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The geometry of free fall and light propagation

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THE GEOMETRY OF FREE FALL AND LIGHT PROPAGATION

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1. Introduction and description of results

In both the special and the general theory of relativity, as well as in similar differential-geometrical theories of space–time, one may distinguish several local geometrical structures assigned to the space–time manifold M: the topological, differential, conformal, projective, affine, and metric structures. In making these distinctions, we follow essentially H. Weyl[1] with some modifications explained later.

Postponing more precise definitions of the different structures to later sections, we recall that a *conformal structure* (of normal-hyperbolic or Lorentzian type†) consists of a field of infinitesimal null cones defined all over M; *a projective structure* consists of a family of curves,‡ called *geodesics*, whose members behave in the second-order infinitesimal neighbourhood of each point of M like the straight lines of an ordinary projective four-space; an *affine structure* differs from a projective structure in that the preferred curves carry preferred *affine parameters* (defined up to linear transformations) such that, infinitesimally, there is an affine geometry around each point of M; and a *metric* assigns to any pair of adjacent points of M a number called its *separation*.

Equivalently, we may list the fundamental operations generating these structures infinitesimally. In the conformal case, it is the construction of the hyperplane element orthogonal to a given line element—the null elements being those contained in their orthocomplements. In the projective case, it is the parallel displacement of a direction D from a point p to an adjacent point q which lies in that same direction D relative to p—the preferred autoparallel lines l being those whose tangent elements are parallel along l. In the affine case, it is the usual Levi-Civita parallel displacement of an

† In this paper, we shall assume throughout that dim $M = 4$ and that the signature is $(+++-)$. However, many statements can be modified so as to hold without these physically motivated restrictions.

‡ We shall use the term 'curve' for what is called a 'one-dimensional submanifold' in current differential-geometric literature. Thus, for us, a curve is not a map from an interval of R into M, although it can locally be obtained as the range of many such maps.

arbitrary vector from a point to any adjacent point—equidistant points (with respect to an affine parameter) on a geodesic being such that their connection vectors are parallel. Finally, in the metric case, it is the construction of all line elements at a point q that are congruent to a given line element at some other point p. All these structures are assumed to be smooth with respect to the differential topology, which also enters basically into the infinitesimal operations.

In the Riemannian space–times of special and general relativity, all these structures are present, and they are intimately related to each other.

From the physical as well as from the geometrical point of view, one may ask which of these structures should be considered as basic and which as derived. Excluding, for well-known reasons, rigid rulers as primitive physical concepts, we may follow Synge[2] in accepting as basic the concepts *particle* and (standard) *clock*, and introduce the metric as the fundamental structure, postulating that whenever x, $x+\mathrm{d}x$ are two nearby events contained in the world line, or history, of a clock then the separation associated with (x, $x+\mathrm{d}x$) equals the time interval as measured by that (and any other suitably scaled) clock. This procedure has two advantages. First, it uses as primitive a physical quantity that can, in fact, be measured locally and with extreme precision, and, secondly, it introduces as the primary geometric structure the metric, from which all the other structures can be obtained in a straightforward manner.

If the aim is a *deduction* of the theory from a few axioms, the *chronometric* approach is indeed very economical. If, however, one wishes to give a *constructive* set of axioms for relativistic space–time geometries, which is to exhibit as clearly as possible the physical reasons for adopting a particular structure and which indicates alternatives, then the chronometric approach does not seem to be particularly suitable, for the following three reasons. It seems difficult to derive *from the behaviour of clocks alone*, without the use of light signals, the *Riemannian* form for the separation,

$$\mathrm{d}s = |g_{ij}\mathrm{d}x^i\mathrm{d}x^j|^{1/2}, \tag{1}$$

rather than some other, first-degree homogeneous, functional form in the $\mathrm{d}x^i$ (as, for instance, the Newtonian form $\mathrm{d}s = g_i\mathrm{d}x^i$). Postulating this form axiomatically, one foregoes the possibility of understanding the reason for its validity. The second difficulty is that if the g_{ij} are defined by means of the chronometric hypothesis, it seems not at all compelling—if we disregard our knowledge of the full theory and try to construct it from scratch—that these chronometric coefficients should determine the behaviour of freely falling particles and light rays, too. Thus the geodesic hypotheses, which are introduced as additional axioms in the chronometric approach, are hardly intelligible; they fall from heaven like eqn (1). Finally, once the geodesic hypotheses have been accepted, it is possible, in the theories of both special and general relativity, to construct clocks by means of freely falling particles and light rays, as shown by Marzke[3] and, differently, by Kundt and Hoffmann.[4] Thus, these hypotheses alone already imply a physical interpretation of the metric in terms of time. The chronometric axiom then appears either as redundant or, if the term 'clock' is interpreted as 'atomic clock', as a link between macroscopic gravitation theory and atomic physics: it claims the equality of gravitational and atomic time. It may be better to test this equality experimentally† or to derive it eventually from a theory that embraces both gravitational and atomic phenomena, rather than to postulate it as an axiom.

