Einstein equations with nonlinear Born–Infeld charge was obtained in [27]. In the presence of the power-law nonlinear Maxwell field [28] and the Born–Infeld-like nonlinear electrodynamics [29–31], charged rotating black holes in the limit of slow rotation parameter have been explored. In the framework of the Einstein–Hilbert action which is supplemented by all possible quadratic and algebraic curvature invariants coupled to a scalar field, a general stationary, slowly rotating black hole, which is solution to a large class of alternative theories of gravity in four dimensions, was obtained [32]. Employing higher order perturbation theory, charged rotating black holes in odd dimensions were obtained in the context of Einstein–Maxwell Lagrangian which is supplemented with a Chern–Simons term [33]. Extremal Myers–Perry black holes coupled to Born–Infeld nonlinear electrodynamics have also been considered in [34,35].

The studies were also extended to include a dilaton field. In the absence of the dilaton potential, the properties of charged rotating dilaton black holes, for an arbitrary dilaton coupling constant, in the small angular momentum limit, in four [36–39] and higher dimensional spacetimes [40] have been studied. In the presence of a Liouville-type dilaton potential, a class of charged slowly rotating dilaton black hole solutions in four [41,42] and higher dimensions [43,44] have been investigated. Because of the presence of one or two term(s) Liouville-type dilaton potential, these solutions [41–44] are neither asymptotically flat nor (A)dS. With an appropriate combination of three Liouville-type dilaton potentials, a class of charged slowly rotating dilaton black hole solution for asymptotically AdS spacetime in four [45] and higher dimensional [46,47] spacetimes have been constructed. Slowly rotating charged Kaluza–Klein black hole solutions of the five-dimensional Einstein–Maxwell-dilaton theory with arbitrary dilaton coupling constant were obtained in [48]. Considering the electric charge as the perturbative parameter, extremal Myers–Perry black holes in dilaton gravity have been studied for linear Maxwell field [49] and nonlinear Born–Infeld theory [50]. The physical properties of rotating Einstein–Maxwell-dilaton black holes in odd dimensions also have been investigated [51]. Till now, charged rotating dilaton black holes coupled to Born–Infeld-like nonlinear electrodynamics have not been constructed. In this paper, we would like to turn the investigation on the slowly rotating dilaton black holes by including the Lagrangian of the exponential form of nonlinear electrodynamics in the action. This is a new step which may shed some light on this issue for further investigations.

Our paper is structured as follows. In the next section, we introduce the basic field equations of Einstein-dilaton gravity in the presence of exponential nonlinear electrodynamics. In Sect. 3, we present a class of slowly rotating charged dilaton black hole solutions of this theory. We also obtain conserved quantities such as the mass, the electric charge, the Hawking temperature, the entropy, the angular momentum, and the gyromagnetic ratio of the solutions. The last section is devoted to summary and conclusion.

2 Basic field equations

The action of four-dimensional Einstein-dilaton gravity coupled to nonlinear electrodynamics (NED) can be written

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu \Phi\partial_\nu\Phi - V(\Phi) + \mathcal{L}(\mathcal{F}, \Phi)\right) \\ -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-\gamma}\,\Theta(\gamma), \tag{1}$$

where R, Φ and $V(\Phi)$ are, respectively, the Ricci scalar the dilaton field and the potential for Φ. In addition, Θ and γ are the trace of the extrinsic curvature and the induced metric on the boundary, respectively. $\mathcal{L}(\mathcal{F}, \Phi)$ can be selected as a new class of dilaton field coupled with the exponential form of NED, whose Lagrangian is [52]

$$\mathcal{L}(\mathcal{F}, \Phi) = 4\beta^2 e^{2\alpha\Phi}\left[\exp\left(-\frac{e^{-4\alpha\Phi}\mathcal{F}}{4\beta^2}\right) - 1\right], \tag{2}$$

where α and β are, respectively, the dilaton and the nonlinearity parameter, and $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant. It is easy to show that in the limit $\beta \to \infty$ (weak field limit of NED), the mentioned $\mathcal{L}(\mathcal{F}, \Phi)$ reduces to the Lagrangian of the standard Maxwell field coupled with a dilaton field, Φ,

$$\mathcal{L}(\mathcal{F}, \Phi) \longrightarrow -\mathcal{F}e^{-2\alpha\Phi}. \tag{3}$$

