Gen Relativ Gravit (2015) 47:93
DOI 10.1007/s10714-015-1935-z

RESEARCH ARTICLE

Multiple field modified gravity and localized energy in teleparallel framework

Habib Abedi[1] · **Mustafa Salti**[2]

Received: 28 April 2015 / Accepted: 20 July 2015 / Published online: 31 July 2015

Abstract This work is devoted to drive an energy-momentum complex (due to matter and fields including gravity) in the realm of modified teleparallel gravity. For this purpose, the Lagrangian of teleparallel theory is extended to a more general form by replacing the torsion scalar with an arbitrary function of it. Furthermore, considering cosmological perturbations, we explicitly calculate energy distribution associated with the Friedmann–Lemaitre–Robertson–Walker spacetime. Finally, we discuss also the coupling case between matter and gravity in the context of teleparallel modified theory.

Keywords Energy-momentum prescriptions · Modified gravity · Teleparallel theory · Dark energy

1 Introduction

The recent observational data have given indication of accelerated expansion of our Universe [1–5]. Astrophysical evidences show that this strange behavior of the Universe is driven by an exotic content with large negative pressure (dark energy). Recently, Planck-2013 observations [5] of the cosmic microwave background have indicated that the Universe is spatially flat and the matter in our Universe is dominated by dark energy (68.3 %) and dark matter (26.8 %). The remaining part (4.9 %) is occupied by other ordinary matters.

✉ Mustafa Salti
musts6@yahoo.com

Habib Abedi
habib.abedi@gmail.com

1 Department of Physics, University of Tehran, North Kargar Ave, Tehran, Iran

2 Physics Department of Science and Literature Faculty, Mersin University, 33343 Mersin, Turkey

There are many candidates given in literature to investigate this enigmatic dark part of the Universe, but still the nature of dark Universe is completely unknown [6]. The cosmological constant is the earliest and simplest suggestion to explain dark energy, but it gives some other difficulties like fine-tuning and cosmic-coincidence puzzle [7,8]. On the other hand, some of other ones have been given to explain the nature of dark Universe: scalar fields (k-essence, q-essence, tachyon, DBI-essence, dilaton) [9], intergalactic gases (Chaplygin gas, Polytropic gas) [10,11], modified theories of gravity [$f(R)$-gravity, $f(T)$-gravity, scalar-tensor and TeVeS theories, Brane-worlds gravity, Horava–Lifshitz theory, Gauss–Bonnet gravity] [12–18] and Unimodular gravity [19,20]. A good review about the mysterious dark energy problem is written by Bamba et al. [21]. Although some of the above candidates usually used new degree of freedom such as scalar field in scalar-tensor theory and non-linearity of $f(R)$ with respect to R that can be written as scalar degree of freedom but the effects of this degree must be screened in local tests of gravity. Thus, the scalar field should be coupled to matter in convenient way, e.g. chameleon, symmetron, Vainshtein mechanisms etc.(for complete review see ref. [22]). Other coupling of gravity with matter was proposed by Harko in ref. [23] by replacing Ricci scalar with $f(R, \theta)$, where θ is trace of matter energy-momentum tensor and some significant results were obtained. Traceless matter, like photon, does not change the theory through a such coupling. Exotic imperfect fluids or quantum effects (conformal anomaly) may make this coupling. The general coupling of matter Lagrangian $\mathcal{L}_m$, with gravity in the presence of scalar fields in the context of General Relativity was studied in ref. [24]. The cosmological constant can be obtained as a function of matter energy-momentum tensor in the case of $f(R, \mathcal{L}_m)$ coupling, known as $\Lambda(\theta)$ gravity, that indicates to change of cosmological constant in environment [25].

Although the recent observations have been consistent with the single field models, the fundamental physics motivates us to generalize methodologies with more than one active scalar field [26,27]. Multi-field theories are different from the single field models in many aspects, these models are studied well in General Relativity [28]. In study of renormalization, these fields must be non-minimally coupled to gravity. The non-canonical kinetic terms also can exist in some models. The kinetic terms coupled to gravity lead to the instability of the theory. However, some other models including higher derivation of scalar field lead to carefully tuned cancelation of instability (for the detailed discussions one can check the ref. [29]).

Einstein's theory of General Relativity is now a hundred years old and there are still many unsolved problems. The localized four-momentum problem is one of the thorny issues which remains unsolved in gravitational theories [30]. To solve this problem there have been many investigations and the first one was attempted by Einstein himself [31]. After that many prescriptions have been given in the literature; Bergmann-Thomson [32], Tolman [33], Weinberg [34], Landau-Lifshitz [35], Papapetrou [36], Moller [37], Qadir-Sharif [38]. Misner et al. [39] indicated that the energy is localizable only for spherical spacetimes. Next, Cooperstock and Sarracino [40] negated this indication and showed that the energy is localizable for all systems. Several papers have showed that different four-momentum prescriptions yield the same result for a given space-time model.

Furthermore, an interesting class of modified theory can be introduced when we modify the action of an equivalent methodology of Einstein's General Relativity based on torsion [41]. This alternative theory was constructed and named by Einstein himself, it is known as Teleparallel equivalent of General Relativity (TEGR) [16–18]. The main idea in teleparallel gravity is to use a geometry with vanishing curvature and non-zero torsion [17]. In the teleparallel theory, the tetrad field (have 16 independent components) are used rather than the metric (have 10 independent components) as the basic entity. Einstein introduced this theory to unify electromagnetism and gravity with the concept of parallelism, but he did not succeed. His idea was revived by Aldrovandi [42]. Inspiring from $f(R)$-gravity, TEGR can also be modified by using an arbitrary function of torsion scalar in the action. The other possible extension of TEGR is the scalar-tensor modification [43,44]. It is known that some modified TEGR theories do not satisfy local Lorentz symmetry [45] (T and $T^{\rho}_{\mu\nu}$ are not scalar under $h^{A}_{\ \mu} = \Lambda^{A}_{\ B}(x)h^{B}_{\ \mu}$ due to $\Gamma^{\rho}_{\mu\nu} = h^{\rho}_{A}\partial_{\nu}h^{A}_{\mu}$). These theories have more degrees of freedom that arise from the local Lorentz violation. When we restrict ourself to assume highly symmetric space-time models such as background level of Friedmann–Lemaitre–Robertson–Walker (FLRW) or spherical symmetry, these degrees of freedom will disappear. Moreover, some works have been devoted to discuss new degrees of freedom for linear (FLRW) perturbations [46–50]. However, ignoring these degrees of freedom in perturbation level may leads to inconsistency [51].