For these reasons, we reject clocks as basic tools for setting up the space–time geometry[6] and propose to use light rays and freely falling particles instead. We wish to show how the full space–time geometry can be synthesized from a few assumptions about light propagation and free fall.

Our method has some similarity to Helmholtz's derivation of the metrics of spaces of constant curvature. According to Helmholtz[7] and Lie,[8] the existence and form of these metrics can be deduced from the qualitative assumption of the free mobility of rigid bodies. Similarly, we attempt to derive the conformal, projective, affine, and metric structures of space–time from some qualitative (incidence and differential-topological) properties of the phenomena of light propagation and free fall that are strongly suggested by experience. Not only the measurement of length but also that of time then appears as a derived operation. All our axioms, which will be stated in the second part of this paper, are *local*; we shall not need to impose any global restrictions on space–time.

Our line of reasoning is illustrated in Fig. 1 and, in summary, is as follows:

(a) The propagation of light determines at each point of space–time the infinitesimal null cone and thus gives it a *conformal structure* $\mathscr{C}$. With respect to this, one can distinguish between time-like, space-like, and null directions, vectors, and curves, respectively, and one can single out as null geodesics those null curves contained in a null hypersurface[9] (see upper right-hand portion of Fig. 1). These null geodesics will be shown to represent light rays.

(b) The motions of freely falling particles determine a family of preferred $\mathscr{C}$-time-like curves, and by assuming this family to satisfy a generalized law of inertia, we show that free fall defines a *projective structure* $\mathscr{P}$ in space–time such that the world lines of freely falling particles are the $\mathscr{C}$-time-like geodesics of $\mathscr{P}$.

(c) The conformal and projective structures thus defined are intimately related, as experience indicates: an ordinary particle (i.e. one with positive

† In recent observational tests of gravitation theories by radar tracking of planetary orbits, atomic time has indeed been used only as an ordering parameter whose relation to gravitational time was to be determined from the observations.[5]

