

Power-law solutions in $f(T)$ gravity

M. R. Setare · F. Darabi

Received: 18 October 2011 / Accepted: 20 June 2012 / Published online: 1 July 2012
© Springer Science+Business Media, LLC 2012

Abstract We have considered an action of the form $T + f(T) + L_m$ describing Einstein's gravity plus a function of the torsion scalar. By considering an exact power-law solution we have obtained the Friedmann equation as a differential equation for the function $f(T)$ in spatially flat universe and obtained the real valued solutions of this equation for some power-law solutions. We have also studied the power-law solutions when the universe enters a Phantom phase and shown that such solutions may exist for some $f(T)$ solutions.

Keywords Power-law · $f(T)$ gravity · Phantom phase

1 Introduction

Recent cosmological observations indicate that our universe is in accelerated expansion. These observations are those which is obtained by SNe Ia [1], WMAP [2], SDSS [3] and X-ray [4]. These observations also suggest that our universe is spatially flat, and consists of about 70% dark energy (DE) with negative pressure, 30% dust matter (cold dark matter plus baryons), and negligible radiation. In order to explain why the cosmic acceleration happens, many theories have been proposed. The simplest candidate of the dark energy is a tiny positive time-independent cosmological constant Λ , for which $\omega = -1$. However, it is difficult to understand why the cosmological

M. R. Setare
Department of Science, Payame Noor University, Bijar, Iran
e-mail: rezakord@ipm.ir

F. Darabi (✉)
Department of Physics, Azarbaijan Shahid Madani University, 53714-161 Tabriz, Iran
e-mail: f.darabi@azaruniv.edu

F. Darabi
Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), 55134-441 Maragha, Iran

constant is about 120 orders of magnitude smaller than its natural expectation (the Planck energy density). This is the so-called cosmological constant problem. Another puzzle of the dark energy is the cosmological coincidence problem: why are we living in an epoch in which the dark energy density and the dust matter energy are comparable? An alternative proposal for dark energy is the dynamical dark energy scenario. The dynamical nature of dark energy, at least in an effective level, can originate from various fields, such as a canonical scalar field (quintessence) [5–7], a phantom field, that is a scalar field with a negative sign of the kinetic term [8–12], or the combination of quintessence and phantom in a unified model named quintom [13–19]. Recently another paradigm has been constructed in the light of the holographic principle of quantum gravity theory, and thus it presents some interesting features of an underlying theory of dark energy [20,21]. This paradigm may simultaneously provide a solution to the coincidence problem [22–30].

It is known that Einstein's theory of gravity may not describe gravity at very high energies. The simplest alternative to general relativity is Brans–Dicke scalar-tensor theory [31]. Modified gravity also provides the natural gravitational alternative for dark energy [32–37]. Moreover, thanks to the different roles of gravitational terms relevant at small and at large curvature, the modified gravity presents natural unification of the early-time inflation and late-time acceleration. It may naturally describe the transition from non-phantom phase to phantom one without necessity to introduce the exotic matter. But among the most popular modified gravities which may successfully describe the cosmic speed-up is $f(R)$ gravity. Very simple versions of such theory like $\frac{1}{R}$ [38–40] and $\frac{1}{R} + R^2$ [41] may lead to the effective quintessence/phantom late-time universe (to see solar system constraints on modified dark energy models refer to [42–44], also general review of reconstruction is given in [45]). Another theory proposed as gravitational dark energy is scalar-Gauss–Bonnet gravity $f(G)$ [46–52] which is closely related with the low-energy string effective action. In this proposal, the current acceleration of the universe may be caused by mixture of scalar phantom and (or) potential/stringy effects. On the other hand, a theory of $f(T)$ gravity has recently been received attention. Models based on modified teleparallel gravity were presented, in one hand, as an alternative to inflationary models [53,54], and on the other hand, as an alternative to dark energy models [55]. In this paper, we show a cosmological power law solution for the acceleration of the universe based on the above mentioned modification of the teleparallel equivalent of General Relativity. We consider $T + f(T)$ gravity model and reconstruct this theory from the cosmological power law solution for the scale factor. We know that the power law solutions are very important in the standard cosmology, because this type of solutions provides a framework for establishing the behaviour of more general cosmological solutions in different histories of our universe, such as radiation dominant, matter dominant or dark energy dominant eras.

