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RESEARCH ARTICLE

Investigations of $f(R)$-gravity counterparts of the general relativistic shear-free conjecture by illustrative examples

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Abstract By adopting a metric based approach and making use of $f(R)$-gravity extended tetrad equations, we have considered three spatially homogeneous metrics in order to investigate the existence of simultaneously rotating and expanding solutions of the $f(R)$-gravity field equations with shear-free perfect fluids as sources. We have shown that the Gödel type expanding universe, as well as a rotating Bianchi-type II spacetime allow no such solutions of the field equations of this modified gravity. On the other hand, we have found that there exist two types of $f(R)$ models in which a shear-free Bianchi-type IX universe can expand and rotate at the same time. The matter content of this universe is described by a perfect fluid having positive or negative pressure, depending on the type of $f(R)$ model and on the cosmological constant; in the particular case of a vanishing cosmological constant we have found that the universe is filled with a pure radiation. Whatsoever the cases, the universe exhibits always coasting anisotropic expansions along three spatial directions evolving like a flat Milne universe, and has a vorticity inversely proportional to cosmic time. A further result is that, due to the nonvanishing of the gravito-magnetic part of the Weyl tensor, this model allows for gravitational waves. Our solution constitutes one more example giving support to that in $f(R)$-gravity there is no counterpart of the general relativistic shear-free conjecture.

Keywords $f(R)$-gravity · Gödel type metric · Bianchi type-II and IX models · Shear-free perfect fluid · Rotation · Tetrad formalism

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1 Introduction

The well-known shear-free perfect fluid conjecture originated from the famous Ellis result in 1967 [1] for pressure free matter and subsequently generalized to include the pressure, states that, in General Relativity (GR), it is impossible to have simultaneous rotation and expansion for a shear-free perfect fluid, satisfying $\mu + p \neq 0$, where μ is the energy density and p is the pressure [2]. The first indications supporting the validity of this conjecture in the case of dust go to a statement of Gödel in 1950 [3], presented without proof, and to the work of Schücking in 1957 [4]; in fact, Gödel claimed that a shear-free dust filled homogeneous Bianchi-type IX cosmology could either expand or rotate, but not both, and Schücking generalized this result to general spatially homogeneous dust [4]. The aforementioned work of Ellis [1] showed that the restriction to spatial homogeneity was unnecessary. This intriguing conjecture has attracted considerable attention during the last fourty years. A number of works [5–20], using either special coordinate systems or particular tetrads or a fully covariant approach together with special assumptions on the equations of state as well as on some kinematic and dynamic quantities, have given strong support to it. Nevertheless, except for these very special cases, the truth or falsity of the conjecture in the general case has still not been established, and at present we know of no counter examples at all. In this connection, we draw the attention of the reader to a recent paper of Ellis [21] in which the deeper understanding of the physical aspects of the shear-free solutions has been explored.

Within the context of general relativistic cosmology, one way to obtain shear-free solutions with simultaneous rotation and expansion is to relax the restriction of a perfect fluid source and introduce a heat flux and/or anisotropic pressure into the energy momentum tensor of the matter content; in other words, the energy momentum tensor must be that of an imperfect fluid form [22]. Another way is to consider more general energy–momentum tensors involving, for example, scalar fields as in inflationary scenarios [23]. On the other hand, we know that in Newtonian theory of gravitation, this conjecture has no counterparts, that is, one can have simultaneously rotating and expanding shear-free perfect fluid solutions in this theory [4,16,24]. The very existence of this fact leads to the suggestion that the shear-free conjecture seems inherent to GR, and consequently one can expect that possible counter examples might exist within the context of alternative theories to GR to describe gravitational phenomena. In fact, as supporting this expectation, an attempt along this line has recently been made by Abebe et al. [25] within the framework of the so-called $f(R)$-gravity [26–32] which has received much attention during the last fifteen years for solving some cosmological issues related to the late-time cosmic acceleration and also to the early-time inflation; in this theory the standard Einstein–Hilbert (EH) action is modified by replacing the curvature scalar R with an arbitrary function $f(R)$, leading then, in general, to fourth order partial differential equations in the metric tensor. The authors of [25], using the 1+3 covariant formalism, have shown that for a stiff fluid in R^3-gravity there exists a flat Milne-universe solution which can rotate and expand simultaneously at the level of linearized perturbation about a Friedmann–Lemaître–Robertson–Walker (FLRW) background. As far as we know, this solution constitutes the first counter example to the shear-free conjecture apart the Newtonian ones. However, this kind of solution exhibits linearization instability; the result may not be the limit of an exact solution in the full non-linear theory, due to the fact that second order terms are discarded during linearization (see, for instance, [33,34]). As a consequence of this fact, one cannot rule out that there is no counterpart of the general relativistic shear-free conjecture in $f(R)$-gravity.

In this paper, we aim to seek whether or not there is a counterpart in $f(R)$-gravity of the general relativistic shear-free conjecture by adopting a metric based approach and making use of the total set of exact non-linear tetrad equations. This approach will lead us to a set of differential equations which will be discussed from the point of compatibility with the parameters of the metric as well as with the crucial assumption $\mu + p \neq 0$. This will provide us with the means to search for the simultaneously rotating and expanding solutions. To achieve this goal, we do not intend to undertake a general attempt by considering general metrics; instead, as a tentative first step in the hope of obtaining easily tractable equations that would lead us to possible analytical solutions, we prefer to start from some relatively simple metric ansatzes. Hence, we will restrict ourselves to particular forms of the shear-free class of spatially homogeneous space-times among which the first will be the Gödel type expanding model [35], and other two will be a specialized form of the rotating Bianchi-type II and IX models. In all of these models the overall expansion will be described by a single time dependent scale factor. We note that a similar approach was already undertaken by Grøn and Soleng in the case of an expanding version of the stationary Gödel metric within the context of GR [22]. The main differences between our approach and the one in [22] are that we consider here, a further generalized form of the Gödel metric, and tetrad equations instead of components of EFE; the advantage of using all of the $f(R)$-gravity extended tetrad equations is that the integrability conditions, i.e., the Bianchi identities, are automatically incorporated within them.

This paper is organized as follows: In Sect. 2, we give the basic equations of $f(R)$-gravity, and the main ingredients of the 1+3 covariant approach and the tetrad formalism in $f(R)$-gravity in order to set up the notation, definitions and relevant formulas that will be used in the sequel. In Sect. 3, three examples of metrics will be applied to $f(R)$-gravity extended tetrad equations, and a compatibility analysis will be undertaken for the so obtained sets of equations. Finally, Sect. 4 contains a summary of the results and conclusions.

We use units in which $c = 1 = 8\pi G$; the symbol ∂ is used for partial differentiation with respect to coordinates, and primes denote derivatives with respect to R. Latin indices range from 0 to 3, and Greek indices, from 1 to 3.

2 Prerequisites

2.1 Field equations of $f(R)$-gravity

$f(R)$-gravity is defined by the following modified Einstein–Hilbert action with matter, in which the Ricci scalar R is replaced by an arbitrary function $f(R)$

$$S = \frac{1}{2}\int d^4x\sqrt{-g}f(R) + S_m, \tag{1}$$

where S_m is the standard action for matter. Variation with respect to the metric g_{ab} leads to the following modified EFE

$$f'R_{ab} - \frac{1}{2}g_{ab}f - \nabla_a\nabla_b f' + g_{ab}\nabla^c\nabla_c f' = T^m_{ab}, \tag{2}$$

where R_{ab} is the Ricci curvature tensor and T^m_{ab} is the energy momentum tensor of the standard matter. Due to the last two terms in the left hand side, these equations are, in general, fourth-order partial differential equations in the metric tensor. When $f(R) = R - 2\Lambda$, Λ being the cosmological constant, Eq. (2) reduces to standard EFE. To make the Einstein tensor G_{ab} explicit, one can rewrite Eq. (2) equivalently in the form of

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R = \frac{T^m_{ab}}{f'} + \frac{1}{f'}\left[\frac{1}{2}g_{ab}(f - Rf') + \nabla_a\nabla_b f' - g_{ab}\nabla^c\nabla_c f'\right] \tag{3}$$

provided that $f' \neq 0$. Then, adopting the effective energy–momentum tensor approach used in [36–38], one may define the following energy–momentum tensors

$$\tilde{T}^m_{ab} = \frac{T^m_{ab}}{f'}, \tag{4a}$$

$$T^R_{ab} = \frac{1}{f'}\left[\frac{1}{2}g_{ab}(f - Rf') + \nabla_a\nabla_b f' - g_{ab}\nabla^c\nabla_c f'\right], \tag{4b}$$

$$T^t_{ab} = \tilde{T}^m_{ab} + T^R_{ab}, \tag{4c}$$

such that the modified EFE (3) take the standard Einstein form $G_{ab} = T^t_{ab}$. Hence, in this approach, one has two effective fluids associated with $\tilde{T}^m_{ab}$ and T^R_{ab} which may be called as "effective matter fluid" and "effective curvature fluid", respectively. These various energy–momentum tensors obey the following conservation equations:

$$\nabla^b T^m_{ab} = 0, \tag{5a}$$

$$\nabla^b T^R_{ab} = \frac{f''}{f'^2}T^m_{ab}\nabla^b R, \tag{5b}$$

$$\nabla^b T^t_{ab} = 0, \tag{5c}$$

where Eq. (5c) results from the contracted second Bianchi Identities $\nabla^b G_{ab} = 0$, and means that the effective total energy–momentum tensor is conserved; Eq. (5b) can be obtained from (3) and (4b), while the Eq. (5a) arises from (4c), as a consequence of (4a), (5b) and (5c). Equation (5a) shows that the usual conservation of the standard matter still holds in this theory.