The rest of paper is organized as follow; the next Section contains some preliminaries. Then, in Sect. 3, we extend the teleparallel theory of gravity to a more general form that contains multiple scalar fields. The purpose of Sect. 4 is studying energy distribution (due to matter plus fields including gravity) associated with the linearly perturbed FLRW spacetime. Next, the Sect. 5 is devoted to investigate scalar fields coupled with matter. Finally, we give conclusions in the last Section.

Notations and conventions Throughout this work, we represent the space-time indices by Greek alphabet (α, β, μ, ν...) and the tangent space indices by Latin alphabet (A, B, C, D...). These indices take the values 0, 1, 2, 3. In addition, we denote that (i, j, k...) and ($a, b, c... = 1, 2, 3$) mean spatial parts, the ($I, J, K... = 1, 2, 3, ..., N$) indices define field space, and the subscript TG describes teleparallel quantities.

2 Preliminaries: $f(T)$-gravity and four-momentum

It is known that vierbein fields $\{\mathbf{h}^{A}(x^{\mu}), \mathbf{h}_{A}(x^{\mu})\}$ construct a teleparallel structure on manifold which forms orthonormal basis of the tangent space and $\mathbf{h}_{A}\cdot\mathbf{h}_{B} = \eta_{AB}$. The vierbein fields $\mathbf{h}_{A}$ with respect to the coordinate basis can be written as $\mathbf{h}_{A} = h^{\mu}_{A}\partial_{\mu}$. For an orthonormal vierbein the metric tensor is defined by the equation

$$g_{\mu\nu} = \eta_{AB}h^{A}_{\ \mu}h^{B}_{\ \nu}, \tag{1}$$

where $\eta_{AB} = diag(+1, -1, -1, -1)$ is the Minkowski metric. The vierbein fields and their dual satisfy the following relations;

$$h^{A}_{\ \mu}h^{\ \mu}_{B} = \delta^{A}_{B}, \quad h^{A}_{\ \mu}h^{\ \nu}_{A} = \delta^{\nu}_{\mu}. \tag{2}$$

Using the parallel transport of vierbein fields and assuming a class of frames in which spin connection is zero, Weitzenböck connection can be defined as

$$\Gamma^{\alpha}_{\mu\nu} := h_A^{\ \alpha}\partial_\nu h^A_{\ \mu} = -h^A_{\ \mu}\partial_\nu h_A^{\ \alpha}. \tag{3}$$

Such connection defines absolute parallel transportation with only torsion effects. The teleparallel's name comes from here; parallel at a distance. The torsion scalar is a specific choice;

$$\begin{aligned} T &= \frac{1}{4}T^{\rho\mu\nu}T_{\rho\mu\nu} + \frac{1}{4}T^{\rho\mu\nu}T_{\nu\mu\rho} - T_{\rho\mu}^{\ \ \rho}T^{\nu\mu}_{\ \ \nu} \\ &= \frac{1}{2}S^{\mu\nu\lambda}T_{\mu\nu\lambda}, \end{aligned} \tag{4}$$

where $T^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu} - \Gamma^{\alpha}_{\mu\nu}$ is the torsion tensor and the tensor

$$S^{\alpha\mu\nu} = \frac{1}{4}\left(T^{\alpha\mu\nu} + T^{\mu\alpha\nu} - T^{\nu\alpha\mu}\right) - \frac{1}{2}\left(g^{\alpha\nu}T^{\beta\mu}_{\ \ \ \beta} - g^{\mu\alpha}T^{\beta\nu}_{\ \ \ \beta}\right), \tag{5}$$

is antisymmetric in the last two indices.

Like we do for the $f(R)$-gravity that generalizes General Relativity, the teleparallel Lagrangian can also be extended to a more general form [$f(T)$-gravity],

$$\mathcal{L}_f = \frac{h}{16\pi G}f(T). \tag{6}$$

In this definition, the special choice $f(T) = T$ reduces f(T)-gravity to the teleparallel theory. Variation with respect to the vierbeins yields the following field equation,

$$\begin{aligned} & f_T(T)\left[\partial_\rho\left(hh_A^{\ \nu}S_\nu^{\ \lambda\rho}\right) - hh_A^{\ \rho}S^{\mu\nu\lambda}T_{\mu\nu\rho}\right] + f_{TT}(T)hh_A^{\ \nu}S_\nu^{\ \lambda\rho}\partial_\rho T \\ & + \frac{1}{2}hh_A^{\ \lambda}f(T) = h\Theta_A^{\ \lambda}, \end{aligned} \tag{7}$$

where $f_T(T) := \frac{df(T)}{dT}$, $f_{TT}(T) := \frac{d^2f(T)}{dT^2}$, and the energy-momentum tensor is

$$\Theta_A^{\ \lambda} := -\frac{1}{h}\frac{\delta\mathcal{L}_m}{\delta h^A_{\ \lambda}}. \tag{8}$$