In addition, the Lagrangian of exponential form of NED may be recovered in the absence of dilaton field, $\alpha \longrightarrow 0$ [53–55],

$$\mathcal{L}(\mathcal{F}, \Phi) \longrightarrow 4\beta^2 \left[\exp\left(\frac{-\mathcal{F}}{4\beta^2}\right) - 1\right]. \tag{4}$$

Before we proceed, let us provide some motivations for considering this form of NED. Considering strong electromagnetic field in the regions near to point-like charges, it was suggested that one may have to use generalized nonlinear Maxwell theory in those regions [56]. Similar behavior may occur in the vicinity of compact objects and therefore it is reasonable to consider NED with an astrophysical motivation [57].

In addition, in quantum electrodynamics (QED) context, it was shown that quantum corrections lead to nonlinear properties of vacuum which affect the photon propagation [58–63]. Besides, within the framework of AdS/CFT, some authors regard the roles of NED on shear viscosity [64,65] and also holographic superconductors [66–68]. Regarding these observations, it is worthwhile to study the effects of NED on the geometrical behavior of black holes.

Although the Born–Infeld theory is a specific model in the context of NED, in recent years there has been a lot of interests on the other types of NED theories. This is mainly due to their emergence in the context of low-energy limit of heterotic string theory or as an effective action for the consideration of effects of loop corrections in QED where quartic corrections of Maxwell field strength appear [69–77]. In particular, a class of exponential form of NED was proposed in the form of (4). This model is significantly richer than that of the Maxwell field, and in the special case $\beta \longrightarrow \infty$, its Lagrangian reduces to that of a linear Maxwell field as

$$\mathcal{L}(\mathcal{F}) = -\mathcal{F} + \frac{\mathcal{F}^2}{8\beta^2} + O\left(\beta^{-4}\right), \tag{5}$$

which is exactly like the Born–Infeld theory. In addition to the black hole solutions of this model, in the context of AdS/CFT correspondence, the effects of the mentioned nonlinear source on strongly coupled dual gauge theory have been investigated [78,79].

It is worth mentioning that although the exponential form of NED does not cancel the divergency of the electric field at $r = 0$, its singularity is much weaker than Einstein–Maxwell theory. This behavior is more natural with respect to the Born–Infeld theory in which the electric field of pointlike charges goes to a constant value. In addition, we should note that considering the $E_8 \times E_8$ heterotic string theory, the $SO(32)$ gauge group has a $U(1)$ subgroup. It has been shown that [77,80] taking into account a constant dilaton, the effective Lagrangian has a Gauss–Bonnet term as well as a quadratic Maxwell invariant in addition to the Einstein–Maxwell Lagrangian. Since, unlike the quadratic Maxwell invariant, the Gauss–Bonnet term becomes a topological invariant and does not give any contribution in four dimensions, it is natural to investigate the Einstein–NED in four dimensions. Motivated by the recent results mentioned above, we take into account the exponential form of NED to obtain four-dimensional slowly rotating dilatonic black hole solutions.

Using the Euler–Lagrange equation, we can obtain gravitational, electromagnetic and scalar field equations with the following explicit forms

$$G_{\mu\nu} = 2\left(\nabla_\mu \Phi\right)\left(\nabla_\nu \Phi\right) - \left((\nabla\Phi)^2 + \frac{V(\Phi)}{2} + \Pi\right) g_{\mu\nu} -2e^{-2\alpha\Phi}\left(F_{\mu\lambda}F_\nu^{\ \lambda} - \frac{\mathcal{F}}{2}g_{\mu\nu}\right)\frac{\partial\mathcal{L}(Y)}{\partial Y}, \tag{6}$$

$$\partial_\mu\left(\sqrt{-g}e^{-2\alpha\Phi}\frac{\partial\mathcal{L}(Y)}{\partial Y}F^{\mu\nu}\right) = 0, \tag{7}$$

$$\nabla^2\Phi = \frac{1}{4}\frac{dV(\Phi)}{d\Phi} + \alpha\Pi, \tag{8}$$

where

$$\Pi = 2\beta^2 e^{2\alpha\Phi}\left[2Y\frac{\partial\mathcal{L}(Y)}{\partial Y} - \mathcal{L}(Y)\right], \tag{9}$$

and $\mathcal{L}(\mathcal{F}, \Phi) = 4\beta^2 e^{2\alpha\Phi}\mathcal{L}(Y)$ with

$$\mathcal{L}(Y) = e^{-Y} - 1, \quad Y = \frac{e^{-4\alpha\Phi}\mathcal{F}}{4\beta^2}. \tag{10}$$