2.2 1+3 Covariant approach in $f(R)$-gravity

Given a preferred vector field **u** which, for definiteness, will be taken as the 4-velocity ($u_a u^a = -1$) of an observer comoving with the fluid flow, one can define a projection tensor $h_{ab} = g_{ab} + u_a u_b$ which, together with u^a, can be used to decompose any spacetime quantity into parts orthogonal and parallel to $\mathbf{u}$. In particular, the covariant derivative of the 4-velocity $\mathbf{u}$, and a general energy–momentum tensor T_{ab} may be split into their basic parts as

$$\nabla_a u_b = -u_a \dot{u}_b + D_a u_b, \quad \text{with } D_a u_b = \sigma_{ab} + \frac{1}{3}\theta h_{ab} + \varepsilon_{abc}\omega^c \tag{6}$$

and

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab}, \tag{7}$$

respectively [39–41]. Here, the overdot denotes covariant time derivative ($u^a\nabla_a$), and D_a is the projected covariant 3-dimensional derivative onto the instantaneous rest space of the observer; ε_{abc} is the spatial permutation tensor. Equation (6) defines the kinematic quantities $\dot{u}_a, \theta, \sigma_{ab}, \sigma, \omega^a, \omega$ associated with the fluid flow; these are: acceleration, expansion scalar, shear tensor, shear scalar, vorticity vector and vorticity scalar, respectively. Equation (7) defines fluid dynamic quantities as measured by an observer flowing with $\mathbf{u}$. These are: the matter–energy density μ, isotropic pressure p, energy flux q_a and the anisotropic stress tensor π_{ab}, defined as

$$\mu = u^a u^b T_{ab}, \quad p = \frac{1}{3}h^{ab}T_{ab}, \quad q_a = -h^b_a u^c T_{bc}, \quad \pi_{ab} = h^c_{\langle a} h^d_{b\rangle} T_{cd}. \tag{8}$$

In the $1+3$ covariant approach to GR, the above fluid kinematic and dynamic quantities together with the electric part E_{ab} and the magnetic part H_{ab} of the Weyl curvature tensor satisfy a set of evolution equations involving time derivatives, and a set of constraint equations involving only D_a derivatives. In the case of $f(R)$-gravity the corresponding covariant equations may be derived directly from the standard GR versions in [41] by simply replacing the energy–momentum tensor terms μ, p, q_a, π_{ab} by μ^t, p^t, q^t_a, π^t_{ab} which, according to Eqs. (4a) and (4c), are given by

$$\mu^t = \frac{1}{f'}\mu^m + \mu^R, \quad p^t = \frac{1}{f'}p^m + \mu^R, \quad q^t_a = \frac{1}{f'}q^m_a + q^R_a, \quad \pi^t_{ab} = \frac{1}{f'}\pi^m_{ab} + \pi^R_{ab}, \tag{9}$$

where each quantity is defined by using the corresponding T^t_{ab}, T^m_{ab} and T^R_{ab} in place of T_{ab} in Eqs. (7) and (8). The total set of $1+3$ covariant equations of the $f(R)$ theory of gravity can be found in slightly various forms in [26,37]. The expressions of dynamic quantities μ^R, p^R, q^R_a and π^R_{ab} of the effective curvature fluid calculated from (4b) and (8) using the 1+3 projection techniques are given in [37]. In the following, we will convert them into the tetrad form.

2.3 Tetrad equations in $f(R)$-gravity

The GR tetrad equations may be translated into the $f(R)$ theory of gravity in the same manner as in the case of $1+3$ covariant equations; we simply have to substitute into the tetrad equations given in [41], the effective total dynamic quantities μ^t, p^t, q^t_α and $\pi^t_{\alpha\beta}$

in place of μ, p, q_α and $\pi_{\alpha\beta}$ wherever they appear. The equations retain their forms, however, there is a notable difference with regard to GR case; the set of equations must be supplemented by two additional evolution equations for μ^m and q^m_α arising from the conservation equation (5a) of the standard matter–energy tensor. These two equations have the same form as the corresponding equations for μ^t and q^t_α derived from the conservation equation (5c) of the effective total energy–momentum tensor [notice that Eq. (5b) is not independent, since it can be obtained from Eqs. (5c) and (5a), using (4a) and (4c)]. In this work we use the evolution and constraint equations given in [41], adapting them to the $f(R)$-gravity case. We note that a part of the overall tetrad equations, might be obtained from the $1+3$ covariant evolution and constraint equations of the $f(R)$-gravity given in [37], by converting them into tetrad forms using suitable transformation formulas described in [42]. We use these formulas here, to establish in particular the following substitutions

$$
\begin{aligned}
&a, b, c \cdots \to \alpha, \beta, \gamma \ldots, \quad \varepsilon_{abc} \to \varepsilon_{\alpha\beta\gamma}, \quad \dot{R} \to e_0 R,\\
&\ddot{R} \to e_0 e_0 R, \quad D_\alpha R \to e_\alpha R, \quad D_\alpha \dot{R} \to e_\alpha e_0 R,\\
&D_a D_b R \to D_\alpha D_\beta R = e_\alpha e_\beta R - \left(a^\lambda \delta_{\alpha\beta} - a_\beta \delta^\lambda_\alpha + \frac{1}{2}\varepsilon_{\alpha\beta\gamma} n^{\lambda\gamma} + \varepsilon^{\gamma\lambda}_{(\alpha} n_{\beta)\gamma} \right) e_\lambda R.
\end{aligned}
\tag{10}
$$

These lead us then to express in tetrad forms the dynamic quantities of the effective curvature fluid as follows:

$$
\mu^R = -\frac{1}{2f'}(f - Rf') + \frac{f''}{f'}(-\theta e_0 R + \delta^{\alpha\beta} e_\alpha e_\beta R - 2a^\alpha e_\alpha R) + \frac{f'''}{f'}\delta^{\alpha\beta} e_\alpha R e_\beta R, \tag{11}
$$

$$
\begin{aligned}
p^R =& \frac{1}{2f'}(f - Rf') + \frac{f''}{f'}\left(\frac{2}{3}\theta e_0 R - \dot{u}^\alpha e_\alpha R + e_0 e_0 R - \frac{2}{3}\delta^{\alpha\beta} e_\alpha e_\beta R + \frac{4}{3} a^\alpha e_\alpha R \right)\\
&+ \frac{f'''}{f'}\left[(e_0 R)^2 - \frac{2}{3}\delta^{\alpha\beta} e_\alpha R e_\beta R \right],
\end{aligned}
\tag{12}
$$

$$
q^R_\alpha = \frac{f''}{f'}\left[-e_\alpha e_0 R + \left(\sigma^\beta_\alpha + \frac{1}{3}\theta \delta^\beta_\alpha + \varepsilon^{\beta\gamma}_\alpha \omega_\gamma \right) e_\beta R \right] - \frac{f'''}{f'} e_0 R e_\alpha R, \tag{13}
$$

$$
\pi^R_{\alpha\beta} = \frac{f''}{f'}\left[-\sigma_{\alpha\beta} e_0 R + e_{\langle\alpha} e_{\beta\rangle} R + (a_{\langle\alpha} \delta^\lambda_{\beta\rangle} + \varepsilon^\lambda_{\gamma\langle\alpha} n^\gamma_{\beta\rangle}) e_\lambda R \right] + \frac{f'''}{f'} e_{\langle\alpha} R \mathrm{e}_{\beta\rangle} R. \tag{14}
$$

3 Metric examples

3.1 Specialization of the metric

In this paper, we will consider three examples of spacetimes belonging to the wide class of rotating spatially homogeneous models which, in a 1-forms basis $\{\omega^a\}$, read

$$
ds^2 = -(\omega^0 - \upsilon_\alpha(t)\omega^\alpha)^2 + \Gamma_{\alpha\beta}(t)\omega^\alpha\omega^\beta, \tag{15}
$$

where $\upsilon_\alpha(t)$ and $\Gamma_{\alpha\beta}(t)$ are functions of the cosmic time t alone. These functions, however, are too general for our purposes, since it is unlikely to obtain tractable equations; so, following [23,35,43], we restrict their functional forms to

$$\upsilon_\alpha(t) = a(t)v_\alpha, \quad \Gamma_{\alpha\beta}(t) = a^2(t)k_\alpha^2\delta_{\alpha\beta} \text{ (no sum on } \alpha), \tag{16}$$

where $a(t)$ is a time dependent cosmic scale factor, k_α and v_α are constant parameters; such that $k_\alpha > 0$ and at least one parameter v_α is nonzero. Note that expression (15) is a very particular form of the general threading form of the spacetime metric [44]

$$ds^2 = -M^2(t,x)(dt - M_\nu(t,x)dx^\nu)^2 + h_{\mu\nu}(t,x)dx^\mu dx^\nu, \tag{17}$$