Here, $\mathcal{L}_m$ is the Lagrangian density of matter fields. Next, considering only the Greek indices, Eq. (7) can be rewritten in the following form

$$f_T G_\alpha^{\ \mu} + \frac{1}{2}\delta_\alpha^{\ \mu}(f - f_T T) + S_\alpha^{\ \mu\nu}\partial_\nu f_T = \Theta_\alpha^{\ \mu}, \tag{9}$$

where we have dropped the explicit dependences of f in T. We set the square of Planck mass as $M_{pl}^2 = 1$ here. Furthermore, the energy-momentum should be symmetric and we have

$$\left(S_{\alpha}{}^{\lambda\nu}g^{\alpha\mu} - S_{\alpha}{}^{\lambda\mu}g^{\alpha\nu}\right)\partial_{\nu} f_T = 0. \tag{10}$$

After considering the Noether theorem and substituting new Lagrangian, one can obtain a new energy-momentum definition for gravitation,

$$ht_{\lambda}^{\ \rho} = f_T(T)ht_{\lambda\mathbf{TG}}^{\rho} - \frac{h}{16\pi G}\delta_{\lambda}^{\ \rho}\left[f(T) - f_T(T)T\right]. \tag{11}$$

Hence, the field equation given by (9) can be written in the following form

$$h\mathcal{T}_{\alpha}{}^{\lambda} = \frac{1}{8\pi G}\partial_{\sigma}[f_T(T)hS_{\alpha}{}^{\lambda\sigma}], \tag{12}$$

where $\mathcal{T}^{\lambda}_{\ \nu} := t^{\lambda}_{\ \nu} + \theta^{\lambda}_{\ \nu}$ is the total energy-momentum of gravitation and matter. This relation leads to the following energy-momentum conservation law,

$$\partial_{\lambda}\left(h\mathcal{T}^{\lambda}_{\ \nu}\right) = 0. \tag{13}$$

The momentum four-vector can be described by an integration over $x^0 = constant$ hypersurface,

$$P_{\mu} = \int \mathcal{T}_{\mu}{}^{0} dxdydz. \tag{14}$$

3 Teleparallel modified gravity with multiple scalar fields

Investigating the presence of scalar fields in gravitational theories is old as General Relativity. Scalar-tensor gravity is one of the first extensions of Einstein's general theory of relativity. In this part of the study, we analyse a very general form of multi-field models in the teleparallel framework with non-canonical kinetic terms. The form which is considered in here

$$S := \int d^4xh\left(\frac{1}{2}f(T, X, \phi^I) + \mathcal{L}_m\right), \tag{15}$$

where $G_{IJ}(\phi^K)$ is a metric on N-dimensional field space and is only a function of fields. The field metric $G_{IJ}(\phi^K)$ determines the kinetic terms,

$$X := \frac{1}{2}G_{IJ}\left(\phi^K\right)\partial_{\mu}\phi^I\partial^{\mu}\phi^J. \tag{16}$$

For the canonical form of kinetic terms, G_{IJ} becomes δ_{IJ}. It is important to mention here that we keep an open mind and do not restrict ourself to consider a flat field space. One can find many studies given in literature that include non-canonical kinetic terms; e.g. Dirac–Born–Infeld (DBI) dark energy and inflation:

$$\mathcal{L} = T + P, \tag{17}$$

where

$$P = -\frac{1}{f(\phi^I)}\left(\sqrt{\mathcal{D}} - 1\right) - V\left(\phi^I\right), \tag{18}$$

and

$$\mathcal{D} := \det\left(\delta^{\nu}_{\mu} + fG_{IJ}\partial^{\mu}\phi^I\partial_{\mu}\phi^J\right). \tag{19}$$

Other direction in field space can be ignored for the radially motion of brane and the Lagrangian (17) is reduced to the single-field DBI-essence [52].

On the other hand, in the ADM formalism, the metric can be generally decomposed as [53,54]

$$ds^2 = \mathcal{N}dt^2 - \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{20}$$

where $\mathcal{N}$ describes lapse function, N^i gives shift vector and γ_{ij} is induced metric. Extrinsic torsion is defined by

$$\Sigma_{ij} = \frac{1}{2\mathcal{N}}\left(\dot{\gamma}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right) \tag{21}$$

where $\mathcal{D}_i$ is 3-covariant derivative with respect to 3-Weitzenböck connection. Now, let us focus on the following action

$$S = \int d^4x F(\phi)\Big(1 + \frac{\zeta}{2}\partial_{\mu}\phi\partial^{\mu}\phi\Big)T, \tag{22}$$

where ζ is a constant. It is known that $\mathcal{N}$ is not a dynamical variable in General Relativity and simplest model of teleparallel gravity. If we use ADM decomposition to rewrite Eq. (22), there will be a term which contains the time derivative of lapse function, i.e. $\zeta\int d^3x dt\sqrt{\gamma}F\dot{\Sigma}^i_i\frac{\dot{\phi}^2}{\mathcal{N}}$. Note that the extrinsic torsion contains the lapse function. Thus the scalar T coupled to X changes the degrees of freedom number, i.e. it makes the lapse function a dynamical variable. We should set $\zeta = 0$ to avoid this problem and related instabilities.