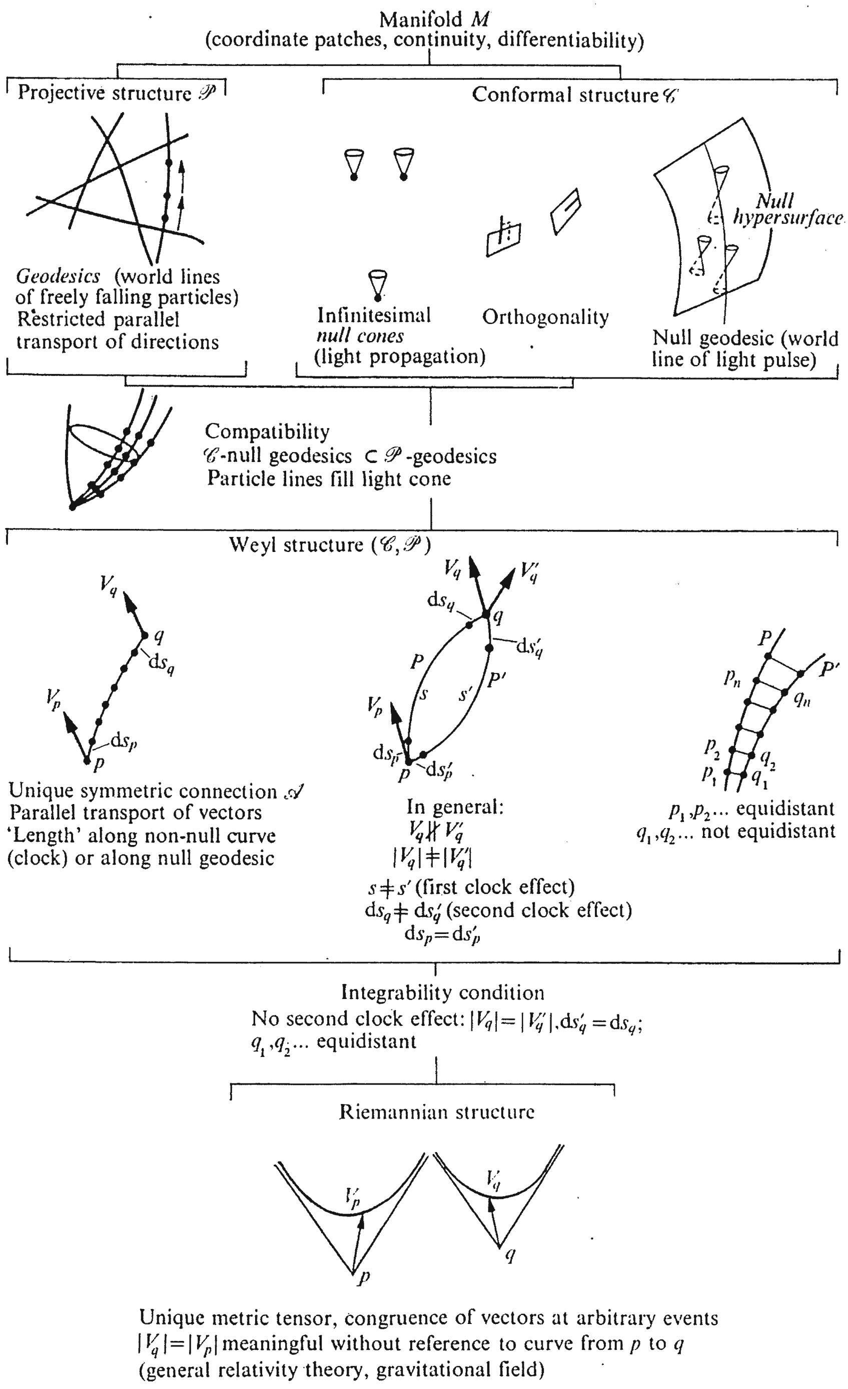

FIG. 1. General scheme of conformal, projective, Weyl, and Riemannian structures.

rest mass), though always slower than light, can be made to chase a photon arbitrarily closely. From this we shall deduce that the conformal and projective structures of space–time are *compatible*, in the sense that every $\mathscr{C}$-null geodesic is also a $\mathscr{P}$-geodesic. We call a manifold M endowed with a compatible pair of $\mathscr{C}$, $\mathscr{P}$ structures a *Weyl space*.

A Weyl space $(M, \mathscr{C}, \mathscr{P})$ possesses a unique *affine structure* $\mathscr{A}$ such that $\mathscr{A}$-geodesics coincide with $\mathscr{P}$-geodesics and $\mathscr{C}$-nullity of vectors is preserved under $\mathscr{A}$-parallel displacement. Conversely, the existence of such an $\mathscr{A}$ for a pair $(\mathscr{C}, \mathscr{P})$ implies that $\mathscr{C}$ and $\mathscr{P}$ determine a Weyl structure. In view of this

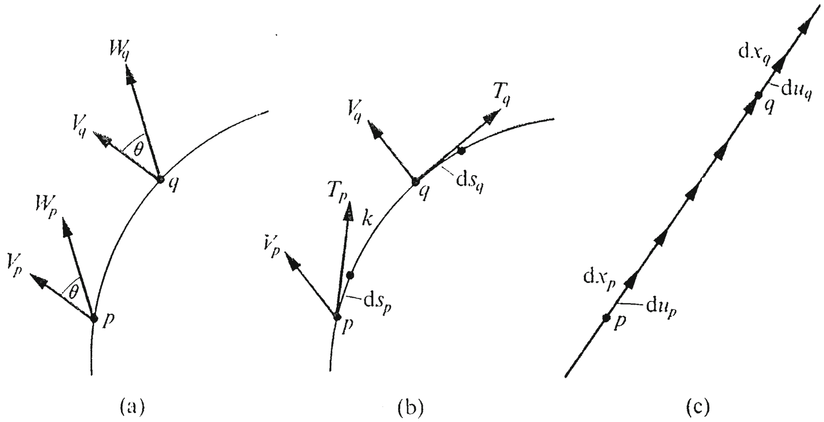

FIG. 2. (a) Parallel transport, (b) length, and (c) affine parameter in a Weyl space.
(a) $\theta = \text{constant}, \dfrac{|V|}{|W|} = \text{constant};$
(b) $\dfrac{\mathrm{d}s}{|V|} = \text{constant}, V_q || V_p, |T| = |V|$ at p and q;
(c) $\mathrm{d}x_q || \mathrm{d}x_p$, $\mathrm{d}u_q = \mathrm{d}u_p$.

theorem, one may say that light propagation and free fall define a Weyl structure on space–time; we symbolize the latter by $(M, \mathscr{C}, \mathscr{A})$.

