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RESEARCH ARTICLE

Fermi and electromagnetic mass

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Abstract Fermi's analysis of the contribution of the electromagnetic field to the inertial mass of the classical electron within special relativity is brought to its logical conclusion, leading to the conservation of the total 4-momentum of the field plus mechanical mass system as seen by the sequence of inertial observers in terms of which the accelerated electron is momentarily at rest.

Keywords Special relativity · Electromagnetic mass · Conservation laws

1 Introduction

In 1921–1923 Enrico Fermi [1–7] wrote his first four scientific papers in a series addressing the question of the contribution of the energy in the Coulomb field of a classical model of the electron to its inertial mass within special relativity. This model had been developed in the first decade of the 1900s by Abraham [8,9] and Lorentz [10] during the same period in which special relativity was being born. Fermi's second paper [2] studied this question within general relativity using a metric introduced by Levi–Civita representing a spacetime reference frame accelerated along one spatial direction. Fermi's third paper [3,11,12] addressed a side issue in this series—the mathematical theory of his Fermi coordinate system and Fermi–Walker transport (both

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extensively employed by Synge in his early textbook on general relativity [13]), the latter of which became a key tool in the theory of general relativity—while the culminating fourth paper written in three versions in Italian and German but never available in English until now [14], though often quoted, has rarely been appreciated nor understood for its actual content. Fermi himself stopped short of considering his result in the fourth paper in the context of his third, namely by considering the electromagnetic contribution to the inertial mass together with a contribution from an additional mass source (mechanical or bare mass). We finish his calculation here. Furthermore although the topic continues to interest people even today as an interesting physics question, the natural completion of his work by applying it to the controversial question of the nature of the 4-momentum integrals for the electromagnetic field has never been correctly considered. We do so here. Details may be found in [15].

An unfortunate complication in this story was the confusion of the entirely separate issue of the stability of the electron with the issue of attributing a unique 4-momentum to its electromagnetic field. Unlike the 4-momentum of a point particle which is a uniquely defined 4-vector at a spacetime point, the 4-momentum of the electromagnetic field in the presence of sources is a nonlocal measurement by an inertial observer which is represented mathematically by an integral over a spacelike hyperplane of constant inertial coordinate time in the observer's associated inertial reference system, and whose result depends on the entire field at such a moment of time. In general this produces a different 4-vector for every inertial observer and for every choice of time in that observer's system of reference. This is a consequence of the nonvanishing divergence of the stress-energy tensor of the electromagnetic field when sources are present, in contrast to the situation for divergence-free such tensors where Gauss's law guarantees that the 4-momentum is independent of the inertial observer and choice of inertial time. Historically the Lorentz transformed components of the rest frame 4-momentum were compared to the components of the distinct 4-momentum seen by an observer in relative motion in the associated inertial coordinate system, but since these are components of two distinct 4-vectors, they cannot agree. It should have been expected that this comparison would fail, but instead this was seen as an apparent problem.

Poincaré [16–20] attempted to restore a unique total 4-momentum result by considering the combined system of the electromagnetic extended charge model with stabilizing stresses that would yield a divergence-free total energy-momentum tensor, thus "closing the system." However, in so doing, he obscured the fact that the electromagnetic field, which gave birth to special relativity through its Lorentz invariance, should make a contribution to the total mass-energy of the electron which is by itself relativistically correct. This perpetuated a basic error with the Abraham–Lorentz model rather than correcting it.

The key to resolving these complications with the model was the notion of rigidity later introduced by Born in 1909 [21,22], the only notion of rigidity that is compatible with special relativity. Fermi understood how to use this condition to invalidate the starting point of the Abraham–Lorentz calculation of the equation of motion for a rigid extended spherically symmetric electron accelerated by an external electromagnetic field—that the total electromagnetic force on the electron at a moment of inertial time in which it is instantaneously at rest be zero—and correct it using his Fermi coordinate system which inserts a Fermi coordinate lapse function factor into the integration of the differential forces over the corresponding time hyperplane. This led to the "correct" mass-energy relationship between the energy of the self-field of the electron and its inertial mass. However, as we will see, it also leads to a conserved total 4-momentum that is naturally associated with the 4-velocity of the electron in the expected way.