3 Slowly rotating charged black holes in 4-dimensions

3.1 Solutions

In this section we look for the slowly rotating nonlinear charged dilatonic black hole solutions in four dimensions. Using series expansion of Kerr and Kerr–Newman spacetimes for small values of rotation parameter, a, and keeping the first order of the angular momentum parameter, one finds that the only term in the metric which is related to the rotation parameter is $g_{t\phi}$. This motivates us to consider the following line element and gauge potential

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} +2aG(r)\sin^2\theta dt d\phi + r^2R^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{11}$$

$$A = h(r)dt + aqC(r)\sin^2\theta d\phi, \tag{12}$$

where the unknown functions $F,\ G,\ R,\ h,\ C$, and the dilaton field Φ depend only on r, and $a,\ q$ are parameters associated with the angular momentum and the electric charge of the black holes. To generalize the four-dimensional static dilaton black holes with the exponential form of NED [52] to the slowly rotating ones, we substitute the metric (11) and the gauge potential (12) into the field Eqs. (6)–(8), and solve these equations up to the linear order of the angular momentum parameter a.

After some algebraic calculations, we find that Eq. (7) leads to two independent differential equations

$$j_1 = E'(r) - 2\alpha\Phi'(r)E(r) + \frac{2\beta^2 E(r)\left[R(r) + rR'(r)\right]}{rR(r)\left[E^2(r)e^{-4\alpha\Phi} + \beta^2\right]} = 0, \tag{13}$$

$$j_2 = E' - \frac{2\alpha\left[\beta^2 + E^2 e^{-4\alpha\Phi}\right]\Phi'}{Ee^{-4\alpha\Phi}} - \frac{\beta^2\left\{2 + rRE\left[3rRG' - 2\left(rRG\right)'\right]\right\}}{qr^2R^2FEC'e^{-4\alpha\Phi}} = 0, \tag{14}$$

where $E(r) = h'(r)$, and prime denotes the first derivative with respect to r. As we see, Eq. (13) is free of $F(r)$ and $G(r)$, and one can obtain $E(r)$ as a function of $\Phi(r)$ and $R(r)$ with the following explicit form

$$E(r) = \frac{qe^{2\alpha\Phi}\exp\left[-\frac{1}{2}L_W\right]}{r^2R^2(r)}, \tag{15}$$

where $L_W = \text{Lambert}W\left(\frac{q^2}{\beta^2r^4R^4(r)}\right)$ is the Lambert W function which satisfies $\text{Lambert}W(x)\exp\left[\text{Lambert}W(x)\right] = x$, and has a convergent series expansion for $|x| < 1$ such that $\text{Lambert}W(x) = x - x^2 + 3x^3/2 - 8x^4/3 + \cdots$ (for more details, see [81,82]). One can use series expansion of $E(r)$ for large β to obtain the electromagnetic field of Maxwell theory in the presence of dilaton field

$$E(r)|_{\text{Large } \beta} = e^{2\alpha\Phi}\left[\frac{q}{r^2R^2(r)} - \frac{q^3}{2\beta^2r^6R^6(r)} + O\left(\frac{q^5}{\beta^4r^{10}R^{10}(r)}\right)\right]. \tag{16}$$

This equation shows that for large values of β, the dominant term of the dilatonic nonlinear electromagnetic field is the same as that in four-dimensional Reissner–Nordström black holes with the dilaton field [90,91].