(d) In a Weyl space, one can define an arc length (unique to within linear transformations) along any non-null curve (i.e. a curve whose tangent is nowhere null) k by requiring that the corresponding tangent vector T be congruent, at each point of k, to a non-null vector V which is parallelly displaced along k (congruence of vectors at a point is, of course, well defined in a conformal space[9]). Intuitively, this means that two infinitesimal line elements of k, $\mathrm{d}s_p$ and $\mathrm{d}s_q$, situated at p and q respectively, are considered as congruent on k if the infinitesimal connection vector belonging to $\mathrm{d}s_q$ arises from that associated with $\mathrm{d}s_p$ by parallel transport from p and q, followed by a rotation (or pseudorotation) at q (see Fig. 2). Applying this to the time-like world line of a particle P (not necessarily freely falling), one obtains a proper time ($\equiv$ arc length) t on P, provided two events on

P have been selected as zero point and unit point of time. The (idealized) Kundt–Hoffmann experiment[4] to measure proper time along a time-like world line in Riemannian space–time by means of light signals and freely falling particles can be used without change to measure proper time t in a Weyl space–time.

The preceding considerations contain the basis of our construction of space–time geometry. Whereas the ideas that light propagation determines a conformal structure $\mathscr{C}$ and free fall defines a projective structure $\mathscr{P}$ on space–time have been clearly spelled out by Weyl,[10] neither he himself nor anybody else, as far as we know, has used these two structures and their compatibility as fundamental, and derived the existence and uniqueness of an affine connection from these data. Rather, although Weyl emphasized repeatedly the fundamental roles of the structures $\mathscr{C}$ and $\mathscr{P}$ from a physical point of view, in geometry he took the affine and metric structures as basic and considered the projective and conformal structures as arising from these by abstraction only, although also geometrically $\mathscr{C}$ and $\mathscr{P}$ are the more primitive (less restrictive) structures.† It seems remarkable that, as we have shown, the analysis of light propagation and free fall leads quite naturally to a Weyl geometry.‡

Our construction also establishes the feasibility of Trautman's formulation of the principle of equivalence.[11] According to this author, the principle states first that the motions of freely falling particles endow space–time with an affine connection and, secondly, that all local physical processes lead to essentially the same connection. In arguing for the first part, which is the only one with which we are concerned here, one should recognize the fact that the free-fall trajectories determine directly (at best) a projective and not an affine structure. Since, however, the totality of free-fall trajectories passing through an event determines the light cone as its boundary, our reasoning shows that the first assertion is, in fact, true; one might even substitute 'Weyl structure ' for 'affine connection' in that assertion. The subtle point is that to obtain the affine connection one has to make use of both the projective and the conformal structure. This is even true in the case of special relativity, if one

† In Weyl's deep, group-theoretical analysis of the uniqueness of the Pythagorean metric (quadratic fundamental form), he took as basic operations the congruent mappings of a tangent space onto itself and the translation of a tangent space into that of a neighbouring point.[1] These operations seem empirically less immediately accessible than the construction of a null cone (which is equivalent to emission of a flash of light) and the drawing of a time-like geodesic (which is equivalent to ejection of a freely falling particle).

‡ Our definition of a Weyl geometry in terms of $\mathscr{C}$ and $\mathscr{P}$ or, equivalently, $\mathscr{C}$ and $\mathscr{A}$, seems to be preferable to Weyl's own description since it uses only unique, intrinsic structures of that geometry. Weyl's linear fundamental form arises only if one chooses arbitrarily a Riemannian metric $\mathscr{M}$ compatible with $\mathscr{C}$ and forms the difference tensor of the (intrinsic) Weyl connection $\mathscr{A}$ and the (arbitrary, non-intrinsic) Riemannian connection of $\mathscr{M}$. The contraction of the difference tensor is twice the Weyl linear fundamental form. Gauge transformations do not occur in our intrinsic formulation.