2 Electromagnetic preliminaries

We follow the conventions of Misner, Thorne and Wheeler [23] for the $-+++$ signature metric $g_{\alpha\beta}$ of Minkowski spacetime, which in inertial coordinates $(x^\alpha) = (t, x^i)$ with the identification $x^0 = t$ has nonzero components $-g_{00} = g_{ii} = 1$ (Greek and Latin indices run from 0 to 3 and 1 to 3 respectively, units are chosen so that $c = 1$) and for the electromagnetic field tensor $F_{\alpha\beta}$, whose stress-energy tensor

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \tag{1}$$

has nonzero divergence

$$T^{\mu\nu}{}_{;\nu} = -F^{\mu}{}_{\nu}J^{\nu}, \tag{2}$$

as a result of the Maxwell equation $F^{\alpha\beta}{}_{;\beta} = 4\pi J^\alpha$, where J^α is the 4-current.

Gauss's law can only be applied to a 4-vector field on Minkowski spacetime, so introduce a covariant constant vector field Q^α, $Q^\alpha{}_{;\beta} = 0$ representing a translation Killing vector field and let $\mathcal{J}^\alpha = Q_\beta T^{\beta\alpha}$, so that $\mathcal{J}^\beta{}_{;\beta} = Q_\alpha T^{\alpha\beta}{}_{;\beta}$. Let R be the spacetime region between two spacelike hyperplanes Σ_1 and Σ_2 oriented by their future-pointing unit normals $u^\alpha_{(1)}$ and $u^\alpha_{(2)}$ which are the 4-velocities of the corresponding inertial observers. Provided that the fields fall off sufficiently fast at spatial infinity so that the closing timelike boundary integral there between the two hyperplanes vanishes, Gauss's law states

$$\int_R \mathcal{J}^\beta{}_{;\beta}\, d^4V = \int_{\Sigma_2} \mathcal{J}^\beta d\Sigma_\beta - \int_{\Sigma_1} \mathcal{J}^\beta d\Sigma_\beta, \tag{3}$$

where for a single hypersurface Σ, the hypersurface volume element is $d\Sigma_\beta = -u_\beta d\Sigma$, so that

$$\int_\Sigma \mathcal{J}^\beta d\Sigma_\beta = \int_\Sigma (-u_\beta \mathcal{J}^\beta)\, d\Sigma \tag{4}$$

is the integral of the future-normal component of the vector field with respect to the intrinsic volume element $d\Sigma = dV_\Sigma$. In inertial coordinates adapted to the 4-velocity u^α so that Σ coincides with a hyperplane of constant inertial time t, this is just $dV_\Sigma = dx^1dx^2dx^3$, and the hyperplane integral is just a triple integral with respect to these spatial coordinates, while the spacetime volume element is then $d^4V = dt\, dx^1 dx^2 dx^3$. For intersecting such hyperplanes Σ_1 and Σ_2 associated with observers in relative motion, R must be oriented oppositely on the two disjoint pieces into which the intersection divides it, with the half for which Σ_2 is the future boundary oriented positively, and the other half oriented negatively (see Fig. 5.3.c of Misner, Thorne and Wheeler [23]). Thus

$$\int_R Q_\alpha T^{\alpha\beta}{}_{;\beta} d^4V = \int_{\Sigma_2} Q_\alpha T^{\alpha\beta} d\Sigma_\beta - \int_{\Sigma_1} Q_\alpha T^{\alpha\beta} d\Sigma_\beta, \tag{5}$$

where if we agree to evaluate these expressions in inertial coordinates where Q_α are constants, then they can be factored out of the equation.