expressed in local coordinates $\{x^i\}$ with basis (dx^i, ∂_i). The metric (15) is obtained by adopting the gauge $M(t,x) \equiv 1$ and assuming that both $M_\nu(t,x)$ and $h_{\mu\nu}(t,x)$ are separable functions of the timelike and spacelike coordinates as $M_\nu(t,x) = \upsilon_\alpha(t)\omega_\nu^\alpha(x)$ and $h_{\mu\nu}(t,x) = \Gamma_{\alpha\beta}(t)\omega_\mu^\alpha(x)\omega_\nu^\beta(x)$. This allows then to define four time independent basis 1-forms $\omega^a = \omega_i^a(x)dx^i$, together with the corresponding dual basis vectors $E_a = \omega_a^i(x)\partial_i \quad (\omega^a(E_b) = \delta_b^a)$ as

$$\begin{array}{l} \omega^0 = dt \\ \omega^\alpha = \omega_\nu^\alpha(x)dx^\nu \end{array} \Leftrightarrow \begin{array}{l} E_0 = \partial_t \\ E_\alpha = \omega_\alpha^\nu(x)\partial_\nu \end{array} \quad \left(\text{with } \omega_\nu^\alpha\omega_\beta^\nu = \delta_\beta^\alpha\right), \tag{18}$$

such that, in this comoving (ω^a, E_a)-frame, (17) takes the form (15). Since it is much easier to work in an orthonormal basis, we pass now from the (ω^a, E_a)-frame to an orthonormal (σ^a, e_a)-frame such that the space-time line element has the Lorentzian form $ds^2 = \eta_{ab}\sigma^a\sigma^b$ where $\eta_{ab} = diag(-1,1,1,1)$. From Eqs. (15) and (16), this can be accomplished by the following obvious choices

$$\begin{array}{l} \sigma^0 = \omega^0 - a(t)v_\alpha\omega^\alpha \\ \sigma^\alpha = a(t)k_\alpha\omega^\alpha \end{array} \Leftrightarrow \begin{array}{l} e_0 = E_0 \\ e_\alpha = v_\alpha k_\alpha^{-1}E_0 + a^{-1}(t)k_\alpha^{-1}E_\alpha \end{array} \tag{19}$$

(there is no summation over the repeated index α in the above equations). Note that this orthonormal tetrad frame is comoving, too, since $e_0 = E_0 = \partial_t = \mathbf{u}$. For such a frame, commutators of the basis vectors e_a are given by Ellis and van Elst in [41].

Having done this preparatory work, we can now present the steps of our calculations. We first calculate kinematics: $\theta, \sigma_{\alpha\beta}, \dot{u}_\alpha \cdots$ using the commutators of basis vectors. These are then substituted into the set of all constraint equations given in [41] and into Eqs. (11)–(14), to get the effective total fluid dynamics μ^t, p^t, q_α^t and $\pi_{\alpha\beta}^t$, and the effective curvature fluid dynamics μ^R, p^R, q_α^R and $\pi_{\alpha\beta}^R$, respectively. To relate these quantities with each other we make use of the following key equations,

$$\mu^t = \frac{1}{f'}\mu^m + \mu^R, \tag{20a}$$

$$p^t = \frac{1}{f'}p^m + p^R, \tag{20b}$$

$$q^t_\alpha = q^R_\alpha, \tag{20c}$$
$$\pi^t_{\alpha\beta} = \pi^R_{\alpha\beta}. \tag{20d}$$

obtained from (9) by assuming a perfect fluid i.e., for $q^m_a = 0 = \pi^m_{ab}$. We then obtain a set of simultaneous equations in the unknowns $a(t)$, μ^m, p^m and $f(R)$. It remains to analyze the compatibility of these equations with each other and with the parameters of the metric, as well as the crucial requirement $\mu^m + p^m \neq 0$. In case of compatibility, one is led to reconstruct $f(R)$-gravity model.

3.2 Examples

3.2.1 Gödel type expanding cosmological model

As a first example to shear-free, rotating and expanding space-times, we consider the following Gödel-like metric, which reads in local coordinates $\{x^i\} = \{t, x, y, z\}$

$$ds^2 = -dt^2 + 2\sqrt{S}a(t)e^{mx}dtdy + a^2(t)(dx^2 + Ke^{2mx}dy^2 + dz^2) \tag{21}$$

where $a(t)$ is a time dependent scale factor and m, S, K are constant parameters that satisfy $m > 0$, $S > 0$, $K \neq 0$ and $S + K > 0$ [35]. The rotating and stationary original Gödel model proposed in 1949 [45] corresponds to $a(t) = a_0 =$ constant, $K = -1/2$, $S = 1$ and $m = \sqrt{2}\omega_0 a_0$ with ω_0 being the constant vorticity parameter. The line element (21) constitutes its natural generalization to the non-stationary case simply by letting a be a function of time. The parameters S and K have been introduced in the coming years as a further generalization in order to discuss some unusual features of the model [35]. In what follows, we will seek whether $f(R)$ gravity field equations allow this kind of spacetime as a solution. In the basis (ω^a, E_a) defined by

$$\omega^a = (dt, dx, e^{mx}dy, dz) \quad \Leftrightarrow \quad E_a = (\partial_t, \partial_x, e^{-mx}\partial_y, \partial_z), \tag{22}$$

the line element (21) brings to the form (15), from which, on using (16), read $v_1 = v_3 = 0$, $v_2 = \sqrt{S}$ and $k_1 = k_3 = 1$, $k_2 = \sqrt{K+S}$. Hence, according to (19), the basis (σ^a, e_a) will be given explicitly as

$$\begin{aligned} \sigma^0 &= dt - \sqrt{S}a(t)e^{mx}dy, \\ \sigma^1 &= a(t)dx, \\ \sigma^2 &= \sqrt{K+S}\,a(t)e^{mx}dy, \\ \sigma^3 &= a(t)dz, \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} e_0 &= \partial_t, \\ e_1 &= a^{-1}(t)\partial_x, \\ e_2 &= \sqrt{\tfrac{S}{K+S}}\partial_t + \tfrac{1}{\sqrt{K+S}}a^{-1}(t)e^{-mx}\partial_y, \\ e_3 &= a^{-1}(t)\partial_z. \end{aligned} \tag{23}$$

Then, using the commutation relations, we get the following kinematics:

$$\sigma_{\alpha\beta} = 0, \quad \theta = 3\frac{\dot a}{a}, \quad \dot u_\alpha = \left(0, \sqrt{\frac{S}{K+S}}\frac{\dot a}{a}, 0\right),$$

$$\omega_\alpha = \left(0, 0, \frac{1}{2}\sqrt{\frac{S}{K+S}}\frac{m}{a}\right) = -\Omega_\alpha, \quad a_\alpha = \left(-\frac{m}{2a}, -\sqrt{\frac{S}{K+S}}\frac{\dot{a}}{a}, 0\right),$$
$$n_{\alpha\beta} = -\frac{m}{2a}\delta^2_\alpha\delta^3_\beta. \tag{24}$$

Here the dot denotes the derivative with respect to t (since $e_0 = \partial_t$). We note that in this spacetime, the fluid with nonvanishing vorticity has shear-free expansion and nonvanishing acceleration when $\dot{a} \neq 0$. We first substitute kinematic quantities given in Eq. (24) into the constraint equations of Ref. [41], making use of e_α in (23) for the frame derivatives. As expected, the constraint equations (C_2) and $(C_J)_\alpha$ are trivially satisfied. From $(C_1)_\alpha$ we get

$$q^t_1 = -\frac{S}{K+S}\frac{m\dot{a}}{a^2}, \quad q^t_2 = 2\sqrt{\frac{S}{K+S}}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right), \quad q^t_3 = 0, \tag{25}$$

while $(C_3)_{\alpha\beta}$ leads to $H_{\alpha\beta} = 0$. On the other hand, due to the vanishing of the shear, the shear evolution equation (eq. (98) in [41]) is converted into a new constraint equation. Using this, together with $(C_G)_{\alpha\beta}$ provide the following $E_{\alpha\beta}$ and $\pi^t_{\alpha\beta}$ components

$$E_{11} = E_{22} = -\frac{1}{2}E_{33} = -\frac{K}{6(K+S)}\frac{m^2}{a^2}, \quad E_{12} = E_{23} = E_{31} = 0, \tag{26}$$

$$\pi^t_{11} = \frac{2S}{3(K+S)}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \frac{2K+S}{6(K+S)}\frac{m^2}{a^2}, \tag{27a}$$

$$\pi^t_{22} = -\frac{4S}{3(K+S)}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \frac{2K+S}{6(K+S)}\frac{m^2}{a^2}, \tag{27b}$$

$$\pi^t_{33} = \frac{2S}{3(K+S)}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + \frac{2K+S}{3(K+S)}\frac{m^2}{a^2}, \tag{27c}$$

$$\pi^t_{12} = \pi^t_{21} = \sqrt{\frac{S}{K+S}}\frac{m\dot{a}}{a^2}, \quad \pi^t_{23} = \pi^t_{31} = 0. \tag{27d}$$