wants to work with local properties only, as Weyl[1] pointed out long ago. In Newtonian space–time the role of the conformal structure is played by the absolute time.[12]

(e) It is now a straightforward matter to formulate additional assumptions that are necessary and sufficient in order that a Weyl space $(M, \mathscr{C}, \mathscr{A})$ be a Riemann space, in the sense that there exists a Riemannian metric $\mathscr{M}$ compatible with $\mathscr{C}$ (i.e., having the same null cones) and having $\mathscr{A}$ as its metric connection. The Riemannian metric is then necessarily unique, up to a constant positive factor. To obtain such conditions, we make use of the fact that $\mathscr{A}$ determines a *curvature tensor* R. Using the equation of geodesic deviation, we show that $(M, \mathscr{C}, \mathscr{A})$ is Riemannian if and only if the proper times t, t' on two arbitrary, infinitesimally close, freely falling particles P, P' are linearly related (to first order in the distance) by Einstein-simultaneity; i.e. if and only if whenever $p_1, p_2, \ldots$ is an equidistant sequence of events on P (ticking of a clock) and $q_1, q_2, \ldots$ is the sequence of events on P' that are Einstein-simultaneous with $p_1, p_2, \ldots$ respectively, then $q_1, q_2, \ldots$ is (approximately) an equidistant sequence on P' (see Fig. 1). Equivalently, we consider parallel transport of a vector V_p from a point p to a point q along two different curves P, P'. The resulting vectors, at q, $\mathbf{V}_q$, and $\mathbf{V}'_q$, will be different, in general (see Fig. 1). If and only if $\mathbf{V}_q$ and $\mathbf{V}'_q$ are congruent for all such figures, the Weyl geometry considered is, in fact, Riemannian. Taking P, P' as world lines of (not necessarily freely falling) particles, one can easily see that the Riemannian property means that whenever two standard clocks (as determined above) associated with P, P' have equal rates at p, then they also have equal rates at q. Both criteria given can, in principle, be tested experimentally. If one adopts either of them one obtains the full space–time structure of general relativity.

In any case, one may consider R as the (intrinsic) gravitational field and devise methods for measuring it, as in the Riemannian case. It is also easy to add a physically meaningful axiom that singles out the space–time of special relativity, either by requiring homogeneity and isotropy of M with respect to $(\mathscr{C}, \mathscr{A})$, or by postulating vanishing relative accelerations between arbitrary, neighbouring, freely falling particles.

It should be clear now that the difficulties inherent in the chronometric approach and listed on pp. 64, 65 are absent from the test-particle approach presented here, and that the latter approach offers a deeper understanding of the space–time geometry than the former. In particular, it seems worth emphasizing that those properties of the 'haystack' of particle trajectories that single out the Riemannian from among the more general Weyl space–times are distinctly more complicated than those that lead to the latter ones.

In the remainder of this paper, we shall state our axioms and outline the proofs of the assertions that have been stated above. A fully rigorous formalization has not yet been achieved, but we nevertheless hope that the main line of reasoning will be intelligible and convincing to the sympathetic reader. Further details, in particular geometrical constructions and a geometrical characterization of the projective curvature tensor, will be published elsewhere.

2. Axioms and proofs

We begin by assuming a set $M = \{p, q, \ldots\}$ of elements called *events*, and two collections $\mathscr{L} = \{L, N, \ldots\}$, $\mathscr{P} = \{P, Q, \ldots\}$ of subsets of M, the members of which are called *light rays* $L, \ldots$ and *particles* $P, \ldots$ respectively. (From now on we shall use 'particle' instead of 'world line of a freely falling particle', for brevity.) Both light rays and particles are to be understood in a classical sense; light rays are small, identifiable wave packets or parts of wave trains such as are used, for example, in radar-echo observations, and particles might be artificial satellites, billiard balls, or some such bodies whose extension and structure can, under suitable circumstances, be neglected.

We now discuss in turn how various structures can be introduced on M by means of $\mathscr{L}$ and $\mathscr{P}$, provided the latter satisfy certain axioms suggested by experience. The (idealized) processes involving particles and light rays are supposed to take place in an otherwise empty region of space–time.