At this point, we focus on our model. Variation of the action (15) with respect to the vierbein fields yields

$$\begin{aligned}&\frac{1}{2}fh_k^{\ \mu} + f_T\big[h^{-1}\partial_{\nu}\left(hh_k^{\ \rho}S_{\rho}^{\ \mu\nu}\right) - h_k^{\ \gamma}S^{\rho\beta\mu}T_{\rho\beta\gamma}\big] + h_k^{\ \rho}S_{\rho}^{\ \mu\nu}\partial_{\nu}f_T\\ &\quad -\frac{1}{2}f_X G_{IJ}\left(\phi^L\right)h_k^{\ \nu}\partial^{\mu}\phi^I\partial_{\nu}\phi^J = \Theta_k^{\ \mu},\end{aligned} \tag{23}$$

where the subscript X denotes derivative with respect to X. Then, the field equation, in Greek indices, transforms into the following form

$$f_T G_{\alpha}^{\ \mu} + \frac{1}{2}\delta_{\alpha}^{\ \mu}\left(f - f_T T\right) + S_{\alpha}^{\ \mu\nu}\partial_{\nu}f_T - \frac{1}{2}f_X G_{IJ}\left(\phi^L\right)\partial^{\mu}\phi^I\partial_{\alpha}\phi^J = \Theta_{\alpha}^{\ \mu}. \tag{24}$$

In addition, vanishing antisymmetric part of the matter energy-momentum tensor leads

$$\big(S_{\alpha}^{\ \lambda\nu}g^{\alpha\mu} - S_{\alpha}^{\ \lambda\mu}g^{\alpha\nu}\big)\partial_{\nu}f_T = 0. \tag{25}$$

On the other hand, the variation of the action defining the multi-field gravity with respect to scalar fields gives

$$\nabla_\alpha \nabla^\alpha \phi^L + \partial_\mu \phi^J \partial^\mu \phi^K \Gamma^L_{KJ} + \frac{\partial_\alpha f_X}{f_X} \partial^\alpha \phi^L - \frac{G^{LI}}{f_X} f_I = 0, \tag{26}$$

where Γ^L_{KJ} is the field space Christoffel symbols, and f_I defines the partial derivation of f with respect to ϕ^I. Considering the Noether theorem, one can find

$$ht^\lambda_\alpha = \frac{\partial \mathcal{L}}{\partial \partial_\lambda h^A_\mu} \partial_\alpha h^A_\mu + \frac{\partial \mathcal{L}}{\partial \partial_\lambda \phi^J} \partial_\alpha \phi^L - \delta^\lambda_\alpha \mathcal{L}, \tag{27}$$

where $\mathcal{L}$ is the gravitational part of Lagrangian density used in Eq. (15). Hence, the gravitational energy-momentum takes the form

$$ht^\lambda_\alpha = hf_T t^\lambda_{\alpha\mathbf{TG}} - \frac{1}{16\pi G} h\delta^\lambda_\alpha (f - f_T T) + \frac{1}{16\pi G} hf_X G_{IJ} \partial^\mu \phi^J \partial_\alpha \phi^J, \tag{28}$$

and, the energy-momentum conservation law can be written as

$$\partial_\nu (h\mathcal{T}^\lambda_\nu) = \partial_\nu \left[h\left(t^\lambda_\nu + \theta^\lambda_\nu\right)\right] = 0, \tag{29}$$

where

$$8\pi Gh\mathcal{T}_\alpha{}^\lambda = \partial_\sigma \left(f_T h S_\alpha{}^{\lambda\sigma}\right). \tag{30}$$

As we see, the result is similar to the one obtained in $f(T)$-gravity. In the case of k-essence with the Lagrangian $f(T, \phi^I, X) = T + P(\phi^I, X)$, which scalar fields minimally coupled to gravity, we get same results of simple model of teleparallel gravity.

4 Cosmological perturbations and energy in FLRW Universe

Now, we study the energy distribution associated with the flat FLRW Universe by considering cosmological perturbations and discuss the effects of extra degrees of freedom. The flat FLRW metric is given by

$$ds^2 = dt^2 - a^2(t)\delta_{ij} dx^i dx^j. \tag{31}$$

Choosing trivial vierbein fields

$$h^A_\mu = diag(1, a, a, a), \tag{32}$$

and considering the field equations of teleparallel multi-field gravity, we obtain

$$3H^2 f_T - \frac{1}{2}(f - f_T T) + \frac{1}{2} f_X G_{IJ}\dot{\phi}^I\dot{\phi}^J = \rho_m, \tag{33}$$

$$-\left(3H^2 + 2\dot{H}\right) f_T + \frac{1}{2}(f - f_T T) - H\partial_0 f_T = p_m, \tag{34}$$

and

$$\ddot{\phi}^L + \Gamma^L_{\ KJ}\dot{\phi}^J\dot{\phi}^K + \left(3H + \frac{\partial_0 f_X}{f_X}\right)\dot{\phi}^L - \frac{G^{LI}}{f_X} f_I = 0. \tag{35}$$

Here, the derivative with respect to the cosmic time t of a function is denoted by a dot. After using the definition $\mathcal{D}_t\dot{\phi}^I := \ddot{\phi}^I + \Gamma^I_{\ KJ}\dot{\phi}^J\dot{\phi}^K$ which describes the acceleration in field space, one can write Eq. (35) in a more compact form;

$$\mathcal{D}_t\left(a^3 f_X\dot{\phi}^I\right) = a^3 f^I, \tag{36}$$

where $\mathcal{D}_t$ acts on quantities as ordinary derivative and does not have any effects on capital Latin letters. Next, the surviving components of $S_\mu^{\ \nu\lambda}$ are

$$S_j^{\ 0i} = -H\delta^i_j. \tag{37}$$

Thus, it is seen that the background energy-momentum distribution associated with the FLRW spacetime in TEGR and its modification version are equal to zero.