Then, upon substitution of the quantities given by Eqs. (23)–(27d) into the constraint equations $(C_4)_\alpha$ and $(C_5)_\alpha$ we obtain, after simplifying and rearranging,

$$e_\alpha\mu^t = \sqrt{\frac{S}{K+S}}\left[-\frac{2S}{K+S}\frac{\dddot{a}}{a} + \frac{\dot{a}}{a}\left(6\frac{\ddot{a}}{a} - \frac{2(3K+2S)\dot{a}^2}{a^2} + \frac{4K+S}{2(K+S)}\frac{m^2}{a^2}\right)\right]\delta^2_\alpha \tag{28}$$

and

$$\mu^t + p^t = -\frac{2(3K+4S)}{3(K+S)}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - \frac{4K-S}{6(K+S)}\frac{m^2}{a^2}, \tag{29}$$

respectively. Finally, from the last constraint equation (C_G) it follows that

$$\mu^t = -\frac{2S}{K+S}\frac{\ddot{a}}{a} + \frac{3K+2S}{K+S}\frac{\dot{a}^2}{a^2} - \frac{4K+S}{4(K+S)}\frac{m^2}{a^2}, \tag{30}$$

for which, Eq. (29) yields

$$p^t = -\frac{2(3K+S)}{3(K+S)}\frac{\ddot{a}}{a} - \frac{3K-2S}{3(K+S)}\frac{\dot{a}^2}{a^2} + \frac{4K+5S}{12(K+S)}\frac{m^2}{a^2}. \tag{31}$$

These are all that could be obtained from the constraint equations. A straightforward calculation, using the above quantities and their frame e_a-derivatives, shows that, except for the two conservation equations for standard matter, all the evolution equations in [41], as well as (28) is fulfilled, so, they do not give any new information. We have now to consider the conservation equations for the standard matter, which then may be written as

$$\dot{\mu}^m = -3\frac{\dot{a}}{a}(\mu^m + p^m), \tag{32}$$

and

$$e_1 p^m = 0, \quad e_2 p^m = -\sqrt{\frac{S}{K+S}}\frac{\dot{a}}{a}(\mu^m + p^m), \quad e_3 p^m = 0, \tag{33}$$

respectively. On the other hand, by taking into account Eqs. (30), (31) for μ^t and p^t, and (36), (37) for μ^R and p^R, together with the expression of the Ricci curvature scalar calculated as

$$R(t) = \mu^t - 3p^t = \frac{6K}{K+S}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \frac{4K+3S}{2(K+S)}\frac{m^2}{a^2}, \tag{34}$$

the key equations (20) imply that μ^m and p^m are both functions of the time t only. Consequently, due to (23), the first and third equations in (33) are trivially satisfied, while the second equation leads to an evolution equation for the pressure as

$$\dot{p}^m = -\frac{\dot{a}}{a}(\mu^m + p^m). \tag{35}$$

We note that such an evolution equation for the isotropic pressure is not present in the general set of evolution equations in [41]. It arises here from (33) which is termed as a constraint equation due to the fact that it contains only spatial derivative. However, this is only in appearance, indeed, e_2 involves also partial temporal derivative, so it cannot be considered as a purely constraint equation. The same remark is also valid for Eq. (28). At this point, we draw the attention of the reader to the comment made by the authors of [44] concerning the trouble in the nomenclature in question.

Let us now return to Eqs. (11)–(14). Using (23) and (24), we obtain the following effective dynamic quantities of the curvature fluid:

$$\mu^R = -\frac{1}{2f'}(f - Rf') + \frac{f''}{f}\left(\frac{S}{K+S}\ddot{R} - \frac{3K+S}{K+S}\frac{\dot{a}}{a}\dot{R}\right) + \frac{f'''}{f'}\frac{S}{K+S}\dot{R}^2, \quad (36)$$

$$p^R = \frac{1}{2f'}(f - Rf') + \frac{f''}{f'}\left(\frac{3K+S}{3(K+S)}\ddot{R} + \frac{6K-S}{3(K+S)}\frac{\dot{a}}{a}\dot{R}\right) + \frac{f'''}{f'}\frac{3K+S}{3(K+S)}\dot{R}^2, \quad (37)$$

$$q_1^R = \frac{1}{2}\frac{f''}{f'}\frac{S}{K+S}\frac{m\dot{R}}{a}, \quad q_2^R = -\sqrt{\frac{S}{K+S}}\left[\frac{f''}{f'}(\ddot{R} - \frac{\dot{a}}{a}\dot{R}) + \frac{f'''}{f'}\dot{R}^2\right], \quad q_3^R = 0, \quad (38)$$

$$\pi_{11}^R = -\frac{1}{2}\pi_{22}^R = \pi_{33}^R = -\frac{1}{3}\frac{S}{K+S}\left[\frac{f''}{f'}(\ddot{R} - \frac{\dot{a}}{a}\dot{R}) + \frac{f'''}{f'}\dot{R}^2\right], \quad (39a)$$

$$\pi_{12}^R = -\frac{1}{2}\frac{f''}{f'}\sqrt{\frac{S}{K+S}}\frac{m\dot{R}}{a}, \quad \pi_{23}^R = \pi_{31}^R = 0. \quad (39b)$$

Before making use of the key equations (20), for clearness and also to save some space, let us introduce two auxiliary functions which incorporate the effect of the $f(R)$ function via its derivatives:

$$X \equiv \frac{\dot{a}}{a^2} + \frac{1}{2}\frac{f''}{f'}\frac{\dot{R}}{a} \text{ and } Y \equiv 2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) + \frac{f''}{f'}\left(\ddot{R} - \frac{\dot{a}}{a}\dot{R}\right) + \frac{f'''}{f'}\dot{R}^2. \quad (40)$$

Then, in terms of these, on using Eqs. (25), (27), (38)–(39b), Eqs. (20) read

$$q_1^t = q_1^R: \quad -\frac{S}{K+S}mX = 0, \quad (41a)$$

$$q_2^t = q_2^R: \quad 0 = -Y, \quad (41b)$$

$$\pi_{11}^t = \pi_{11}^R: \quad -\frac{2K+S}{6(K+S)}\frac{m^2}{a^2} = -\frac{1}{3}\frac{S}{K+S}Y, \quad (42a)$$

$$\pi_{22}^t = \pi_{22}^R: \quad -\frac{2K+S}{6(K+S)}\frac{m^2}{a^2} = \frac{2S}{3(K+S)}Y, \quad (42b)$$

$$\pi_{33}^t = \pi_{33}^R: \quad \frac{2K+S}{3(K+S)}\frac{m^2}{a^2} = -\frac{S}{3(K+S)}Y, \quad (42c)$$

$$\pi_{12}^t = \pi_{12}^R: \quad \sqrt{\frac{S}{K+S}}\,mX = 0. \quad (42d)$$

On the other hand, instead of writing Eqs. (20a and b) separately, let us consider their following combination

$$\mu^t + p^t = \frac{1}{f'}(\mu^m + p^m) + \mu^R + p^R, \quad (43)$$

which, on using (30), (31), (36) and (37), can be written as

$$-\frac{1}{6}\frac{4K-S}{(K+S)}\frac{m^2}{a^2} = \frac{1}{f'}(\mu^m + p^m) + \frac{1}{3}\frac{3K+4S}{(K+S)}Y. \quad (44)$$

At this stage we also remark that Eqs. (32) and (35) may be combined to give

$$(\mu^m + p^m)^{\cdot} = -4\frac{\dot{a}}{a}(\mu^m + p^m), \tag{45}$$

which, in turn, assuming $\dot{a} \neq 0$ and $\mu^m + p^m \neq 0$, integrates to

$$\mu^m + p^m = \frac{C}{a^4}, \tag{46}$$

where C is a constant of integration.

We now possess a set of equations (32), (34), (35), (40), (41a)–(42d), (44) and (46) for the unknowns $a(t)$, $\mu^m(t)$, $p^m(t)$, and $f(R)$, which is closed but clearly overdetermined. To ensure compatibility of the equation (41b) with (42a,b,c), it is immediately seen that, the parameters K and S must obey the condition $2K + S = 0$ which, taking into account the requirement $K + S > 0$, leads to the condition that $K < 0$. We have now, in the above system of Eqs. (41a)–(42d), only two independent equations which are $X = 0$ and $Y = 0$. To investigate their consistency with a time dependent scale factor $a(t)$, let us take the time derivative of X. Using the definition (40), we get after a straightforward calculation

$$0 = \dot{X} = \frac{1}{a}\left(\frac{1}{2}Y - 3\frac{\dot{a}^2}{a^2}\right) = -3\frac{\dot{a}^2}{a^3}, \tag{47}$$

which in turn leads to $\dot{a} = 0$ implying that $a = \text{constant} \equiv a_0$. Hence, we arrive at the conclusion that a time-dependent scale factor $a(t)$ is not allowed as a solution of $f(R)$ field equations. Before closing this subsection, it would be interesting to look for the existence of the stationary Gödel solution in $f(R)$-gravity. Then setting $a(t) = \text{constant} \equiv a_0$, it follows that above equations greatly simplify; we have from Eq. (34), $R = -\frac{4K+3S}{2(K+S)}\frac{m^2}{a_0^2}$ and from (24), $\omega_3 \equiv \omega_0 = \frac{1}{2}\sqrt{\frac{S}{K+S}}\frac{m}{a_0}$. On the other hand, from (32) and (35), we get $\dot{\mu}^m = 0 = \dot{p}^m$, both of which integrate to $\mu^m = \text{constant}$ and $p^m = \text{constant}$. Then using the constancy of R we immediately see that Eqs. (41a) and (41b) are trivially satisfied, while Eqs. (42) lead to the compatibility condition $2K + S = 0$ as before, for which R and ω_0 simplify to $R = -m^2/a_0^2$ and $\omega_0 = m/(\sqrt{2}a_0)$, respectively. Then, putting all these into Eqs. (20a and b) and in their combination (44), we get