2 Field equations for $[T + f(T) + L_m]$ gravity

The action for the theory of modified gravity based on a modification of the teleparallel equivalent of General Relativity, namely $f(T)$ theory of gravity, coupled with matter L_m is given by Bengochea [55], Linder [56] and Cai [57]

$$S = \frac{1}{16\pi G} \int d^4x e [T + f(T) + L_m], \tag{1}$$

where $e = \det(e^i_\mu) = \sqrt{-g}$. The teleparallel Lagrangian T is defined as follows

$$T = S_\rho^{\mu\nu} T^\rho_{\mu\nu}, \tag{2}$$

where

$$T^\rho_{\mu\nu} = e^i_\nu (\partial_\mu e^i_\nu - \partial_\nu e^i_\mu),$$

$$S_\rho^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}_\rho + \delta^\mu_\rho T^{\theta\nu}_\theta - \delta^\nu_\rho T^{\theta\mu}_\theta),$$

and $K^{\mu\nu}_\rho$ is the contorsion tensor

$$K^{\mu\nu}_\rho = -\frac{1}{2} (T^{\mu\nu}_\rho - T^{\nu\mu}_\rho - T_\rho^{\mu\nu}).$$

The field equations are obtained by varying the action with respect to vierbein e^i_μ as follows

$$\begin{aligned} e^{-1} \partial_\mu (e S_i^{\mu\nu}) (1 + f_T) - e_i^\lambda T^\rho_{\mu\lambda} S_\rho^{\nu\mu} f_T + S_i^{\mu\nu} \partial_\mu (T) f_{TT} \\ - \frac{1}{4} e_i^\nu (1 + f(T)) = 4\pi G e_i^\rho T_\rho^\nu, \end{aligned} \tag{3}$$

where $f_T = f'(T)$ and $f_{TT} = f''(T)$. Now, we take the usual spatially-flat metric of Friedmann–Robertson–Walker (FRW) universe, in agreement with observations

$$ds^2 = dt^2 - a(t)^2 \sum_{i=1}^3 (dx^i)^2, \tag{4}$$

where $a(t)$ is the scale factor as a one-parameter function of the cosmological time t . Moreover, we assume the background to be a perfect fluid. Using the Friedmann–Robertson–Walker metric and the perfect fluid matter in the teleparallel Lagrangian (2) and the field equations (3), one obtains

$$T = -6H^2, \tag{5}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{1}{6}f - 2H^2 f_T, \tag{6}$$

$$\dot{H} = -\frac{4\pi G(\rho + p)}{1 + f_T - 12H^2 f_{TT}}, \tag{7}$$

where ρ and p denote the matter density and pressure respectively, and the Hubble parameter H is defined by $H = \dot{a}/a$.

In the FRW universe, the energy conservation law can be expressed as the standard continuity equation

$$\dot{\rho} + 3H(\rho + p) = \dot{\rho} + 3H(1 + w)\rho = 0, \tag{8}$$

where ρ is the matter energy density and $p = w\rho$ is the equation of state relating pressure p with energy density.

3 Exact matter dominant power-law solutions

We now assume an exact power-law solution for the field equations

$$a(t) = a_0 t^m, \tag{9}$$

where m is a positive real number. From the assumption (9) and the continuity equation (8), we obtain

$$\rho(t) = \rho_0 t^{-3m(1+w)}. \tag{10}$$

Moreover, using the assumption (9), Eq. (5) leads us to the following result

$$T = -6 \frac{m^2}{t^2} < 0. \tag{11}$$

By using Eqs. (5), (10) and (11) in Eq. (6), we obtain the Friedmann equation

$$\frac{T}{3} f_T + \frac{1}{6}(T - f) + \frac{8\pi G}{3} \rho_0 m^{-3m(1+w)} \left(-\frac{1}{6} T\right)^{\frac{3}{2}m(1+w)} = 0. \tag{12}$$

This is a differential equation for the function $f(T)$. The general solution of this equation is obtained as

$$f(T) = C_1 \sqrt{T} - \frac{2^{4-\frac{3}{2}m(1+w)} (3)^{-\frac{3}{2}m(1+w)} m^{-3m(1+w)} (-T)^{\frac{3}{2}m(1+w)}}{3m(1+w) - 1} \pi G \rho_0 - T, \tag{13}$$

where C_1 is an arbitrary constant of integration. Because of $T < 0$, in order to avoid of imaginary function $f(T)$, with no loss of generality we may assume the constant $C_1 = 0$. Hence, the function $f(T)$ becomes

$$f(T) = \frac{2^{4-\frac{3}{2}m(1+w)} (3)^{-\frac{3}{2}m(1+w)} m^{-3m(1+w)} (-T)^{\frac{3}{2}m(1+w)}}{1 - 3m(1+w)} \pi G \rho_0 - T. \tag{14}$$

We note that having a finite real valued solution of $f(T)$ requires also one of the followings:

- (1) $m > 0$ for any value of ω ,
- (2) $m < 0$ for $-3m(1 + \omega) \neq$ half integer,
- (3) $m = 0$,

all subject to the condition $3m(1 + \omega) \neq 1$. While the first case leads to an expanding universe, the second case describes a contracting universe, and the third case describes an static universe. Therefore, power law solutions exist for the function (14) subject to the first and second conditions above.