The inertial coordinate components of the 4-momentum of the electromagnetic field as seen by an inertial observer with 4-velocity u^α at a moment of time t in the observer rest frame represented by a spacelike time coordinate hyperplane Σ (for which u^α is in fact the future-pointing timelike unit normal vector field) is given by the integral formula

$$P(\Sigma)^\alpha = \int_\Sigma T^{\alpha\beta} d\Sigma_\beta. \tag{6}$$

In inertial coordinates where $u^\alpha = \delta^\alpha{}_0$ this gives the energy and momentum as the integral of the local energy density and the Poynting vector respectively

$$P(\Sigma)^0 = \int_\Sigma T^{00} dV_\Sigma,\ P(\Sigma)^i = \int_\Sigma T^{0i} dV_\Sigma. \tag{7}$$

While the contracted pair of indices in the integral (6) can be evaluated in any coordinates, one can integrate over an object with a free index only if that index is expressed in some inertial coordinate system where it makes sense to compare 4-vectors at different spacetime points in the flat spacetime due to the path independence of parallel transport. In such coordinates we then have from Eqs. (2), (5) and the definition (6)

$$\int_R -F^\alpha{}_\beta J^\beta d^4V = P(\Sigma_2)^\alpha - P(\Sigma_1)^\alpha. \tag{8}$$

When $J^\alpha = 0$, the left hand side is zero, showing that the 4-momentum vector functional is independent of the hyperplane and defines a single 4-vector which represents the conserved 4-momentum of the free electromagnetic field.

3 Lagrangian equations in Fermi coordinates

The Born rigidity condition requires that the charge and mass density profiles of an electron model be time-independent in the Fermi coordinate system adapted to a world

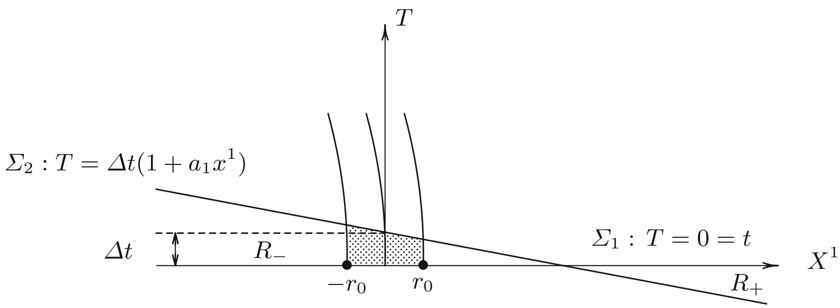

Fig. 1 Inertial Cartesian coordinates (T, X^1) with Fermi coordinates (t, x^1) such that $t = 0 = T$ coincide, showing an $X^2 = 0 = X^3$ cross-section of the world tube of an electron sphere instantaneously at rest at the origin at $T = 0$ but accelerated in the negative $x^1 = X^1$ direction ($a_1 < 0$). At a successive Fermi time Δt later, the Fermi time hyperplanes intersect to the right of the world tube (equivalent to the assumption $|a_1| r_0 < 1$). The spacetime region within the electron world tube between the two slices (*shaded* in this plane cross-section) occurs in the Gauss's law application to the wedge between the two time slices, namely $R = R_- \cup R_+$, two regions which are separated from each other by a plane of constant x^1 within the hypersurface $t = 0$ shown as the intersection point in this diagram; R_- must be positively oriented, but R_+ negatively oriented for Eq. (3)

line within the localized matter distribution. The constant Fermi time hyperplanes in such a coordinate system are orthogonal to this world line at their point of intersection, representing the local rest space of the associated comoving observer at that point of the world line. In fact the Fermi time coordinate lines are always orthogonal to the Fermi time coordinate hyperplanes; for this reason these coordinates are often known as Fermi normal coordinates.

The classical model of the nonrotating electron assumes a spherically symmetric distribution of mass and charge within a sphere of radius r_0 of the central world line in such a coordinate system, where the metric line element has the form

$$ds^2 = -N_F^2 dt^2 + \delta_{ij} dx^i dx^j, \quad N_F = 1 + a_i x^i \tag{9}$$

and a_i are the Fermi coordinate components of the 4-acceleration a^α of the central world line $x^i = 0$, where the proper time derivative along the time lines $d/d\tau = N_F^{-1} d/dt$ reduces to the Fermi coordinate time derivative; in the Fermi coordinates one has $a^\alpha = \delta^\alpha{}_i a^i$. The spacetime volume element is $d^4V = N_F\, dt\, dV$, where $dV = dx^1 dx^2 dx^3$ is the spatial volume element. See Misner, Thorne and Wheeler [23] for details of this coordinate system. Figure 1 shows a 2-dimensional cross-section of two successive Fermi time hyperplanes for a central world line decelerating along the x^1 direction, and the interpretation of the Fermi lapse function for an infinitesimal increment Δt of Fermi time.