$$\frac{1}{2}\frac{m^2}{a_0^2} = \frac{1}{f'}\mu^m - \frac{1}{2f'}(f - Rf'), \tag{48}$$

$$\frac{1}{2}\frac{m^2}{a_0^2} = \frac{1}{f'}p^m + \frac{1}{2f'}(f - Rf'), \tag{49}$$

$$\frac{m^2}{a_0^2} = \frac{1}{f'}(\mu^m + p^m), \tag{50}$$

respectively. The last equation immediately integrates to $f(R) = c_1 R + c_2$, where $c_1 = (\mu^m + p^m)a_0^2/m^2$ and c_2 is a constant of integration, which can be obtained from Eqs. (48) and (49) as $c_2 = \mu^m - p^m$. In the GR limit, we have $f(R) = R - 2\Lambda$, so $c_1 = 1$ and $c_2 = -2\Lambda$, leading then respectively to the relationships $\mu^m + p^m = m^2/a_0^2 = 2\omega_0^2$ and $\mu^m - p^m = -2\Lambda$, which have already been found by Ellis [1]; this is the Gödel's universe, generalized to include pressure. We thus conclude that even the stationary Gödel universe is not allowed as solution of the $f(R)$-gravity field equations, provide that $f(R) \neq R +$ constant. We note that, the values of the parameters K and S do not affect our conclusions provided that they satisfy $2K + S = 0$ and $K < 0$.

3.2.2 Rotating Bianchi-type II model

The second space-time we consider is the rotating Bianchi type-II model, which is given in the basis $(\omega^a; E_a)$ defined by the following coordinate realizations [46]

$$\omega^a = (dt, dx - zdy, dy, dz) \Leftrightarrow E_a = (\partial_t, \partial_x, z\partial_x + \partial_y, \partial_z). \tag{51}$$

Thus, according to expressions (19) the orthonormal $(\sigma^a; e_a)$-frame reads explicitly

$$\begin{array}{l} \sigma^0 = dt - v_1 a(t)dx + (v_1 z - v_2)a(t)dy \\ \quad -v_3 a(t)dz, \\ \sigma^1 = k_1 a(t)dx - k_1 a(t)zdy, \\ \sigma^2 = k_2 a(t)dy, \\ \sigma^3 = k_3 a(t)dz, \end{array} \Leftrightarrow \begin{array}{l} e_0 = \partial_t, \\ e_1 = \frac{v_1}{k_1}\partial_t + \frac{1}{k_1 a(t)}\partial_x, \\ e_2 = \frac{v_2}{k_2}\partial_t + \frac{z}{k_2 a(t)}\partial_x + \frac{1}{k_2 a(t)}\partial_y, \\ e_3 = \frac{v_3}{k_3}\partial_t + \frac{1}{k_3 a(t)}\partial_z. \end{array} \tag{52}$$

Then, the kinematics obtained from the commutators are as follows:

$$\sigma_{\alpha\beta} = 0, \quad \theta = 3\frac{\dot{a}}{a}, \quad \dot{u}_\alpha = \frac{v_\alpha}{k_\alpha}\frac{\dot{a}}{a}, \quad \omega_\alpha = \left(\frac{v_1}{2k_2k_3}\frac{1}{a}, 0, 0, \right) = -\Omega_\alpha,$$
$$a_\alpha = -\frac{v_\alpha}{k_\alpha}\frac{\dot{a}}{a}, \quad n_{\alpha\beta} = diag\left(-\frac{k_1}{k_2k_3}\frac{1}{a}, 0, 0\right). \tag{53}$$

Following the similar procedure as in the previous subsection, we calculate effective total dynamic quantities from the constraint equations, and effective dynamic quantities of the effective curvature fluid from Eqs. (11)–(14). Then, on inserting for these expressions into the key equations (20), we get the following set of equations in terms of X and Y:

$$\mu^t + p^t = \frac{1}{f'}(\mu^m + p^m) + \mu^R + p^R:$$
$$-\frac{k_1^2 - 5v_1^2}{6k_2^2k_3^2}\frac{1}{a^2} = \frac{1}{f'}(\mu^m + p^m) + \frac{1}{3}\left(3 + \frac{v_1^2}{k_1^2} + \frac{v_2^2}{k_2^2} + \frac{v_3^2}{k_3^2}\right)Y, \tag{54}$$
$$q_\alpha^t = q_\alpha^R: \ \frac{v_1}{k_1}\frac{k_1^2}{2k_2^2k_3^2}\frac{1}{a^2} = \frac{v_1}{k_1}Y, \tag{55a}$$
$$\frac{v_3 v_1}{k_2 k_3^2}X = \frac{v_2}{k_2}Y, \tag{55b}$$

$$\frac{v_1 v_2}{k_2^2 k_3} X = \frac{v_3}{k_3} Y, \tag{55c}$$

$$\pi^t_{\alpha\beta} = \pi^R_{\alpha\beta} : \quad \frac{2k_1^2 - v_1^2}{k_2^2 k_3^2} \frac{1}{a^2} = \left(\frac{2v_1^2}{k_1^2} - \frac{v_2^2}{k_2^2} - \frac{v_3^2}{k_3^2} \right) Y, \tag{56a}$$

$$-\frac{1}{2} \frac{2k_1^2 - v_1^2}{k_2^2 k_3^2} \frac{1}{a^2} = \left(\frac{2v_2^2}{k_2^2} - \frac{v_3^2}{k_3^2} - \frac{v_1^2}{k_1^2} \right) Y, \tag{56b}$$

$$-\frac{1}{2} \frac{2k_1^2 - v_1^2}{k_2^2 k_3^2} \frac{1}{a^2} = \left(\frac{2v_3^2}{k_3^2} - \frac{v_1^2}{k_1^2} - \frac{v_2^2}{k_2^2} \right) Y, \tag{56c}$$

$$\frac{k_1}{k_2 k_3} \frac{v_3}{k_3} X = \frac{v_1 v_2}{k_1 k_2} Y, \tag{56d}$$

$$0 = \frac{v_2 v_3}{k_2 k_3} Y, \tag{56e}$$

$$\frac{k_1}{k_2 k_3} \frac{v_2}{k_2} X = \frac{v_3 v_1}{k_3 k_1} Y. \tag{56f}$$

To seek compatibility of these equations with each other, we consider three cases:

1. $v_1 \neq 0, v_2 \neq 0, v_3 \neq 0$: Equation (56e) implies $Y = 0$ which, inserted for Y into (55a) leads to $k_1^2/(2k_2^2 k_3^2 a^2) = 0$, whence $k_1 = 0$, which contradicts $k_1 \neq 0$.
2. $v_1 \neq 0, v_2 \neq 0, v_3 = 0$: In this case, Eqs. (55a) reduces to $k_1^2/(2k_2^2 k_3^2 a^2) = Y$ and (55b), to $Y = 0$. Then, we have again a contradiction since $k_1 \neq 0$.
3. $v_1 \neq 0, v_2 = 0, v_3 = 0$: In this case, Eq. (55a) reduces to

$$\frac{k_1^2}{2k_2^2 k_3^2} \frac{1}{a^2} = Y, \tag{57}$$

and Eqs. (56a), (56b) and (56c) reduce to a single one

$$\frac{2k_1^2 - v_1^2}{k_2^2 k_3^2} \frac{1}{a^2} = \frac{2v_1^2}{k_1^2} Y, \tag{58}$$

while the remaining equations are identically fulfilled. Inserting for Y from Eq. (55a) into (56a) we get $v_1^2 = k_1^2$ as a compatibility condition. Now, it remains only to consider Eq. (54). Under the condition $v_1^2 = k_1^2$, it becomes

$$\frac{2}{3} \frac{k_1^2}{k_2^2 k_3^2} \frac{1}{a^2} = \frac{1}{f'} (\mu^m + p^m) + \frac{4}{3} Y, \tag{59}$$

which, on using Eq. (55a) for Y, simplifies to

$$\frac{1}{f'} (\mu^m + p^m) = 0, \tag{60}$$

then contradicting the assumption $(\mu^m + p^m) \neq 0$.