By inserting (14) in the action (1), we find that the standard Einstein gravity will automatically be recovered when $f(T) = 0$. This happens provided that $\frac{3}{2}m(1 + \omega) = 1$ and

$$\rho_0 = \frac{3m^{3m(1+\omega)}}{8\pi G} = \frac{3m^2}{8\pi G}, \quad (15)$$

which guarantees the positivity requirement of ρ_0 .

4 Exact Phantom phase power-law solutions

One may also study the power-law solutions where the universe enters a phantom phase leading to a Big Rip singularity. For this case, the general class of Hubble parameters and cosmological solutions are defined as

$$H(t) = \frac{m}{t_s - t}, \quad (16)$$

$$a(t) = a_0(t_s - t)^{-m}, \quad (17)$$

where t_s is the so called ‘‘Rip time’’ at future singularity. It is easy to show that all the results in Sect. 4 follow from Sect. 2 by replacing m by $-m$.

Demanding a Big Rip during the phantom phase, as the cosmic time t approaches t_s , requires $m \geq 1$ in (17). Therefore, power law solutions for the Phantom phase exist for the corresponding function $f(T)$.

5 Conclusion

In the present paper we have considered a $T + f(T) + L_m$ action which describes Einstein’s gravity plus a function of the torsion scalar. Then, by considering an exact power-law solution for the field equations we have obtained the Friedmann equation in spatially flat universe. The Friedmann equation appears as a differential equation for the function $f(T)$. We obtained the solution of this equation and showed that our model with this solution for $f(T)$ has power-law solution of the type $a(t) = a_0 t^m$. We have also studied the power-law solutions when the universe enters a Phantom phase. By considering such power-law solution for the field equations, the corresponding Friedmann equation and the solution $f(T)$ is simply obtained by comparing with the results obtained in non-Phantom phase and replacing m by $-m$. It is shown that the power-law solution of the type $a(t) = a_0(t_s - t)^{-m}$ also exists in the phantom phase for this $f(T)$ solution.

Acknowledgments This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha (RIAAAM) under research project No. 1/2360.