Now we focus on other field Eqs. (6) and (8). It is easy to show that Eq. (8) with the metric (11) and the gauge potential (12) reduces to

$$\begin{aligned} j_3 = {} & 32\alpha\beta^2 rRe^{2\alpha\Phi}\left[\left(X + \frac{1}{2}\right)e^{-X} - \frac{1}{2}\right] + 4rFR\Phi'' \\ & + \left[8rFR' + 4R\left(2F + rF'\right)\right]\Phi' - rR\frac{dV}{d\Phi} = 0, \end{aligned} \tag{17}$$

where

$$X = \frac{-E^2e^{-4\alpha\Phi}}{2\beta^2}. \tag{18}$$

In addition, calculations show that the nonzero components of the gravitational field Eq. (6) with the metric (11) and the gauge potential (12) are

$$
\begin{aligned}
j_{tt} &= rRF'' + 2F'(R + rR') + 8\beta^2 rRe^{2\alpha\Phi}\left[1 - \left(2X + 1 + \frac{E^2 e^{-4\alpha\Phi}}{\beta^2}\right)e^{-X}\right] \\
&\quad + rRV = 0, \qquad (19) \\
j_{rr} &= j_{tt} - 4F\left(rR\Phi'^2 + rR'' + 2R'\right) = 0, \qquad (20) \\
j_{\phi\phi} &= 8\beta^2 R^2 e^{2\alpha\Phi}\left[\left(X + \frac{1}{2}\right)e^{-X} - \frac{1}{2}\right] - F\left(RR'\right)' \\
&\quad - \frac{RR'(4F + rF')}{r} - R^2\left(\frac{F}{r^2} + \frac{F'}{r} + \frac{V}{2}\right) + \frac{1}{r^2} = 0, \qquad (21) \\
j_{\phi t} &= GF'' - FG'' + \frac{2G}{r^2R^2} + 8e^{-2\alpha\Phi - X}E^2\left(\frac{qC'F}{E} - G\right) = 0, \qquad (22) \\
j_{t\phi} &= 2r^2GRR'' - r^2R^2G'' + 2G\left(4rRR' + r^2R'^2 + R^2\right) \\
&\quad + 8qC'r^2ER^2e^{-2\alpha\Phi - X} = 0. \qquad (23)
\end{aligned}
$$

Now we should obtain exact analytical solutions of the field Eqs. (14), (17) and (19)–(23) up to the linear order of rotation parameter for arbitrary α and β. We follow the method of [52] to consider a dilaton potential contains two Liouville type terms with the following form

$$V(\Phi) = 2\Lambda_0 e^{2\xi_0\Phi} + 2\Lambda e^{2\xi\Phi}, \qquad (24)$$

where Λ_0, ξ_0, Λ and ξ are constants. In order to solve Eqs. (14), (17) and (19)–(23), we use a suitable ansatz such that

$$R(r) = e^{\alpha\Phi}. \qquad (25)$$

Considering Eqs. (19) and (20), one finds that $j_{tt} - j_{rr} = 0$ leads to a differential equation for the dilaton field $\Phi(r)$ with the following solutions

$$\Phi(r) = \frac{\alpha}{1 + \alpha^2}\ln\left(\frac{b}{r}\right), \qquad (26)$$

where b is an integration constant. Now, we are in a position to obtain $F(r)$, $G(r)$, and $C(r)$. Inserting $E(r)$, $R(r)$, $V(\Phi)$, and $\Phi(r)$ in Eqs. (14), (17) and (19)–(23), we find

$$F(r) = \left[(1 + \alpha^2)\int r^{\frac{-2\alpha^2}{\alpha^2+1}} H(r)dr - m\right] r^{\frac{\alpha^2-1}{\alpha^2+1}}, \qquad (27)$$

$$H(r) = 4q\beta\left(\frac{1}{\sqrt{L_W}} - \sqrt{L_W}\right) - \left[\Lambda_0\left(\frac{b}{r}\right)^{\frac{2\alpha\xi_0}{\alpha^2+1}} + \Lambda\left(\frac{b}{r}\right)^{\frac{2\alpha\xi}{\alpha^2+1}} + 4\beta^2\left(\frac{b}{r}\right)^{\frac{2\alpha^2}{\alpha^2+1}}\right]r^2 + \left(\frac{b}{r}\right)^{\frac{-2\alpha^2}{\alpha^2+1}}, \tag{28}$$

with

$$\Lambda_0 = \frac{\alpha^2}{\left(\alpha^2-1\right)b^2}, \quad \xi_0 = \frac{1}{\alpha}, \quad \xi = \alpha, \tag{29}$$

where m is the integration constant which is related to the Arnowitt–Deser–Misner (ADM) mass of the black hole. Considering Eqs. (22) and (23), one can obtain

$$C(r) = \frac{r^2(1+\alpha^2)G' - 2rG}{8q^2(1+\alpha^2)}\left(\frac{b}{r}\right)^{\frac{2\alpha^2}{\alpha^2+1}}. \tag{30}$$

In order to find the explicit form of $G(r)$, we set

$$G(r) = N(r)F(r) + B(r), \tag{31}$$

and insert it to Eqs. (22) and (23) with the mentioned $C(r)$.