The time lines are the world lines of the elements of the charged matter distribution, having Fermi coordinate 4-velocity components

$$U^\alpha = \frac{dx^\alpha}{d\tau} = \frac{1}{N_F} \frac{dx^\alpha}{dt} = \frac{1}{N_F} \delta^\alpha{}_0, \tag{10}$$

from which one obtains the acceleration $a^\alpha = DU^\alpha/dt|_{x^i=0}$ of the central world line. Let ρ and $\rho_{(\mathrm{me})}$ be the spherically symmetric charge and mechanical or bare mass densities, which are functions only of the radius $r = (\delta_{ij} x^i x^j)^{1/2}$ and which vanish outside $r = r_0$. The Abraham–Lorentz spherical shell model assumes a delta function distribution at $r = r_0$: $\rho_{\text{shell}} = \delta(r - r_0) e/(4\pi r_0^2)$, where e is the total charge of the electron; one may also easily consider a uniform density distribution within the sphere of radius r_0. Let $de = \rho\, dV$ and $dm_{(\text{me})} = \rho_{(\text{me})} dV$ be the elements of the charge and mechanical mass distributions, so that $e = \int \rho\, dV$ is the total charge and $m_{(\text{me})} = \int \rho_{(\text{me})}\, dV$ is the total mechanical mass (also called bare mass). The 4-current is then $J^\alpha = \rho\, U^\alpha$.

The action for the electromagnetic field together with the matter distribution considered by Fermi in his third paper [3] is

$$S = \int_R \left(-\frac{1}{16\pi} F^{\alpha\beta} F_{\alpha\beta} + A_\alpha J^\alpha \right) d^4x - \int d\tau\, dm_{(\text{me})}, \tag{11}$$

where the second integral in the Lagrangian is the integral with respect to the differential of mechanical mass of the line integral over the world line of the matter element, while the second term in the first integral here can be similarly expressed as

$$\begin{aligned} \int \rho A_\alpha U^\alpha N_F\, dt\, dV &= \int \rho A_\alpha \frac{dx^\alpha}{d\tau} N_F\, dt\, dV \\ &= \int \rho A_\alpha \frac{dx^\alpha}{dt}\, dt\, dV = \int A_\alpha dx^\alpha\, de, \end{aligned} \tag{12}$$

showing that it is a parametrization-independent line integral integrated over the charge distribution. The region R of integration is assumed to be a cylindrical region with respect to the Fermi coordinate system between two fixed Fermi times, over an arbitrary time-independent spatial region B in the Fermi coordinate system. The action is a function of the 4-potential of the electromagnetic field, in terms of which $F_{\alpha\beta} = dA_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$, and of the world lines of the matter distribution, which are the time lines of the Fermi coordinate system. Varying the action with respect to A_α yields the remaining Maxwell's equations.

The first term in the action is independent of the world lines. Varying the world lines such that $\delta x^\alpha = \delta^\alpha{}_i \delta x^i$ leads to the Lagrangian equations of motion for the central world line of the rigid charged matter distribution. Varying the 4-current term with respect to the world lines, as shown by Fermi [3–6], ignoring a boundary term which arises from an integration by parts in time, leads to

$$\begin{aligned} &\int_{t_1}^{t_2} \left(\int_B F_{\alpha\beta} \frac{dx^\alpha}{dt} de \right) \delta x^\beta\, dt \\ &\quad = \int_{t_1}^{t_2} \left(\int_B F_{\alpha\beta} U^\alpha N_F\, de \right) \delta x^\beta\, dt \\ &\quad = \int_{t_1}^{t_2} \left(\int_B E(U)_i N_F\, de \right) \delta x^i\, dt. \end{aligned} \tag{13}$$

where $E(U)^\alpha = F^\alpha{}_\beta U^\beta$ is the electric field as seen by the Fermi coordinate observer with 4-velocity U^α. The variation of the mechanical mass term yields $-\int_{t_1}^{t_2} m_{(\mathrm{me})} a_i \, \delta x^i \, dt$.