3.2.3 Rotating Bianchi-type IX model

This model is defined in the basis (E_a, ω^a) by [46]

$$\omega^a = (dt, \cos y \cos z dx - \sin z dy, \cos y \sin z dx + \cos z dy, -\sin y dx + dz),$$
$$E_a = \left(\partial_t, \frac{\cos z}{\cos y}\partial_x, -\sin z\partial_y + \tan y \cos z\partial_z, \frac{\sin z}{\cos y}\partial_x + \cos z\partial_y + \tan y \sin z\partial_z, \partial_z\right), \tag{61}$$

from which, according to Eq. (19), one can construct an orthonormal basis (σ^a, e_a). Then, following the same procedure as for the previous models, we find:

$$\sigma_{\alpha\beta} = 0, \quad \theta = 3\frac{\dot{a}}{a}, \quad \dot{u}_\alpha = \frac{v_\alpha}{k_\alpha}\frac{\dot{a}}{a}, \quad \omega_\alpha = \left(\frac{v_1}{2k_2k_3}\frac{1}{a}, \frac{v_2}{2k_3k_1}\frac{1}{a}, \frac{v_3}{2k_1k_2}\frac{1}{a}\right) = -\Omega_\alpha,$$
$$a_\alpha = -\frac{v_\alpha}{k_\alpha}\frac{\dot{a}}{a}, \quad n_{\alpha\beta} = diag\left(-\frac{k_1}{k_2k_3}\frac{1}{a}, -\frac{k_2}{k_3k_1}\frac{1}{a}, -\frac{k_3}{k_1k_2}\frac{1}{a}\right). \tag{62}$$

$$R = 6\left[1 - \left(\frac{v_1^2}{k_1^2} + \frac{v_2^2}{k_2^2} + \frac{v_3^2}{k_3^2}\right)\right]\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)$$
$$-\frac{1}{2}\left[\frac{k_1^2 - v_1^2}{k_2^2k_3^2} + \frac{k_2^2 - v_2^2}{k_1^2k_3^2} + \frac{k_3^2 - v_3^2}{k_1^2k_2^2} - 2\left(\frac{1}{k_1^2} + \frac{1}{k_2^2} + \frac{1}{k_3^2}\right)\right]\frac{1}{a^2}, \tag{63}$$

$$q_\alpha^t = q_\alpha^R : \ \frac{v_2v_3}{k_1}\left(\frac{1}{k_2^2} - \frac{1}{k_3^2}\right)X + \frac{k_1v_1}{2k_2^2k_3^2}\frac{1}{a^2} = \frac{v_1}{k_1}Y, \tag{64a}$$

$$\frac{v_3v_1}{k_2}\left(\frac{1}{k_3^2} - \frac{1}{k_1^2}\right)X + \frac{k_2v_2}{2k_3^2k_1^2}\frac{1}{a^2} = \frac{v_2}{k_2}Y, \tag{64b}$$

$$\frac{v_1v_2}{k_3}\left(\frac{1}{k_1^2} - \frac{1}{k_2^2}\right)X + \frac{k_3v_3}{2k_1^2k_2^2}\frac{1}{a^2} = \frac{v_3}{k_3}Y, \tag{64c}$$

$\pi_{\alpha\beta}^t = \pi_{\alpha\beta}^R$:

$$\left(\frac{2k_1^2 - v_1^2}{k_2^2k_3^2} - \frac{2k_2^2 - v_2^2}{2k_3^2k_1^2} - \frac{2k_3^2 - v_3^2}{2k_1^2k_2^2} + \frac{2}{k_1^2} - \frac{1}{k_2^2} - \frac{1}{k_3^2}\right)\frac{1}{a^2} = \left(\frac{2v_1^2}{k_1^2} - \frac{v_2^2}{k_2^2} - \frac{v_3^2}{k_3^2}\right)Y, \tag{65a}$$

$$\left(\frac{2k_2^2 - v_2^2}{k_3^2k_1^2} - \frac{2k_3^2 - v_3^2}{2k_1^2k_2^2} - \frac{2k_1^2 - v_1^2}{2k_2^2k_3^2} + \frac{2}{k_2^2} - \frac{1}{k_3^2} - \frac{1}{k_1^2}\right)\frac{1}{a^2} = \left(\frac{2v_2^2}{k_2^2} - \frac{v_3^2}{k_3^2} - \frac{v_1^2}{k_1^2}\right)Y, \tag{65b}$$

$$\left(\frac{2k_3^2 - v_3^2}{k_1^2k_2^2} - \frac{2k_1^2 - v_1^2}{2k_2^2k_3^2} - \frac{2k_2^2 - v_2^2}{2k_3^2k_1^2} + \frac{2}{k_3^2} - \frac{1}{k_1^2} - \frac{1}{k_2^2}\right)\frac{1}{a^2} = \left(\frac{2v_3^2}{k_3^2} - \frac{v_1^2}{k_1^2} - \frac{v_2^2}{k_2^2}\right)Y, \tag{65c}$$

$$\left(\frac{k_1}{k_2k_3} - \frac{k_2}{k_3k_1}\right)\frac{v_3}{k_3}X - \frac{v_1v_2}{2k_1k_2k_3^2}\frac{1}{a^2} = \frac{v_1v_2}{k_1k_2}Y, \tag{65d}$$

$$\left(\frac{k_2}{k_3k_1}-\frac{k_3}{k_1k_2}\right)\frac{v_1}{k_1}X-\frac{v_2v_3}{2k_2k_3k_1^2}\frac{1}{a^2}=\frac{v_2v_3}{k_2k_3}Y, \tag{65e}$$

$$\left(\frac{k_3}{k_1k_2}-\frac{k_1}{k_2k_3}\right)\frac{v_2}{k_2}X-\frac{v_3v_1}{2k_3k_1k_2^2}\frac{1}{a^2}=\frac{v_3v_1}{k_3k_1}Y, \tag{65f}$$

$$\mu^t+p^t=\frac{1}{f'}(\mu^m+p^m)+\mu^R+p^R:$$

$$-\frac{1}{6}\left[\frac{k_1^2-5v_1^2}{k_2^2k_3^2}+\frac{k_2^2-5v_2^2}{k_3^2k_1^2}+\frac{k_3^2-5v_3^2}{k_1^2k_2^2}-2\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)\right]\frac{1}{a^2}$$

$$=\frac{1}{f'}(\mu^m+p^m)+\frac{1}{3}\left(3+\frac{v_1^2}{k_1^2}+\frac{v_2^2}{k_2^2}+\frac{v_3^2}{k_3^2}\right)Y \tag{66}$$

One must also have in mind that Eqs. (32), (35) and (46) are valid for this model, too.

Now, we undertake the compatibility analysis by considering again three cases:

1. $v_1\neq 0, v_2\neq 0, v_3\neq 0$: We first note that if $k_2=k_3$, then Eqs. (64a) and (65e) lead to $Y=\frac{k_1^2}{2k_2^2k_3^2}\frac{1}{a^2}>0$ and $Y=-\frac{1}{2k_1^2}\frac{1}{a^2}<0$, respectively, whence a contradiction since Y cannot be vanishing when $k_1\neq 0$. Similarly, one cannot have $k_3=k_1$, otherwise one would have from (64b) and (65f) $Y>0$ and $Y<0$ simultaneously. On the other hand, if $X=0$, then using again (64a) and (65e) we get the similar contradiction. Consequently, as compatibility conditions, one must have $k_1\neq k_2\neq k_3$ and $X\neq 0$. We seek now whether or not there is any condition on v_α's. Subtracting Eqs. (65e) and (65f) from (65d) and dividing side by side we get after rearranging

$$\left(k_2^2-k_3^2\right)^2v_1^2+\left(k_3^2-k_1^2\right)^2v_2^2+\left(k_1^2-k_2^2\right)^2v_3^2=0. \tag{67}$$

   This equation is satisfied if and only if for $v_1=v_2=v_3=0$, contradicting our starting assumption that $v_1\neq 0, v_2\neq 0, v_3\neq 0$.
2. $v_1\neq 0, v_2\neq 0, v_3=0$: In this case the first three equations of the set (64a)–(64c) reduce to

$$\frac{k_1^2}{2k_2^2k_3^2}\frac{1}{a^2}=Y,\quad \frac{k_2^2}{2k_3^2k_1^2}\frac{1}{a^2}=Y,\quad \frac{v_1v_2}{k_3}\left(\frac{1}{k_1^2}-\frac{1}{k_2^2}\right)X=0. \tag{68}$$

   They are compatible with each other if and only if $k_1=k_2$ and $Y>0$. But, for $k_1=k_2$, Eq. (65d) becomes $-1/(2k_3^2a^2)=Y<0$, then contradicting $Y>0$.
3. $v_1\neq 0, v_2=0, v_3=0$: In this case Eqs. (64b), (64c), (65d) and (65f) are trivially satisfied, so the surviving equations are (64a), (65a,b,c,e) which simplify to

$$\frac{k_1^2}{2k_2^2k_3^2}\frac{1}{a^2}=Y, \tag{69}$$

$$\left(\frac{2k_1^2-v_1^2}{k_2^2k_3^2}-\frac{k_2^2}{k_3^2k_1^2}-\frac{k_3^2}{k_1^2k_2^2}+\frac{2}{k_1^2}-\frac{1}{k_2^2}-\frac{1}{k_3^2}\right)\frac{1}{a^2}=\frac{2v_1^2}{k_1^2}Y, \tag{70a}$$

$$\left(\frac{2k_2^2}{k_3^2k_1^2} - \frac{k_3^2}{k_1^2k_2^2} - \frac{2k_1^2 - v_1^2}{2k_2^2k_3^2} + \frac{2}{k_2^2} - \frac{1}{k_3^2} - \frac{1}{k_1^2}\right)\frac{1}{a^2} = -\frac{v_1^2}{k_1^2}Y, \tag{70b}$$

$$\left(\frac{2k_3^2}{k_1^2k_2^2} - \frac{2k_1^2 - v_1^2}{2k_2^2k_3^2} - \frac{k_2^2}{k_3^2k_1^2} + \frac{2}{k_3^2} - \frac{1}{k_1^2} - \frac{1}{k_2^2}\right)\frac{1}{a^2} = -\frac{v_1^2}{k_1^2}Y, \tag{70c}$$

$$\left(\frac{k_2}{k_3k_1} - \frac{k_3}{k_1k_2}\right)\frac{v_1}{k_1}X = 0. \tag{70d}$$

In order that Eqs. (69) and (70d) be compatible with each other one must have either a) $Y > 0$ and $k_2 = k_3$ or b) $Y > 0$ and $X = 0$.