In this spacetime model, all of the background quantities are time dependent and indicates that the Universe is homogeneous and isotropic. However, inhomogeneities of our actual Universe have been already presented. Perturbation theory is widely applicable for the seed of structure in the Universe. The most general form of perturbed FLRW metric is

$$\begin{aligned} ds^2 = (1+2\psi)dt^2 + 2a(G_i + \partial_i B)dtdx^i - a^2\big[(1-2\varphi)\delta_{ij} + \partial_i\partial_j E \\ + \partial_{(i}C_{j)} + h_{ij}\big]dx^i dx^j, \end{aligned} \tag{38}$$

where ψ, B, φ and E are scalar perturbations, G_i and C_i with the conditions $\partial^i G_i = 0 = \partial^i C_i$ are vector perturbations and h_{ij} with $h^i_{\ i} = 0 = \partial^i h_{ij}$ is the tensor part of perturbations. It is known that linear order scalar, vector and tensor perturbations evolve independently. The teleparallel formalism violates local Lorentz symmetry, thence it gives six extra degrees of freedom. We used the same perturbation as in ref. [55]. Extra degrees can be described by α, α^i and $B^i_{\ j}$ where $\partial_i\alpha^i = 0$, $B^{ij} = -B^{ji}$ and $\partial_i\partial_j B^{ij} = 0$ (B^{ij} has one scalar and two vector degrees). These two scalar and four vector degrees will not appear in metric formalism.

Now, we can assume that vierbein fields are

$$h^0_{\ \mu} = \delta^0_\mu(1+\psi) + a\delta^i_\mu\partial_i(B+\alpha) + a\delta^i_\mu(G_i + \alpha_i), \tag{39}$$

$$h^a_{\ \mu} = a\delta^a_\mu(1-\varphi) + a\delta^i_\mu(\partial_i\partial^a E + \partial^a C_i + h^a_i) + a\delta^i_\mu B^a_i + \delta^0_\mu(\partial^a\alpha + \alpha^a). \tag{40}$$

It is known that the action must be invariant under the gauge transformations. The coordinate systems of two different gauges are related to each other by the following transformation

$$x^{\mu} \rightarrow \tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}, \tag{41}$$

and, the vierbein fields transform as

$$h^{A}_{\ \tilde{\mu}}(\tilde{x}) = \frac{\partial x^{\nu}}{\partial \tilde{x}^{\mu}} h^{A}_{\ \nu}. \tag{42}$$

Applying the gauge transformation, we get the following vierbein perturbation components defined in the new gauge

$$\begin{aligned}
\tilde{\psi} &= \psi - \dot{\xi}^{0}, \\
a\tilde{E} &= aE + a\dot{\xi} - \xi^{0}, \\
\tilde{\alpha} &= \alpha - \dot{\xi}, \\
a\tilde{B} &= aB - \xi, \\
\tilde{\varphi} &= \varphi + H\xi^{0}, \\
\tilde{G}_i &= G_i + \dot{\xi}^{i(v)}, \\
\tilde{\alpha}_i &= \alpha_i - \dot{\xi}_i^{(v)}, \\
a\tilde{C}_i &= aC_i - \xi_i^{(v)}, \\
a\tilde{B}_{ij} &= aB_{ij} - \xi_{j,i}^{(v)} + \xi_{i,j}^{(v)}, \\
\tilde{h}_{ij} &= h_{ij}.
\end{aligned} \tag{43}$$

Here, it has been used $a\xi^{i} := \xi_i^{(v)} + \xi_{,i}$ with $\partial^{i}\xi_i^{(v)} = 0$. Choosing $\xi = aB$, $\xi^{0} = aE + a(aB)^{\cdot}$ and $\xi_i^{(v)} = aC_i$, one can omit B, E and C_i. For scalar perturbations, we have

$$h^{0}_{\ \mu} = \delta^{0}_{\mu}(1+\psi) + a\delta^{i}_{\mu}\partial_i(B+\alpha), \tag{44}$$

$$h^{a}_{\ \mu} = a\delta^{a}_{\mu}(1-\phi) + a\delta^{i}_{\mu}\left(\partial_i\partial^{a}E + B^{a}_{\ i}\right) + \delta^{0}_{\mu}\partial^{a}\alpha, \tag{45}$$

and, the case $B = 0 = E$ gives

$$g_{\mu\nu} = diag((1+2\psi), -a^{2}(1+2\varphi)\delta_{ij}), \tag{46}$$

which is known as conformal Newtonian or Longitudinal gauge (it means that we choose an observer attached to the unperturbed frame).

Moreover, the surviving components of torsion tensor are

$$\begin{aligned}
T^{0}_{\ i0} &= \partial_i\psi - a\partial_0\partial_i\alpha, \\
T^{i}_{\ j0} &= (\dot{\varphi} - H)\delta^{i}_{j} - \partial_0 B^{i}_{\ j} + a^{-1}\partial_j\partial^{i}\alpha, \\
T^{i}_{\ jk} &= \partial_k\left(\delta^{i}_{j}\varphi - B^{i}_{\ j}\right) - \partial_j\left(\delta^{i}_{k}\varphi - B^{i}_{\ k}\right).
\end{aligned} \tag{47}$$

Here, we need only the following components of superpotential $S_{\mu}^{\ \alpha\nu}$

$$S_0^{\ 0i} = \frac{1}{a^2}\left(\partial^i \varphi + \frac{1}{2}\partial_j B^{ij}\right). \tag{48}$$

After using $\partial_i \partial_j B^{ij} = 0$, it is found that

$$\mathcal{T}_0^{\ 0} = \frac{1}{8\pi G a^2}\bar{f}_T \triangle\varphi \tag{49}$$

where $\triangle := \partial_i \partial^i$. One can easily check that the background scalar torsion is equal to $\bar{T} = -6H^2$. Thus, we see that the energy density is independent of extra degrees of teleparallel framework. Now, the total energy distribution can be obtained by using Eq. (14)

$$E = P_0 = \frac{\bar{f}_T}{8\pi G a^2}\oint d\overrightarrow{A}.\overrightarrow{\nabla}\varphi. \tag{50}$$

where surface integral denotes to two dimensional boundary of the Universe. We need to define a background vector or tensor for contraction with the vector and tensor perturbations in linear order, but it is impossible. Therefore, the vector and tensor parts of perturbations given in Eqs. (39) and (40) in linear order do not contribute to the energy, and the study of scalar perturbations is sufficient.