If the variations δx^i are arbitrary functions of the Fermi time, vanishing at the endpoint Fermi times to justify ignoring the integration by parts boundary term, then one obtains the Fermi condition, now amended by the re-insertion of the mechanical mass term

$$\int_B \rho E(U)_i N_F \, dV - m_{(\mathrm{me})} a_i = 0. \tag{14}$$

This condition with $m_{(\mathrm{me})} = 0$, as assumed by Fermi in his fourth paper and by Abraham and Lorentz in their purely electromagnetic model of the electron, differs from the Abraham–Lorentz starting condition for their derivation of the equations of motion only by the additional factor of the Fermi coordinate lapse function in the integral, see Jackson [27] who reproduces their calculation. However, the first term in (14) reversed in sign is exactly the Gauss integral integrand for the integral over t in Eq. (8), namely

$$\int_R -F^\alpha{}_\beta J^\beta d^4 V = -\int_{t_1}^{t_2} \left(\int_B \rho \delta^\alpha{}_i E(U)^i N_F \, dV \right) dt, \tag{15}$$

so that using the Fermi condition to replace the expression in parentheses, the latter becomes

$$\int_R -F^\alpha{}_\beta J^\beta d^4 V = -\int_{t_1}^{t_2} m_{(\mathrm{me})} \delta^\alpha{}_i a^i \, dt. \tag{16}$$

However, to evaluate this vector integral we need to express its components in an inertial coordinate system where we can utilize the relation $a^\alpha = DU^\alpha/d\tau = dU^\alpha/d\tau$, remembering that the Fermi coordinate time is the proper time along the central world line

$$\begin{aligned} m_{(\mathrm{me})} \int_{t_1}^{t_2} a^\alpha \, d\tau &= m_{(\mathrm{me})} \int_{t_1}^{t_2} \frac{dU^\alpha}{d\tau} \, d\tau \\ &= m_{(\mathrm{me})} U^\alpha |_{t_1}^{t_2} = p^\alpha_{(\mathrm{me})} |_{t_1}^{t_2}, \end{aligned} \tag{17}$$

which are the inertial coordinate components of the mechanical 4-momentum of the rigid matter distribution. Gauss's law (8) using (15)–(17) then becomes

$$-p^\alpha_{(\mathrm{me})} |_{t_1}^{t_2} = P(\Sigma_{t_2})^\alpha - P(\Sigma_{t_1})^\alpha, \tag{18}$$

or

$$p^{\alpha}_{(\mathrm{me})}(t_1) + P(\Sigma_{t_1})^{\alpha} = p^{\alpha}_{(\mathrm{me})}(t_2) + P(\Sigma_{t_2})^{\alpha}, \tag{19}$$

showing that the total 4-momentum of the system as seen by the Fermi coordinate comoving observer is independent of the Fermi time, a result apparently overlooked until now. Aharoni [24], who put out a new edition of his textbook on special relativity in 1965 in order to explain Fermi's work on this particular problem after its rediscovery, got very close to this result with his postulated self-force introduced in his reinterpretation of Fermi's results—but he missed it by neglecting to consider Gauss's law for the electromagnetic field with sources. Attention had been brought to Fermi's work in 1960 by Rohrlich's discussion of the 4-momentum integral for the electromagnetic field of the unaccelerated spinless classical electron [25] without knowledge of Fermi's work or of the same conclusions reached earlier in 1949 by Kwal [26], who was also unaware of Fermi's work.

It should also be noted that Nodvik generalized this model to include spin by adding Euler angles describing the orientation of the electron spin axis with respect to a Fermi–Walker propagated orthonormal frame along the central world line [28], as more recently updated and extended by Appel and Kiessling [29,30] and reviewed by Spohn [31]. The bare mass contribution to the Lagrangian is then modified by the Lorentz gamma factor of the motion of the elementary elements of the mass distribution with respect to the central world line, described by the intrinsic "gyration" angular velocity of the electron. However, the resulting discussion becomes extremely complicated and very difficult to follow for those of us who are not experts in advanced classical electrodynamics.