To obtain consistent solutions, we regard the following ansatz

$$N(r) = 1, \tag{32}$$

$$B(r) = -\frac{1+\alpha^2}{1-\alpha^2}\left(\frac{b}{r}\right)^{\frac{-2\alpha^2}{\alpha^2+1}}. \tag{33}$$

We find that, considering the above ansatz, the metric (11), the gauge potential (12), and the dilaton field (26) satisfy all field equations.

In the $a = 0$ limit, above solutions reduce to the static dilaton black holes with the exponential form of NED [52]. In addition, obtained solutions are consistent with special case $\beta \longrightarrow \infty$ [42]. For more clarifications, we consider the large values of β to obtain slowly rotating dilatonic Maxwell black hole solutions with a correction of NED. Using series expansion of $F(r)$ for large β leads to

$$F(r) \approx \left(\frac{1+\alpha^2}{1-\alpha^2} + \frac{2(1+\alpha^2)q^2}{r^2}\right)\left(\frac{b}{r}\right)^{\frac{-2\alpha^2}{1+\alpha^2}} - mr^{\frac{\alpha^2-1}{1+\alpha^2}} + \frac{\left(1+\alpha^2\right)^2\Lambda r^2}{\alpha^2-3}\left(\frac{b}{r}\right)^{\frac{2\alpha^2}{1+\alpha^2}} - \frac{\left(1+\alpha^2\right)^2 q^4}{2\beta^2 r^6(\alpha^2+5)}\left(\frac{b}{r}\right)^{\frac{-6\alpha^2}{1+\alpha^2}} + O(\beta^{-4}), \tag{34}$$

where the last term on the right hand side is the leading nonlinear correction to the dilatonic Maxwell solutions. Moreover, it is easy to show that in the absence of dilaton field ($\alpha \longrightarrow 0$), Eq. (34) reduces to

$$F(r) \approx 1 - \frac{m}{r} + \frac{2q^2}{r^2} - \frac{\Lambda r^2}{3} - \frac{q^4}{10\beta^2 r^6} + O\left(\beta^{-4}\right), \tag{35}$$

which represents Einstein–Maxwell–AdS black holes in the limit of $\beta \rightarrow \infty$.

3.2 Physical properties and conserved quantities

Now, we are in a position to investigate the geometric properties of the solutions (11). Because of the presence of the dilaton field, the asymptotic behavior of these solutions are neither flat nor (A)dS. Looking for the curvature singularity, we find that the Ricci and the Kretschmann scalars are the same as those in the static case [52], and therefore there is an essential singularity located at the origin. In addition, it is shown that, similar to the case of the static solution [52], depending on the nonlinearity parameter, $g^{rr} = F(r)$ gives two horizons, one extreme horizon, or one non-extreme horizon, and therefore the singularity can be covered with an event horizon r_+.

Calculations show that the surface gravity and the event horizon area do not change up to the linear order of a, and thus, we can obtain the the Hawking temperature and the entropy of slowly rotating dilatonic black hole solutions as

$$T_+ = \frac{1}{2\pi}\sqrt{-\frac{1}{2}(\nabla_\mu \chi_\nu)(\nabla^\mu \chi^\nu)} = \frac{F'(r_+)}{4\pi}, \tag{36}$$

$$S = \frac{\mathcal{A}}{4}, \tag{37}$$

where the Killing vector χ is the null generator of the event horizon, and $\mathcal{A}$ is the horizon area. Then we obtain

$$T_+ = -\frac{\alpha^2+1}{4\pi} r_+^{\frac{\alpha^2-1}{\alpha^2+1}} \left[\frac{b^{-\frac{2\alpha^2}{\alpha^2+1}}}{\alpha^2-1} + b^{\frac{2\alpha^2}{\alpha^2+1}} (4\beta^2+\Lambda) r_+^{\frac{2(1-\alpha^2)}{\alpha^2+1}} + 4q\beta r_+^{-\frac{2\alpha^2}{\alpha^2+1}} \left(\sqrt{L_{W+}} - \frac{1}{\sqrt{L_{W+}}} \right) \right], \tag{38}$$

$$S = \frac{\omega b^{\frac{2\alpha^2}{1+\alpha^2}} r_+^{\frac{2}{1+\alpha^2}}}{4}, \tag{39}$$

where we have used $F(r_+) = 0$, $L_{W+} = \mathrm{Lambert}W\left(\frac{q^2}{\beta^2 r_+^4 R^4(r_+)}\right)$, and ω is the area of a unit two-sphere.