References

1. Perlmutter, S. et al.: *Astrophys. J.* **517**, 565 (1999)
2. Bennett, C.L. et al.: *Astrophys. J. Suppl.* **148**, 1 (2003)
3. Tegmark, M. et al.: *Phys. Rev. D* **69**, 103501 (2004)
4. Allen, S.W. et al.: *Mon. Not. R. Astron. Soc.* **353**, 457 (2004)
5. Ratra, B., Peebles, P.J.E.: *Phys. Rev. D* **37**, 3406 (1988)
6. Wetterich, C.: *Nucl. Phys. B* **302**, 668 (1988)
7. Zlatev, I., Wang, L.M., Steinhardt, P.J.: *Phys. Rev. Lett.* **82**, 896 (1999)
8. Caldwell, R.: *Phys. Lett. B* **545**, 23 (2002)
9. Caldwell, R.R., Kamionkowski, M., Weinberg, N.N.: *Phys. Rev. Lett.* **91**, 071301 (2003)
10. Nojiri, S., Odintsov, S.D.: *Phys. Lett. B* **562**, 147 (2003)
11. Onemli, V.K., Woodard, R.P.: *Phys. Rev. D* **70**, 107301 (2004)
12. Setare, M.R., Sadeghi, J., Amani, A.R.: *Phys. Lett. B* **666**, 288 (2008)
13. Setare, M.R.: *Phys. Lett. B* **641**, 130 (2006)
14. Zhao, W., Zhang, Y.: *Phys. Rev. D* **73**, 123509 (2006)
15. Setare, M.R., Sadeghi, J., Amani, A.R.: *Phys. Lett. B* **660**, 299 (2008)
16. Sadeghi, J., Setare, M.R., Banijamali, A., Milani, F.: *Phys. Lett. B* **662**, 92 (2008)
17. Setare, M.R., Saridakis, E.N.: *Phys. Lett. B* **668**, 177 (2008)
18. Setare, M.R., Saridakis, E.N.: [arXiv:0807.3807 [hep-th]]
19. Setare, M.R., Saridakis, E.N.: *JCAP* **0809**, 026 (2008)
20. 't Hooft, G.: [arXiv:gr-qc/9310026]
21. Susskind, L.: *J. Math. Phys.* **36**, 6377 (1995)
22. Li, M.: *Phys. Lett. B* **603**, 1 (2004)
23. Hu, B., Ling, Y.: *Phys. Rev. D* **73**, 123510 (2006)
24. Li, H., Guo, Z.K., Zhang, Y.Z.: *Int. J. Mod. Phys. D* **15**, 869 (2006)
25. Setare, M.R.: *Phys. Lett. B* **642**, 1 (2006)
26. Setare, M.R.: *Phys. Lett. B* **642**, 421 (2006)
27. Setare, M.R., Zhang, J., Zhang, X.: *JCAP* **0703**, 007 (2007)
28. Setare, M.R.: *Phys. Lett. B* **648**, 329 (2007)
29. Setare, M.R.: *JCAP* **0701**, 023 (2007)
30. Setare, M.R.: *Eur. Phys. J. C* **50**, 991 (2007)
31. Brans, C., Dicke, C.H.: *Phys. Rev.* **124**, 925 (1961)
32. Nojiri, S., Odintsov, S.D., Stefancic, H.: *Phys. Rev. D* **74**, 086009 (2006)
33. Nojiri, S., Odintsov, S.D.: *J. Phys. A* **40**, 6725 (2007)
34. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Zerbini, S.: *Phys. Rev. D* **75**, 086002 (2007)
35. Nojiri, S., Odintsov, S.D.: *J. Phys. Conf. Ser.* **66**, 012005 (2007)
36. Setare, M.R.: *Int. J. Mod. Phys. D* **17**, 2219 (2008)
37. Setare, M.R.: *Astrophys. Space Sci.* **326**, 27 (2010)
38. Capozziello, S.: *Int. J. Mod. Phys. D* **11**, 483 (2002)
39. Capozziello, S., Carloni, S., Troisi A.: arXiv:astro-ph/0303041
40. Carroll, S.M., Duvvuri, V., Trodden, M., Turner, S.: *Phys. Rev. D* **70**, 043528 (2004)
41. Nojiri, S., Odintsov, S.D.: *Phys. Rev. D* **68**, 123512 (2003)
42. Nojiri, S., Odintsov, S.D.: 0707.1941v2 [hep-th]
43. Nojiri, S., Odintsov, S.D.: 0710.1738v2 [hep-th]
44. Cognola, G., Elizalde, E., Nojiri, S., Odintsov, S.D., Sebastiani, L., Zerbini, S.: 0712.4017v1 [hep-th]
45. Nojiri, S., Odintsov, S.D.: *Phys. Rep.* **505**, 59 (2011)
46. Nojiri, S., Odintsov, S.D., Sasaki, M.: *Phys. Rev. D* **71**, 123509 (2005)
47. Nojiri, S., Odintsov, S.D., Sami, M.: *Phys. Rev. D* **74**, 046004 (2006)
48. Carter, B.M.N., Neupane, I.P.: *Phys. Lett. B* **638**, 94 (2006)
49. Carter, B.M.N., Neupane, I.P.: *JCAP* **0606**, 004 (2006)
50. Moffat, J.W., Toth, V.T.: arXiv:0710.0364 [astro-ph]

51. Elizalde, E., Myrzakulov, R., Obukhov, V.V., Sáez-Gómez, D.: *Class. Quantum Gravity* **27**, 095007 (2010)
52. Myrzakulov, R., Sáez-Gómez, D., Tureanu, A.: *Gen. Relativ. Gravit.* (2011, in press). arXiv:1009.0902 [gr-qc]
53. Ferraro, R., Fiorini, F.: *Phys. Rev. D* **75**, 084031 (2007)
54. Ferraro, R., Fiorini, F.: *Phys. Rev. D* **78**, 124019 (2008)
55. Bengochea, G., Ferraro, R.: *Phys. Rev. D* **79**, 124019 (2009)
56. Linder, E.V.: *Phys. Rev. D* **81**, 127301 (2010)
57. Cai, Y.-F., Chen, S.-H., Dent, J.B., Dutta, S., Saridakis, E.N.: arxiv:1104.4349v2