5 Coupling with matter

Up to now we consider geometry and scalar fields minimally coupled to matter. However, such coupling can exist for some reasons. In this section we study the energy-momentum complex in some case of non-minimal coupling. In the scalar-tensor theory, the local gravity constraints imply that the mass of scalar field should be large. On the other hand, for the cosmic evolution it must be light. Coupling of scalar fields with matter part of Lagrangian can solve this tricky problem due to the dependence of effective potential in energy density. Inspiring the chameleon action defined in General Relativity, we can couple matter fields with scalar fields using the following action

$$S := \int d^4xh\left(\frac{1}{2}f\left(T, X, \phi^I\right) + \mathcal{L}_m\left[\tilde{h}_k^{\ \mu}, \Psi\right]\right), \tag{51}$$

where $\tilde{h}_k^{\ \mu}$ has a conformal relation with $h_k^{\ \mu}$;

$$\tilde{h}_k^{\ \mu} := A(\phi)h_k^{\ \mu}, \qquad \tilde{g}^{\mu\nu} = A^2(\phi)g^{\mu\nu}. \tag{52}$$

The chameleonic coupling leads to the following field equations;

$$f_T G_\alpha^{\ \mu} + \frac{1}{2}\delta_\alpha^{\ \mu}(f - f_T T) + S_\alpha^{\ \mu\nu}\partial_\nu f_T - \frac{1}{2}f_X G_{IJ}\left(\phi^L\right)\partial^\mu\phi^I\partial_\alpha\phi^J = \tilde{\Theta}_\alpha^{\ \mu}, \tag{53}$$

and

$$\nabla_\alpha \nabla^\alpha \phi^L + \partial_\mu \phi^J \partial^\mu \phi^K \Gamma^L_{KJ} + \frac{\partial_\alpha f_X}{f_X} \partial^\alpha \phi^L - \frac{G^{LI}}{f_X} \left(f_I + \beta_I \tilde{\Theta} \right) = 0, \tag{54}$$

where we set $\beta_I := \frac{d}{d\phi} \ln A(\phi)$ and defined $\tilde{\Theta}_\alpha^{\ \mu} := \tilde{h}^A_{\ \alpha} \tilde{\Theta}_A^{\ \mu}$ with $\tilde{\Theta}_A^{\ \lambda} := -\frac{1}{\tilde{h}} \frac{\delta \mathcal{L}_m}{\delta \tilde{h}^A_{\ \lambda}}$. Field equation (53) is similar to Eq. (24), thus our previous definition of energy-momentum leads to the same conservation law, i.e. $\partial_\nu (h \mathcal{T}^\lambda_{\ \nu}) = 0$.

One can suppose that the gravity part of action is coupled with the trace of matter energy-momentum tensor. Inspiring from $f(R, \theta)$, we consider more general form of teleparallel which contains scalar fields, trace of matter energy-momentum tensor and torsion scalar as follow

$$S := \int d^4 x h \left(\frac{1}{2} f \left(T, X, \phi^I, \Theta \right) + \mathcal{L}_m \right). \tag{55}$$

Then, variation with respect to the vierbein fields of this action yields

$$\begin{aligned} & f_T G_\alpha^{\ \mu} + \frac{1}{2} \delta_\alpha^{\ \mu} (f - f_T T) - \frac{1}{2} f_X G_{IJ} \left(\phi^L \right) \partial^\mu \phi^I \partial_\alpha \phi^J + S_\alpha^{\ \mu\nu} \partial_\nu f_T \\ & \quad = \Theta_\alpha^{\ \mu} \left(1 - \frac{1}{2} f_\Theta \right) - \frac{1}{2} f_\Theta B_\alpha^{\ \mu}, \end{aligned} \tag{56}$$

where

$$B_k^{\ \mu} := h^l_{\ \alpha} \frac{\delta \Theta^l_{\ \alpha}}{\delta h^k_{\ \mu}}. \tag{57}$$

In the case of perfect fluid assumption, we have $\Theta^a_{\ \alpha} = (\rho + p) u^a u_\alpha + p h^a_{\ \alpha}$, thus we get $B^a_{\ \alpha} = \Theta^a_{\ \alpha} - 2 p h^a_{\ \alpha}$. On the other hand, variation with respect to scalar fields gives another field equation

$$\nabla_\alpha \nabla^\alpha \phi^L + \partial_\mu \phi^J \partial^\mu \phi^K \Gamma^L_{KJ} + \frac{\partial_\alpha f_X}{f_X} \partial^\alpha \phi^L - \frac{G^{LI}}{f_X} f_I = 0, \tag{58}$$

and, with the definition of four-momentum, field equation leads

$$\partial_\mu (h \mathcal{T}^\mu_{\ \nu}) = \partial_\mu \left(\frac{1}{2} f_\Theta \left(B^\mu_\nu + \Theta^\mu_\nu \right) \right) \neq 0. \tag{59}$$

Hence, the general form that includes the existence of Θ in the gravity part of Lagrangian yields a non-conservation case of our prescription. As the last model, lets consider a bit more complicated modification of teleparallel theory with an assumption

$$S := \int d^4 x \frac{h}{2} f \left(T, \phi^I, X, \mathcal{L}_m \right), \tag{60}$$

where f is an arbitrary function of torsion scalar, matter Lagrangian density and scalar fields. Then, variation with respect to the vierbein fields of the action (60) yields a new field equation;

$$\begin{aligned}f_T G_\alpha^{\ \mu} + \frac{1}{2}\delta_\alpha^{\ \mu}\left(f - f_T T - f_{\mathcal{L}_M}\mathcal{L}_M\right) + S_\alpha^{\ \mu\nu}\partial_\nu f_T \\ - \frac{1}{2} f_X G_{IJ}\left(\phi^L\right)\partial^\mu\phi^I\partial_\alpha\phi^J = \frac{1}{2} f_{\mathcal{L}_M}\Theta_\alpha^{\ \mu},\end{aligned} \tag{61}$$

where the subscript $\mathcal{L}_M$ denotes to derivation with respect to $\mathcal{L}_M$. Considering the Noether theorem for the gravitational energy-momentum and Eq. (60) gives a quantity that we cannot distinguish between matter and gravity parts, and we encounter more complication to define an energy momentum complex. Thus, it is better to use our approach to extend energy-momentum in a simple extension of TEGR.