Here, we would like to calculate the conserved quantities such as the finite mass and the angular momentum of the slowly rotating black hole solutions we just found.

There are several approaches for calculating the mass of the black holes. For example, for asymptotically AdS solutions one can use the counterterm method inspired by (A)dS/CFT correspondence [83–87]. Another way for calculating the mass is through the use of the substraction method of Brown and York [88,89]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. In our case, due to the presence of the non-trivial dilaton field, the asymptotic behavior of the solutions are neither flat nor (A)dS, and therefore, we have used the reference background metric to calculate the mass. Since g_{tt} does not change up to first order of rotation parameter, it is reasonable to expect that the mass is independent of the rotation parameter (for small a). According to the substraction method of [88,89], if we write the metric of static spherically symmetric spacetime in the form [90,91]

$$ds^2 = -W^2(r)dt^2 + \frac{dr^2}{V^2(r)} + r^2 d\Omega^2, \tag{40}$$

and the matter action contains no derivatives of the metric, then the quasilocal mass is given by [90,91]

$$\mathcal{M} = rW(r)\left(V_0(r) - V(r)\right). \tag{41}$$

Here $V_0(r)$ is an arbitrary function which determines the zero of the energy for a background spacetime and r is the radius of the spacelike hypersurface boundary. It was argued that the *ADM* mass M is the $\mathcal{M}$ determined in (41) in the limit $r \to \infty$ [90,91]. It is a matter of calculations to show that the mass of the slowly rotating black holes is obtained as [52]

$$\mathcal{M} = \frac{\omega b^{\frac{2\alpha^2}{1+\alpha^2}} m}{8\pi\left(1+\alpha^2\right)}. \tag{42}$$

Next, we calculate the angular momentum and the gyromagnetic ratio of these rotating dilaton black holes which appear in the limit of slow rotation parameter. The angular momentum of the dilaton black hole can be calculated through the use of the quasi-local formalism of the Brown and York [88,89]. According to the quasilocal formalism, the quantities can be constructed from the information that exists on the boundary of a gravitating system alone. Such quasilocal quantities will represent information about the spacetime contained within the system boundary, just like the Gauss's law. In our case the finite stress-energy tensor can be written as

$$T^{ab} = \frac{1}{8\pi}\left(\Theta^{ab} - \Theta\gamma^{ab}\right), \tag{43}$$

which is obtained by variation of the action (1) with respect to the boundary metric γ_{ab}. To compute the angular momentum of the spacetime, one should choose a spacelike surface $\mathcal{B}$ in $\partial\mathcal{M}$ with metric σ_{ij}, and write the boundary metric in ADM form

$$\gamma_{ab}dx^a dx^b = -N^2 dt^2 + \sigma_{ij}\left(d\varphi^i + V^i dt\right)\left(d\varphi^j + V^j dt\right),$$

where the coordinates φ^i are the angular variables parameterizing the hypersurface of constant r around the origin, and N and V^i are the lapse and shift functions respectively. When there is a Killing vector field ξ on the boundary, then the quasilocal conserved quantities associated with the stress tensors of Eq. (43) can be written as

$$Q(\xi) = \int_{\mathcal{B}} d^2\varphi \sqrt{\sigma} T_{ab} n^a \xi^b, \tag{44}$$

where σ is the determinant of the metric σ_{ij}, ξ and n^a are the Killing vector field and the unit normal vector on the boundary $\mathcal{B}$. For boundaries with rotational ($\varsigma = \partial/\partial\varphi$) Killing vector field, one obtains the quasilocal angular momentum

$$J = \int_{\mathcal{B}} d^2\varphi \sqrt{\sigma} T_{ab} n^a \varsigma^b, \tag{45}$$

provided the surface $\mathcal{B}$ contains the orbits of ς. After a few calculations, the angular momentum J associated with the spacelike Killing vector field $\partial/\partial\phi$ at infinity can be obtained as

$$J = \frac{\omega \left(3 - \alpha^2\right) b^{\frac{2\alpha^2}{1+\alpha^2}} ma}{24\pi \left(1 + \alpha^2\right)}. \tag{46}$$