The advantage of our presentation for the spinless model is that it retains the elegance and simplicity of the work initiated by Fermi himself while remaining at the comprehension level of the standard reference text for classical electrodynamics by Jackson [27]. This allows the central idea of much more sophisticated analyses to be accessible to the general audience, an idea which is not explicitly described in the leading books on this subject [31–35].

4 The unaccelerated electron

For the case of an unaccelerated distribution of charge $a_i = 0$ when the exterior electromagnetic field vanishes and $N_F = 1$, Fermi's condition reduces to equating to zero the total electric force on the electron from its own Coulomb field

$$\int_B \rho E(U)_i \, dV = 0, \tag{20}$$

in which case the volume integral in Gauss's law vanishes and the total 4-momentum $P(\Sigma)^{\alpha}$ in the electromagnetic field is independent of time in the Fermi coordinate system, as expected since the state of the system is static in that inertial reference frame. However, although the 4-momentum of the Coulomb field is time-independent for any inertial observer, different inertial observers in relative motion measure different 4-vectors for this 4-momentum. Because this was misunderstood, and it is natural to want to associate a 4-vector representing the 4-momentum of the Coulomb field of the electron which is aligned with the 4-velocity of its central world line, people looked for solutions.

Poincaré [16–19] introduced stresses needed to balance the electromagnetic stresses in the charge distribution for the case of zero mechanical mass soon after the Abraham–Lorentz model was developed, but unfortunately retained their mistaken nonrelativistic notion of rigidity for the accelerated electron, mixing up the separate issue of the stability of this model with the lack of consistency within special relativity. By adding a nonunique ad hoc stress-energy tensor to cancel out the divergence of the electromagnetic one, he re-established the existence of a conserved 4-momentum at the cost of being inconsistent with special relativity, deriving an inertial mass for the electromagnetic field related to the energy W by the relation $\frac{4}{3}W/c^2$ instead of W/c^2.

Decades later in 1949 Kwal [26] essentially realized that in order to have a unique 4-momentum associated with the Coloumb field of the unaccelerated electron, one simply had to restrict the time hyperplane in the 4-momentum integral to one associated with the electron's inertial rest frame, although he was not sophisticated enough to actually talk about the region of integration and only examined the volume element for the hyperplane integration. In fact one can simply insert a projection along the rest frame 4-velocity into the contracted pair of indices in the 4-momentum integral definition to enforce this result for any time hyperplane. In inertial coordinates the components of this adjusted 4-momentum are

$$P_{\text{Kwal}}(\Sigma)^{\alpha} = \int\limits_{\Sigma} T^{\alpha\beta}(-U_{\beta}U^{\delta})\,d\Sigma_{\delta}, \tag{21}$$

where as above U^{α} is the 4-velocity of the rest frame of the electron and u^{α} is the 4-velocity of the inertial observer associated with the time slice Σ. However,

$$-U^{\delta}\,d\Sigma_{\delta} = -U^{\delta}u_{\delta}\,d\Sigma = \gamma(U,u)\,d\Sigma = d\Sigma_{\text{rest}}, \tag{22}$$

leads to the differential of volume $d\Sigma_{\text{rest}}$ on the tilted hyperplanes associated with a different rest frame time hypersurface at each point of Σ, a differential whose Lorentz contraction $d\Sigma_{\delta} = \gamma(U,u)^{-1}d\Sigma_{\text{rest}}$ by the relative gamma factor $\gamma(U,u) = -u_{\delta}U^{\delta}$ yields the original differential. This corresponds to integrating over the the corresponding region of Σ_{rest} related by moving to it from Σ along the rest frame time lines. Thus we have

$$P_{\text{Kwal}}(\Sigma)^{\alpha} = \int\limits_{\Sigma} T^{\alpha\beta}U_{\beta}\,d\Sigma_{\text{rest}}. \tag{23}$$