6 Conclusions

There are many important issues introduced in literature such as cosmological missing matter problem and four-momentum localization in curved spacetimes, since the advent of general theory of relativity. These problems are still in doubt and have non-specific solutions [56]. We could ask why should one investigate four-momentum distribution associated with a spacetime. The corresponding answer is one can get an excellent idea of the spacetime by knowing four-momentum distribution. The study of four-momentum complexes are useful due to it predicts how astrophysical gravitational lensing phenomena would appear in a given spacetime [56]. The energy distribution analysis helped Virbhadra to discover great lensing phenomena [57–61].

In the present work, we mainly generalized the teleparallel theory of gravity to a more general form: the modified teleparallel multi-field gravity. First, we extended the energy-momentum (due to matter plus fields including gravity) to the one defined in $f(T)$ gravity, then it is obtained for the modified multi-field theory in teleparallel framework. In both cases, our definition of energy-momentum leads to a conservation law which is similar to the one given in teleparallel gravity. But, the case of coupling with matter causes a problem, it gives a non-conservation law.

Furthermore, we calculated the energy distribution associated with the FLRW Universe (here we also considered the linear cosmological perturbations). It is shown that this quantity is independent of extra degrees of freedom which appear in teleparallel formalism. Albrow [62] and Tryon [63] assumed that the net energy value of Universe may be equal to zero. The subject of four-momentum of closed and open Universes was initiated by an interesting work [64,65]; in this paper Cooperstock and Israelit found the zero value of energy density for any homogenous isotropic Universe described by a Friedmann–Robertson–Walker metric in the context of general relativity. This interesting study influenced some authors [66–69]. In our calculations, if we neglect the scalar field contributions, the multi-field gravity can be reduced to the teleparallel equivalent of General Relativity. This special assumption supports the viewpoints of

Albrow-Tryon [62,63] and agrees with the previous works of Cooperstock-Israelit [65], Rosen [66], Johri et al. [67], Banerjee-Sen [68], Xulu [69] and Vargas [70].

In the presence of dark contents, we find a surviving energy component for the four-momentum distribution. Due to the contribution of dark components, the energy value we find contradicts the assumption of Albrow [62] and Tryon [63]. In perturbation level the spacetime is not spatially flat. The Ricci scalar of hypersurface $t = constant$ in perturbation level is given by $R^{(3)} = \frac{4}{a^2}\triangle\varphi$. Thus, the non-vanishing energy distribution becomes

$$\mathcal{T}_0{}^0 = \frac{1}{32\pi G}\bar{f}_T R^{(3)}. \tag{62}$$

The extra degrees of freedom does not appear in this quantity. In background and linear order we can conclude the energy of the closed Universe vanishes. The Universe also can have vanishing energy in spatially non-flat model due to $R^{(3)} = 0$ in linear order. Note that, to describe a spatially flat Universe, Riemann tensor needs to vanish.

On the other hand, our results must be extended to some more complicated cases in a more general form by considering another approach. Further work towards this investigation is required and in progress, it is very lengthy and time taking. Our approach is not convenient for the general matter-gravity coupling, but it is significant to mention here that the chameleon coupling is good.

Acknowledgments We would like to thank K.S. Virbhadra and T. Harko for valuable suggestions, O. Aydogdu and K. Sogut for carefully reading manuscript and the referee for giving such constructive comments which substantially helped improving the quality of the paper.

References

1. Supernovae Search Team Collaboration: Astron. J. **116**, 1009 (1998)
2. Boomerang Collaboration: Nature **404**, 955 (2000)
3. Supernovae Cosmology Project Collaboration: Astrophys. J. **517**, 565 (1999)
4. Supernovae Cosmology Project Collaboration: Astrophys. J. **598**, 102 (2003)
5. Planck Collaboration: e-Print: 1303.5072 (2013)
6. Chakraborty, S., Biswas, A.: Astrophys. Space Sci. **343**, 791 (2013)
7. Sharif, M., Saleem, R.: Mod. Phys. Lett. A **27**, 1250187 (2012)
8. Copeland, E.J., Sami, M., Tsujikawa, S.: Int. J. Mod. Phys. D **15**, 1753 (2006)
9. Zlatev, I., Wang, L., Steinhardt, P.J.: Phys. Rev. Lett. **82**, 896 (1999)
10. Padmanabhan, T.: Phys. Rep. **380**, 235 (2003)
11. Cai, Y.F., Saridakis, E.N., Setare, M.R., Xia, J.Q.: Phys. Rep. **493**, 1 (2010)
12. Sotiriou, T.P., Faraoni, V.: Rev. Mod. Phys. **82**, 451 (2010)
13. De Felice, A., Tsujikawa, S.: Living Rev. Rel. **13**, 3 (2010)
14. Sahni, V., Shtanov, Y.: JCAP **11**, 014 (2003)
15. Horava, P.: Phys. Rev. D **79**, 084008 (2009)
16. Hehl, F.W., Von Der Heyde, P., Kerlick, G.D., Nester, J.M.: Rev. Mod. Phys. **48**, 393 (1976)
17. Hayashi, K., Shirafuji, T.: Phys. Rev. D **19**, 3524 (1979)
18. Mikhail, M.I., Wanas, M.I.: Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. **356**, 471 (1977)
19. Padilla, A., Saltas, I.D.: e-Print:1409.3573 [gr-qc]
20. Fiol, B., Garriga, J.: JCAP **1008**, 015 (2010)
21. Bamba, K., Capoziello, S., Nojiri, S., Odintsov, S.D.: Astrophys. Space Sci. **342**, 155 (2012)
22. Joyce, A., Jain, B., Khoury, J., Trodden, M.: Phys. Rep. **568**, 1–98 (2015)
23. Harko, T., Lobo, F.S.N., Nojiri, S., Odintsov, S.D.: Phys. Rev. D **84**, 024020 (2011)