We see that, up to the linear order of the angular momentum parameter a, the effect of nonlinearity parameterized by β does not appear in the angular momentum (46) and the mass (42), and these conserved quantities coincide with those of the four-dimensional slowly rotating Einstein–Maxwell-dilaton black holes [42].

Next, we calculate the gyromagnetic ratio of these rotating nonlinear charged black holes. One of the important subjects about the 4-dimensional charged black hole in the Einstein gravity is that it can be assigned a gyromagnetic ratio $g = 2$ just like the electron in Dirac theory. Here we want to know how does the value of the gyromagnetic ratio change for slowly rotating nonlinear charged black holes in four dimensions. The magnetic dipole moment for this slowly rotating black hole is

$$\mu = Qa, \tag{47}$$

where $Q = q\omega/(4\pi)$ denotes the electric charge of the black hole. Therefore, the gyromagnetic ratio is given by

$$g = \frac{2\mu\mathcal{M}}{QJ} = \frac{6}{3 - \alpha^2}. \tag{48}$$

It was shown that the dilaton modifies the gyromagnetic ratio of the charged rotating black hole solutions [42]. Our result here confirms their arguments. One may also note that the nonlinear parameter β does not modify the gyromagnetic ratio of the black holes in the limit of slow rotation. In the absence of a nontrivial dilaton ($\alpha = 0$) , the gyromagnetic ratio reduces to

$$g = 2, \tag{49}$$

which is the gyromagnetic ratio of the 4-dimensional Kerr–Newman black holes [11,12].

4 Summary and conclusion

In this paper, we have presented a new stationary solution of Einstein-dilaton gravity in the presence of exponential form of nonlinear electrodynamics and in the limit of small angular momentum. This is the generalization of the corresponding static and spherically solution of [52] to include a small amount of rotation parameter. Our strategy for constructing these solutions was the perturbative technique suggested in [39]. We first studied charged black hole solutions of Einstein-dilaton gravity coupled to exponential form of NED. Then, we considered the effect of adding a small amount of rotation parameter a to the black hole. We discarded any terms involving a^2 or higher power in a. Using series expansion of Kerr–Newman spacetimes for small values of rotation parameter, a, shows that the only term in the metric changes to $O(a)$ is $g_{t\phi}$. Similarly, the dilaton does not change to $O(a)$ and A_ϕ is the only component of the gauge potential that change to $O(a)$. In the limiting case where $\beta \to \infty$, the obtained solution reduces to the slowly rotating dilaton black holes of Einstein–Maxwell-dilaton gravity presented in [42].

We computed the Hawking temperature and the entropy of black holes, which do not change to $O(a)$ from the static case. We calculated the angular momentum J and the gyromagnetic ratio g which appear up to the linear order of the angular momentum parameter a. We found that the nonlinearity of the electrodynamics do not affect the mass, the angular momentum, and the gyromagnetic ratio of the spacetime, while in contrast, the dilaton field modifies the mass and the angular momentum as well as the gyromagnetic ratio of the rotating black holes. We obtained the gyromagnetic ratio as $g = 6/(3 - \alpha^2)$, where α measures the strength of the dilaton-electromagnetic coupling. For $\alpha = 0$, we obtained $g = 2$, which is the gyromagnetic ratio of the Kerr–Newman black holes in four dimensions.

It is worth mentioning that our work is the quite one generalization of the slowly rotating Einstein–Maxwell-dilaton black holes [44] to include the exponential form of NED. However, even in this special case, we can still learn some new physics by studying the slowly rotating black holes and taking into account the linear rotation parameter. For example, the angular momentum and the gyromagnetic ratio appear in this case and one can consider the effects of dilaton and nonlinear electrodynamics on these quantities. Besides, if we consider the solutions with higher order of rotating parameter, we can expect that some conserved quantities such as mass may changes and there are physical consequences derived from the interaction between the rotation and the exponential form of NED.

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