(a) $Y > 0$ and $k_2 = k_3$: Eqs. (70a)–(70c) reduce then, to the following single equation:

$$\left(2k_1^4 - 2k_1^2k_2^2 - k_1^2v_1^2\right)\frac{1}{a^2} = 2v_1^2k_2^4Y, \tag{71}$$

which, on substituting for Y from Eq. (69), leads to another compatibility condition

$$v_1^2 = k_1^2 - k_2^2, \tag{72}$$

which in turn implies that $k_1 > k_2 = k_3$. Bearing in mind that we are left with only one nontrivial equation (69), let us now consider the remaining key equation (66) which then, on using (72) for v_1^2, reduces to

$$\frac{1}{6}\frac{4k_1^2 - k_2^2}{k_2^4}\frac{1}{a^2} = \frac{1}{f'}(\mu^m + p^m) + \frac{1}{3}\frac{4k_1^2 - k_2^2}{k_1^2}Y. \tag{73}$$

This equation in turn leads, on using Eq. (69) for Y, to

$$\frac{1}{f'}(\mu^m + p^m) = 0, \tag{74}$$

contradicting then the assumption $\mu^m + p^m \neq 0$.

(b) $Y > 0$ and $X = 0$: In this subcase, the surviving equations are (69), (70a), (70b) and (70c). Subtracting Eq. (70c) from (70b) we get

$$3\left(k_2^2 - k_3^2\right)\left(k_2^2 + k_3^2 - k_1^2\right) = 0, \tag{75}$$

which implies $k_2 = k_3$ or $k_1^2 = k_2^2 + k_3^2$. The case $k_2 = k_3$ together with $Y > 0$ was studied previously. So we have only to consider the latter case: $k_1^2 = k_2^2 + k_3^2$ together with $k_2 \neq k_3$. Then, Eq. (70a), on substituting for Y from (69), leads that

$$v_1^2 = \frac{2k_2^2k_3^2}{k_2^2 + k_3^2}. \tag{76}$$

Consequently, the required conditions for the compatibility of the above set of nine equations (64a)–(65f) can be summarized as follows: $Y = Y(t) > 0$, $X = 0$, $k_2 \neq k_3$, $k_1^2 = k_2^2 + k_3^2$ and $v_1^2 = \frac{2k_2^2 k_3^2}{k_2^2 + k_3^2}$. By making use of these conditions in Eq. (66), it follows that

$$\frac{1}{f'}(\mu^m + p^m) = -\frac{1}{2}\frac{(k_2^2 - k_3^2)^2}{k_2^2 k_3^2 (k_2^2 + k_3^2)}\frac{1}{a^2} < 0. \tag{77}$$

On the other hand, from Eq. (46), we know that $\mu^m + p^m = \frac{C}{a^4}$ when $\dot{a} \neq 0$. Then, using this in the above equation, we find

$$f' = -\frac{2Ck_2^2 k_3^2 (k_2^2 + k_3^2)}{(k_2^2 - k_3^2)^2}\frac{1}{a^2}. \tag{78}$$

Thus, we are led now to determine the functional form of $f(R)$, i.e., to reconstruct an $f(R)$-gravity. Under the above conditions the expression (63) reduces to

$$R = 6\frac{k_2^4 + k_3^4}{(k_2^2 + k_3^2)^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) + \frac{3}{k_2^2 + k_3^2}\frac{1}{a^2}. \tag{79}$$

On the other hand, consider the time derivative of the equation $X = 0$, which was already given by Eq. (47) as

$$0 = \dot{X} = \frac{1}{a}\left(\frac{1}{2}Y - 3\frac{\dot{a}^2}{a^2}\right), \tag{80}$$

from which we get

$$Y = 6\frac{\dot{a}^2}{a^2}. \tag{81}$$

Comparing this to Eq. (69), it is seen that

$$\dot{a}^2 = \frac{1}{12}\frac{k_2^2 + k_3^2}{k_2^2 k_3^2} = \text{constant}, \tag{82}$$

which integrates to

$$a(t) = At, \text{ with } A = A(k_2, k_3) = \frac{1}{2\sqrt{3}}\frac{\sqrt{k_2^2 + k_3^2}}{k_2 k_3}, \tag{83}$$

where we have set the constant of integration to zero. This in turn implies that $\ddot{a} = 0$; then, taking into account also Eq. (82), Eq. (79) simplifies to

$$R = \frac{1}{2}\frac{k_2^4 + k_3^4 + 6k_2^2 k_3^2}{(k_2^2 + k_3^2)k_2^2 k_3^2}\frac{1}{a^2}. \tag{84}$$

Using this, Eq. (78) becomes

$$f'(R) = -\frac{4Ck_2^4k_3^4(k_2^2+k_3^2)^2}{(k_2^2-k_3^2)^2(k_2^4+k_3^4+6k_2^2k_3^2)}R, \tag{85}$$

which then integrates to give

$$f(R) = -C\Upsilon^2R^2 - 2\Lambda, \quad \text{with } \Upsilon^2 = \frac{2k_2^4k_3^4(k_2^2+k_3^2)^2}{(k_2^2-k_3^2)^2(k_2^4+k_3^4+6k_2^2k_3^2)}, \tag{86}$$

where the integration constant is taken as -2Λ. Thus, it appears that we have succeeded to find a time dependent scale factor and to reconstruct $f(R)$-gravity model. To know details of μ^m and p^m, we notice first that Eq. (32) and (35) integrate to give, on using (46), that $\mu^m = 3C/(4a^4) -$ constant and $p^m = C/(4a^4) +$ constant. To get the value of the constant, let us make use of the key equations (20a and b) which can be written as $\mu^m = f'(\mu^t - \mu^R)$ and $p^m = f'(p^t - p^R)$. Then after some straightforward calculations, using the dynamic quantities μ^t, μ^R, p^t and p^R and the aforementioned requirements together with (83) and (86) it follows that

$$\mu^m(t) = \frac{3C}{4a^4(t)} - \Lambda, \quad p^m(t) = \frac{C}{4a^4(t)} + \Lambda, \tag{87}$$

from which we get a barotropic equation of state (EoS) $p^m = p^m(\mu^m)$, as

$$p^m(t) - \Lambda = \frac{1}{3}(\mu^m(t) + \Lambda). \tag{88}$$

As a final remark, we note that in this case the gravito-electric and gravito- magnetic fields have the following values:

$$E_{\alpha\beta} = 0, \quad H_{23} = \frac{1}{2\sqrt{2}}\frac{k_2^2-k_3^2}{k_2k_3(k_2^2+k_3^2)}\frac{1}{a^2(t)} \tag{89}$$

At this stage let us come back to GR case. Taking then $f(R)$ as $f(R) = R - 2\Lambda$, we have $f' = 1$ and $f'' = f''' = 0$, so X and Y reduce to $X = \frac{\dot{a}}{a^2}$ and $Y = 2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right)$, where we continue to suppose $a = a(t)$. At this stage, we must point out that we only have to consider here the case 3, since in cases 1 and 2, the incompatibility of equations have been arisen regardless of the specific forms of the auxiliary functions X and Y. Consequently, the corresponding analysis applies equally well to the case of GR. Thus, we start from the case 3. First, we observe that the defining relations $X = 0$ and $Y > 0$ of the subcase 3.(b) are not compatible with the above expressions of the X and Y, since the former gives $a =$ constant for which Y becomes zero contradicting $Y > 0$. Therefore, in this subcase, even a stationary model is not permitted. Consider now the subcase 3.(a), for which the defining requirements are: $Y > 0$ and $k_2 = k_3$;

and where, in addition, one has $v_1^2 = k_1^2 - k_2^2$. Then, combining the expression of Y with Eq. (69) we obtain the following second order differential equation

$$2\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = \frac{k_1^2}{2k_2^4}\frac{1}{a^2}, \tag{90}$$

which can be straightforwardly solved to give

$$a(t) = \frac{k_1}{2k_2^2 c_1}\cosh(c_1 t + c_2), \tag{91}$$

where c_1 and c_2 are two constants of integration. Hence, it seems that we have arrived to find a time dependent scale factor $a(t)$, meaning that we have an expanding, shear-free and rotating model in GR. But, unfortunately, this is not the case, since the perfect fluid is still subject to Eq. (74), thereby violating the crucial condition that $\mu^m + p^m \neq 0$, as can be verified directly using $\mu^m = 3k_2^2/k_1^2 - \Lambda$ and $p^m = -3k_2^2/k_1^2 + \Lambda$, calculated following similar way as in obtaining Eq. (87). This special form of the Bianchi-type IX model constitutes another concrete example supporting the validity of the shear-free conjecture in GR.

Before passing to the discussion of our results, in order to relate our work to the linear analysis presented in Ref. [25], let us investigate also what one would obtain if one has used instead of the full non-linear tetrad equations, the linearized ones about a FLRW background. Omitting the details, it results that the terms v_1^2 are discarded from the equations, and we have that $k_1 = k_2 = k_3 \equiv k$ as compatibility conditions of the set of Eqs. (69)–(70d) with $Y = 1/(2k^2a^2)$. Consequently, for a barotropic EoS $p^m = w\mu^m$, we have from Eq. (66)

$$\frac{1}{3}{}^3R = \frac{(1+w)\mu^m}{f'} + Y, \tag{92}$$

where $\mu^m = \mu_0 a^{-3(1+w)}$ and ${}^3R = 3/(2k^2a^2)$, ${}^3R$ being the 3-curvature. It follows that, Eq. (92) reduce to $(1+w)\mu^m/f' = 0$, leading to $\mu^m + p^m = 0$, provided that $\mu^m \neq 0$.