However, if we express the components of this equation in rest frame inertial coordinates where the system is static, the components of the 4-vector integrand $T^{\alpha\beta}U_{\beta}$ are independent of the rest frame time coordinate and so have the same values along each rest frame time line, so the integral is equivalent to integrating over any time hyperplane $\Sigma_{\rm rest}$ in the rest frame with respect to the actual differential of volume on that rest frame time hyperplane and the result is the unique 4-vector $P(\Sigma_{\rm rest})^\alpha$. Kwal essentially only describes replacing the factor $-u_\beta\, d\Sigma$ by $-U_\beta\, d\Sigma_{\rm rest}$ in the integral, without using the time translation invariance in the rest frame to relate the different time hyperplane regions of integration. A decade later in 1960 Rohrlich [25] came to the same conclusion without being aware of the work of Kwal or Fermi or being explicit about the time translation invariance needed to evaluate the rest frame 4-momentum on time slices not associated with the rest frame. Jackson describes in detail Rohrlich's discussion in the Second and Third Editions [27], and provides an alternative explanation for getting the same result on other time slices using the invariants of the electromagnetic field tensor. Unfortunately Rohrlich claims the Abraham–Lorentz definition of the observer-dependent 4-momentum of the electromagnetic field is wrong in the case of the very special case of an unaccelerated electron, which is simply not the case. When the integral is restricted to a bounded region of a constant inertial time hyperplane, this integral is essential in describing the transport of energy and momentum in and out of the region for any configuration of charges, currents and electromagnetic fields. See Sect. 6.7 on Poynting's Theorem in Jackson's Third Edition [27].

Explicit evaluation of the electromagnetic 4-momentum with respect to an inertial observer with 4-velocity u^α on a constant inertial time hyperplane Σ in the shell model of the electron shows that it can be expressed in the form [15]

$$
\begin{aligned}
P(\Sigma)^\alpha &= P(\Sigma_{\rm rest})^\alpha + \frac{1}{3} P(\Sigma_{\rm rest})_\beta U^\beta \nu(u,U)^\alpha \\
&= W\left(U^\alpha - \frac{1}{3}\nu(u,U)^\alpha\right),
\end{aligned}
\tag{24}
$$

where $\nu(u, U)^\alpha$ is the relative velocity of the moving frame compared to the rest frame as seen in the rest frame and $P(\Sigma_{\rm rest})^\alpha = WU^\alpha$, and W is the rest frame energy of the Coulomb field defined explicitly in the next section. The second term on the right hand side of this equation (orthogonal to the first term) shows the explicit dependence of the 4-momentum on the observer 4-velocity u^α. See Fig. 2.

Schwinger [40] has considered a special 1-parameter family of internal stress-energy tensors compatible with this shell model, resulting in a total stress-energy tensor which is divergence-free and hence the total 4-momentum is a single conserved 4-vector. Among these is the choice corresponding to $h = -1$ in the notation of the Third Edition of Jackson [27] where this tensor is proportional to the orthogonal projection $g_{\alpha\beta} + U_\alpha U_\beta$ into the local rest spaces of the electron sphere and hence does not contribute at all to the rest frame evaluation of the total 4-momentum, which therefore equals the 4-momentum of the electromagnetic field alone, the first term on the right hand side of Eq. (24). In any other inertial frame, the integral of the internal stress-energy tensor inside the electron sphere therefore exactly cancels the extra velocity-dependent term in that equation to yield the same total 4-momentum 4-vector.

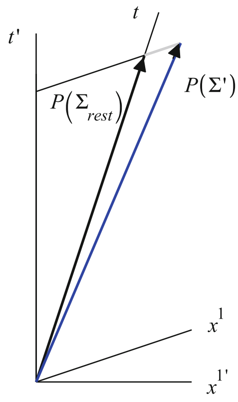

Fig. 2 The relationship between the rest frame 4-momentum $P(\Sigma_{\text{rest}})$ and the 4-momentum $P(\Sigma')$ observed in an inertial frame in relative motion, referred to the corresponding inertial coordinate axes for the case of motion along the x^1 direction