24. Harko, T., Lobo, F.S.N., Minazzoli, O.: Phys. Rev. D **87**, 047501 (2013)
25. Poplawski, N.J.: e-Print: gr-qc/0608031
26. Lyth, D.H., Riotto, A.: Phys. Rep. **314**, 1 (1999)
27. Mazumdar, A., Rocher, J.: Phys. Rep. **497**, 85 (2011)
28. Wands, D.: Lect. Notes Phys. **738**, 275 (2008)
29. Kobayashi, T., Yamaguchi, M., Yokoyama, J.: Prog. Theor. Phys. **126**, 511 (2011)
30. Salti, M., Acikgoz, I.: Phys. Scr. **87**, 045006 (2013)
31. Einstein, A.: Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **17**(217), 224 (1928)
32. Bergmann, P.G., Thomson, R.: Phys. Rev. **89**, 400 (1953)
33. Tolman, R.C.: Relativity, Thermodynamics and Cosmology, p. 227. Oxford University Press, London (1934)
34. Weinberg, S.: Gravitation and Cosmology: Principle and Applications of General Theory of Relativity. Wiley, New York (1972)
35. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields, p. 280. Pergamon Press, Oxford (1977)
36. Papapetrou, A.: Proc. R. Irish Acad. A **52**, 11 (1948)
37. Moller, C.: Ann. Phys. **4**, 347 (1958)
38. Qadir, A., Sharif, M.: Phys. Lett. A **167**, 331 (1992)
39. Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. W.H. Freeman and Co., New York (1973)
40. Cooperstock, F.I., Sarracino, R.S.: J. Phys. A **11**, 877 (1978)
41. Kofinas, G., Saridakis, E.N.: Class. Quant. Gravity **31**, 175011 (2914)
42. Aldrovandi, R., Pereira, J.G.: Teleparallel Gravity, An Introduction. Springer, Berlin (2013)
43. Geng, C.Q., Lee, C.C., Saridakis, E.N., Wu, Y.P.: Phys. Lett. B **704**, 384 (2011)
44. Geng, C.Q., Wu, Y.P.: JCAP **04**, 033 (2013)
45. Li, B., Sotiriou, T.P., Barrow, J.D.: Phys. Rev. D **83**, 064035 (2011)
46. Wu, Y.P., Geng, C.Q.: JHEP **11**, 142 (2012)
47. Izumi, K., OngV, Y.C.: JCAP **06**, 029 (2013)
48. Zhenga, R., Huang, Q.G.: JCAP **03**, 002 (2011)
49. Li, B., Sotitiou, T.P., Barrow, J.D.: Phys. Rev. D **83**, 104017 (2011)
50. Gu, J.A., Lee, C.C., Geng, C.Q.: Phys. Lett. B **718**, 722 (2013)
51. Zheng, R., Huang, Q.-G.: JCAP **03**, 002 (2011)
52. Martin, J., Yamaguchi, M.: Phys. Rev. D **77**, 123508 (2008)
53. Wald, R.M.: General Relativity. University Press, Chicago (1984)
54. Wu, Y.-P., Geng, C.-Q.: Phys. Rev. D **86**, 104058 (2012)
55. Wu, Y.-P., Geng, C.-Q.: AIP Conf. Proc. **1545**, 7 (2013)
56. Sahoo, P.K., Mahanta, K.L., Goit, D., Sihna, A.K., Xulu, S.S., Das, U.R., Prasad, A., Prasad, R.: Chin. Phys. Lett. **32**(2), 020402 (2015)
57. Virbhadra, K.S.: Phys. Rev. D **41**, 1086 (1990)
58. Virbhadra, K.S.: Phys. Rev. D **60**, 104041 (1999)
59. Rosen, N., Virbhadra, K.S.: Gen. Relativ. Gravit. **25**, 429 (1993)
60. Chamorro, A., Virbhadra, K.S.: Pramana J. Phys. **45**, 181 (1995)
61. Aguirregabiria, J.M., Chamorro, A., Virbhadra, K.S.: Gen. Relativ. Gravit. **28**, 1393 (1996)
62. Albrow, M.G.: Nature **241**, 56 (1973)
63. Tryon, E.P.: Nature **246**, 396 (1973)
64. Cooperstock, F.I.: Gen. Relativ. Gravit. **26**, 323 (1994)
65. Cooperstock, F.I., Israelit, M.: Found. Phys. **25**, 631 (1995)
66. Rosen, N.: Gen. Rel. Gravit. **26**, 319 (1994)
67. Johri, V.B., Kalligas, D., Singh, G.P., Everitt, C.W.F.: Gen. Relativ. Gravit. **27**, 323 (1995)
68. Banerjee, N., Sen, S.: Pramana J. Phys. **49**, 609 (1997)
69. Xulu, S.S.: Int. J. Theor. Phys. **30**, 1153 (2000)
70. Vargas, T.: Gen. Rel. Gravit. **36**, 1255 (2004)