We notice that the linearized equation (92) is, in fact, just the tetrad version of the Eq. (55) or (67) of Ref. [25]. This can be seen as follows. Using the following linearized identity for shear-free congruences

$$curlcurl\omega_a = \frac{1}{3}{}^3R\omega_a - D^2\omega_a, \tag{93}$$

and suitable constraint equations, one can express $D^2\omega_a$ as

$$D^2\omega_a = \frac{1}{3}{}^3R\omega_a - \frac{4}{3}\dot{\theta}\omega_a + \eta_a^{bc}D_b q_c^t. \tag{94}$$

Then, converting the last term into tetrad form as

$$\eta_a^{bc}D_b q_c^t \rightarrow \varepsilon_\alpha^{\beta\gamma}(e_\beta - a_\beta)q_\gamma^t - n_\alpha^\gamma q_\gamma^t, \tag{95}$$

we get

$$D^2\omega_1 = \left(\frac{1}{3}{}^3R - \frac{1}{k^2a^2}\right)\omega_1 = \left(\frac{1}{3}{}^3R - 2Y\right)\omega_1 = -\frac{1}{3}{}^3R\omega_1, \tag{96}$$

which, on substituting into Eq. (55) or (67) of Ref. [25], gives the desired result.

The above considerations show that in the context of $f(R)$-gravity, Eq. (55) or (67) of Ref. [25] do not admit a solution which does not violate the Ellis condition $\theta\omega = 0$, provided that $\mu^m + p^m \neq 0$ and $\mu^m \neq 0$. The above result gives a concrete illustration of the fact that the non-existence of a linearized solution does not necessarily imply the non-existence of an exact solution of the full non-linear field equations. Conversely, one cannot infer from the existence of a linearized solution the existence of an exact solution; since, as stated in the Introduction, it may happen that this linearized solution may not be a particular solution or a limit of the exact solution (on the doubts about the linearization procedure see [33,34] and especially, Sec. 4.5 of Ref. [21]). However, despite these perils one cannot say that the linearization procedure is useless. For instance, it can serve to guess the existence of a solution which, of course, has to be checked by using the full non-linear field equations [47]. In this sense, for more general spacetimes, making use of a metric approach together with tetrad equations can accomplish this goal.

4 Conclusions

In this paper, adopting a metric based approach we have sought whether or not the general relativistic shear-free conjecture has a counterpart within the $f(R)$-gravity framework. We had tentatively chosen as illustrative examples, three shearless, rotating and expanding spatially homogeneous spacetimes and investigated whether these are allowed as solutions of the $f(R)$-gravity field equations with a perfect fluid matter as source. The rotation is introduced in these metrics via the threading shift 1-form while the overall expansion is expressed in terms of a single time dependent scale factor. In order to calculate the field equations for a given metric, a direct and practical method is to use exterior differential calculus. However, we have followed a somewhat different but an equivalent method by making use of the already established general relativistic orthonormal tetrad evolution and constraint equations given in [41], but extending them to the $f(R)$-gravity case. Through the adoption of the effective fluid approach developed in [36–38], we have been able to directly translate the general relativistic tetrad equations into the $f(R)$-gravity case, thereby, have avoided to deal with fourth order equations. A further advantage of making use of the total set of $f(R)$-gravity extended tetrad equations is that the integrability conditions are automatically incorporated into the set. As a result, we have obtained a closed although overdetermined set of equations in the unknowns $a(t)$, μ^m, p^m and $f(R)$. Under the requirement that $\mu^m + p^m \neq 0$, the analysis of the compatibility of this set of the equations with respect to parameters k_α and v_α, as well as to the assumption of a time dependent scale factor provided us the following results.

It has been shown by the authors of [22] that an expanding type Gödel universe is not allowed in GR. Here we have shown that a further generalized version of the Gödel universe, by introducing parameters K and S, is also not allowed as solution in $f(R)$-gravity. Moreover, as a byproduct, we have found that even a stationary rotating Gödel universe does not exist in this modified theory whatsoever the functional form of the $f(R)$ provided that $f(R) \neq R - 2\Lambda$. Rotating Bianchi-type II model is not allowed in $f(R)$-gravity, either. This is due to the fact that, in cases $v_1 \neq 0, v_2 \neq 0, v_3 \neq 0$ and $v_1 \neq 0, v_2 \neq 0, v_3 = 0$, the set of equations leads to a contradiction by virtue of the requirement $k_1 \neq 0$, while in the case $v_1 \neq 0, v_2 = 0, v_3 = 0$, the compatibility with $\mu^m + p^m \neq 0$ is not ensured.

As a last example we have considered rotating type Bianchi-IX model. For this model, we have found that the overall compatibilities of equations are ensured only in the case of $v_1 \neq 0, v_2 = 0, v_3 = 0$, under the requirements $k_2 \neq k_3, k_1^2 = k_2^2 + k_3^2$ and $v_1^2 = 2k_2^2 k_3^2/(k_2^2 + k_3^2)$. Then, we have been able to obtain solutions for $a(t)$, μ^m and p^m as given in Eqs. (83) and (87), respectively, and to reconstruct the $f(R)$-gravity as shown in (86). In what follows, we discuss these solutions without entering into the discussions either on energy conditions in $f(R)$-gravity [48,49] or on viability of an $f(R)$ model.

Firstly, we note that, in Eq. (46) C may take on negative values since it simply represents an arbitrary integration constant. Then, depending on the values of μ^m and p^m in the combination $\mu^m + p^m$, C may be either positive or negative. By adopting the natural assumption that $\mu^m > 0$ and by considering Eq. (87) together with (83), we see that, according to the signs of C and Λ, four cases occur; then, denoting a critical time by $t_* \equiv (1/A)\,|C/(4\Lambda)|^{1/4}$ obtained by making use of the assumption $\mu^m > 0$ and Eqs. (83) and (87), it is straightforward to see that:

1. $C > 0, \Lambda > 0$: in this case, we have $\mu^m(t < \sqrt[4]{3}t_*) > 0$ and $p^m(t < \sqrt[4]{3}t_*) > 0$, namely, a positive pressure,
2. $C > 0, \Lambda < 0$: the condition $\mu^m(t) > 0$ is ensured for all t and we have either $p^m(t < t_*) > 0$ or $p^m(t > t_*) < 0$, thus we have a positive or negative pressure depending on t,
3. $C < 0, \Lambda > 0$: this case is unphysical, since $\mu^m(t) < 0$ for all t,
4. $C < 0, \Lambda < 0$: in this case, we have $\mu^m(t > \sqrt[4]{3}t_*) > 0$ and $p^m(t > \sqrt[4]{3}t_*) < 0$, that is a negative pressure.

A negative pressure refers, in general, to an unrealistic form of matter, such as scalarons, strings or Chaplygin gas, giving rise to the violation of various energy conditions. However, they are in current usage for modeling the matter content of the universe. Here, a negative pressure arises in both cases 2 and 4, while a positive pressure arises in both cases 1 and 2. It is also possible to discuss the sign of the pressure with respect to Λ, using the EoS given by Eq. (88); however, we only indicate here that in the particular case of a vanishing Λ, the EoS reduces to $p^m = (1/3)\mu^m$ showing that the universe is filled with pure radiation. On the other hand, we see that, in all the above cases, irrespectively of the sign of the pressure, the scale factor $a(t)$ exhibits an expansion behavior linear in t, like that of a flat Milne model in GR. Thus, we have a coasting universe model, expanding anisotropically in three directions with different scale factors $k_1 At$, $k_2 At$ and $k_3 At$ and rotating about the e_1 axis with a time dependent vorticity of magnitude $\omega_1(t) = \sqrt{6}k_2k_3/(k_2^2 + k_3^2)t^{-1}$.

Let us now return to the $f(R)$-model given in Eq. (86). We see that the sign of the R^2-term depends on the sign of the constant C; it is negative when $C > 0$, leading to an "$f(-R^2)$-gravity", and positive when $C < 0$, leading then to an "$f(+R^2)$-gravity". In the latter case, we have a negative pressure (case 4), while in the former case, we have a positive pressure (cases 1 and 2), however, a negative pressure is possible, either (case 2). Thus, a physically realistic ordinary matter having a positive pressure holds only in the $f(-R^2)$ model. All that can be said, about the unfamiliar minus sign, is that, at theoretical level, there are no a priori reasons to reject this model.

Another interesting situation to point out is that, according to Eq. (89), this shear-free rotating and expanding Bianchi-type IX model is purely gravito-magnetic solution [50], that is, $E_{\alpha\beta} = 0 \neq H_{\alpha\beta}$ giving rise to gravitational waves.

We conclude that in $f(R)$-gravity theory there are situations where a shear-free perfect fluid could have simultaneous rotation and expansion, thereby showing that there is no counterpart of the general relativistic shear-free conjecture in this modified theory of gravity. In this context, it would be of interest to find further examples other than our Bianchi-type IX model and the one found in [25].

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