5 Equations of motion for the rigid charge distribution

The actual equations of motion for the central world line of the rigid charge distribution can be evaluated in the quasi-stationary limit of small enough and sufficiently slowly changing acceleration that one can linearize Eq. (17) with respect to the acceleration and ignore its time derivatives (thus neglecting radiation reaction terms) as described in detail in Jackson [27] for the case of zero mechanical mass, without the Fermi lapse factor. First one must separate out the self-field due to the charge distribution from the external field in which the electron is moving, assuming that the latter is essentially constant over the charge distribution so that it may be factored out of the integral. The self-field is defined through the retarded time integrals of the 4-potential over the charge distribution in Lorentz gauge. One then has

$$\int E^{(\text{self})}(U)_i N_F \, de + \int E^{(\text{ext})}(U)_i N_F \, de - m_{(\text{me})} a_i = 0. \tag{25}$$

The lowest order contribution to the self-force in this approximation as shown by Fermi is $-m_{(\text{em})} a_i$, where the inertial mass coefficient $m_{(\text{em})}$ is a constant equal to the total energy W of the Coulomb field of the charge distribution, defined by

$$W = \frac{1}{2} \int \int d^3\mathbf{x} d^3\mathbf{x}' \, \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tag{26}$$

in the notation of Jackson [27], expressed as seen by an inertial observer at a time when the electron is momentarily at rest. The Fermi coordinate lapse factor in the integrand of the self-field integral in (25) corrects the Abraham-Lorentz result $m_{(\text{em})} = \frac{4}{3} W$ to conform with the Einstein mass-energy relation $E = mc^2$ (with $c = 1$), as we show next.

Then since the first term on the left hand side and the right hand side of Eq. (25) are proportional to the acceleration, the second term must be first order in the acceleration, so keeping only first order terms, one can ignore the Fermi lapse factor in the second term which becomes

$$\int E^{(\mathrm{ext})}(U)_i N_F \, de = E^{(\mathrm{ext})}(U)_i \int de = eE^{(\mathrm{ext})}, \tag{27}$$

leading to the Lorentz force law in the Fermi frame for which the spatial velocity is zero

$$(m_{(\mathrm{me})} + m_{(\mathrm{em})})a_i = eE_i^{(\mathrm{ext})}. \tag{28}$$

In other words the mass formula for the electromagnetic contribution to the inertial mass is

$$m = m_{(\mathrm{me})} + m_{(\mathrm{em})}. \tag{29}$$

For the spherical shell model, one easily finds $m_{(\mathrm{em})} = e^2/(2r_0)$, which compares very nicely with the Reissner–Nordstrom irreducible mass formula [36]

$$m = m_{(\mathrm{irred})} + e^2/(2r_+) \tag{30}$$

for the gravitational mass m of a static spherically symmetric charge distribution of total charge e and outer horizon radius r_+ within general relativity.

6 Conclusions

The classical theory of the electron and related issues has attracted the attention of many of the great physicists of the past century, and has been the subject of many articles and a few books that continue to appear, most of which seem not to reflect Fermi's simple argument, although a relatively recent article by Kolbenstved [37] offers an alternative explanation of that argument. For a complete list of such references see [15], as well as the recent analysis by Boughn and Rothman [38] of a related problem considered by Fermi in his fifth paper [39]. Ultimately the problematic issues of a finite-sized classical electron were sidestepped by the point particle model and renormalization techniques introduced in the quantum theory. However, as recently as the past decade, new results in the classical theory have appeared [29–31,33–35], but which still leave this loose end of Fermi's work unaddressed.

Gauss's law for stress-energy tensors with nonzero divergence is straightforward to consider yet, until now no one has connected up Fermi's results with the question of the 4-momentum in the electromagnetic field of the classical electron model, an issue which arose after he lost interest in the problem. Doing so has provided a useful pedagogical example omitted in all textbooks on general relativity or electrodynamics and has led to the following satisfying result. While in the case of the unaccelerated electron, there is no selection mechanism to pick out the obvious candidate for the 4-momentum aligned with the 4-velocity of the rigid electron other than the alignment itself, for the accelerated electron it is only the instantaneous rest frame observer which leads not only to aligning the 4-momentum of the electromagnetic self-field with the 4-velocity, but also to a 4-momentum conservation law for the total 4-momentum.

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