Differential topology

We sketch first how one might introduce a differential topology on M by means of $\mathscr{L}$ and $\mathscr{P}$. The reason that we do not take this structure for granted is that differentiability plays a crucial role in our introduction of the null cones (pp. 72–6) and in the infinitesimal version of the law of free fall (p. 77). (It should be realized that the representation of light in special relativity by means of ordinary cones rather than by hypersurfaces of hour-glass shape as in Fig. 3 depends partly on a particular choice of differential structure.) The assumptions introduced below are minimal requirements.

We accept in accordance with Axiom L_1 (p. 72) that there are 'figures' in M of the type shown in Fig. 4; i.e., a light signal L is emitted from a particle P at p towards another particle Q where it is reflected at q and arrives back on P at p'. If p varies on P, then so does p'; the map $e : p \to p'$ will be called an *echo* on P from Q. Similarly, let the map $m : p \to q$ be called a *message* from P to Q.

AXIOM D_1. *Every particle is a smooth,*† *one-dimensional manifold; for any pair P, Q of particles, any echo on P from Q is smooth and smoothly invertible.*

AXIOM D_2. *Any message from a particle P to another particle Q is smooth.*

Axiom D_1 characterizes the differential structure of one particle P; any permissible local coordinate t on P may be thought of as the time shown by a (non-metric, possibly irregular) clock associated with P. Axiom D_2 asserts that the 'times' on different particles are smoothly related by light signals.

Let P and P' be two particles, e an event; in accordance with Axiom L_1

† C^3 seems sufficient for our purposes.

(p. 72), let e be connected with P by two light rays, and similarly with P' (see Fig. 5). With respect to local coordinates on P and P', e determines four numbers, u, v, u', v'. If e varies, we obtain a map $x_{PP'}: e \to (u, v, u', v')$ from M to $\mathscr{R}^4$. Making use of such maps, which one might call *radar coordinate systems*, we further assume

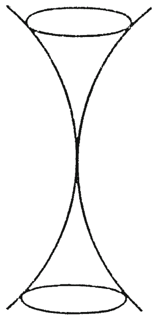

FIG. 3. Deformed light 'cone'.

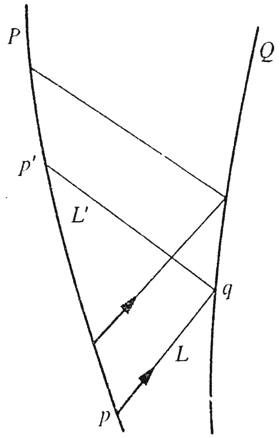

FIG. 4. Echo and message.

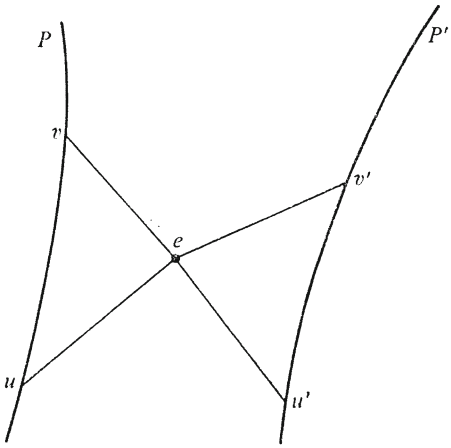

FIG. 5. Radar coordinates.

AXIOM D_3. *There exists a collection of triplets* (U, P, P') *where* $U \subset M, P, P' \in \mathscr{P}$ *such that the system of maps* $x_{PP'}|_U$ *is a smooth atlas for* M.† *Every other map* $x_{QQ'}$ *is smoothly related to the local coordinate systems of that atlas.*

With this axiom (which is a theorem in Minkowski space–time), we have endowed M with a differential topology; henceforth, M will denote the ensuing smooth manifold.

† C^3 seems sufficient for our purposes.

It follows immediately from our axioms that every particle is a smooth curve in M.

Let M_p denote the tangent (vector) space of M at p; according to Axiom D_3, $\dim M = \dim M_p = 4$. We shall write D_p for the projective three-space canonically associated with M_p; its elements are called *directions at p*. Using these notions, we finally require, again referring to processes as illustrated in Fig. 4,

AXIOM D_4. *Every light ray is a smooth curve in M. If $m : p \to q$ is a message from P to Q, then the initial direction of L at p depends smoothly on p along P.*

Light propagation and conformal structure

Experience indicates with high accuracy[13] that light emitted from a source S between two events a and b on S's history reaches a detector D between events c and d that are arbitrarily close to each other, provided a and b are sufficiently close and there is no matter between S and D. Moreover, the reception event c depends on a only and not on the motion of S or on any characteristics of the light pulse (spectrum, polarization, intensity). We therefore lay down the following axiom:

AXIOM L_1. *Any event e has a neighbourhood V such that each event p in V can be connected within V to a particle P by at most two light rays. Moreover, given such a neighbourhood and a particle P though e, there is another neighbourhood $U \subset V$ such that any event p in U can, in fact, be connected with P within V by precisely two light rays L_1, L_2, and these intersect P in two distinct events e_1, e_2 if $p \notin P$. If t is a coordinate on $P \cap V$ with $t(e) = 0$, then $g: p \to -t(e_1)t(e_2)$ is a function of class C^2 on U.* (See Fig. 6.)

We need an additional axiom to restrict further the set of all light rays passing through an event. Considering again a situation as illustrated in Fig. 6, we wish to characterize the configuration that is generated in the vicinity of p if p is kept fixed, but P and with it L_1 and L_2 are moved around arbitrarily in a neighbourhood of p. We require, using the concepts introduced before Axiom D_4,

AXIOM L_2. *The set L_e of light-directions at an (arbitrary) event e separates $D_e - L_e$ into two connected components. In M_e the set of all non-vanishing vectors that are tangent to light rays consists of two connected components.*

This axiom is intended to express the separation of non-light-like directions into time-like ones and space-like ones and the possibility of distinguishing between 'future' and 'past' light vectors. (Only the distinction between two classes matters; we do not introduce any intrinsic difference between future and past here.)

Let us consider some properties of the function g introduced in axiom L_1 for given P, e, and t. By definition, $g(p) = 0$ (for $p \in U$) if and only if p lies on a light ray through e. If it were true that $g_{,a}(e) \neq 0$, then the equation $g(p) = 0$ would define, near e, a smooth hypersurface through e that would contain all light rays through e. Since this contradicts the second part of axiom L_2, we conclude that $g_{,a}(e) = 0$. This last property of g implies that $g_{ab} := g_{,ab}(e)$ defines a tensor at e.

If we differentiate twice the equation $g\{x^a(s)\} = 0$ that holds for any light ray $L = \{x^a(s)\}$ through e and evaluate the result at e, we obtain

LEMMA 1. *The tangent vector* $\mathbf{T}$ *at* e *of any light ray* L *through* e *satisfies*

$$g_{ab}T^aT^b = 0. \tag{2}$$

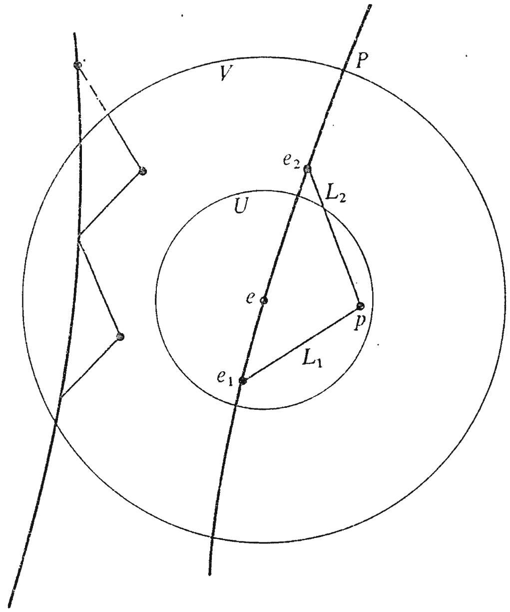

FIG. 6. Illustration of Axiom L_1.

If $x^a(t)$ is the parameter representation of P in any permissible coordinate system (see p. 71), then $g\{x^a(t)\} = -t^2$; hence, $g_{ab}K^aK^b = -2$, $\mathbf{K}$ being the tangent vector of P at e with respect to t. Thus,

$$g_{ab} \neq 0. \tag{3}$$

According to Lemma 1 and inequality (3), L_e is contained in the quadric Q defined in D_e by eqn (2). Axiom L_2 now picks out that or those quadrics Q which contain a subset L_e with the topological properties formulated in that axiom. Testing one by one the possible quadrics in projective three-space, one readily finds[14] that the quadratic form $g_{ab}\xi^a\xi^b$ is non-degenerate and normal hyperbolic, and $L_e = Q$. (The last assertion derives from the fact that no proper subset of any quadric has the required topological properties.) Consequently, eqn (2) characterizes the set of *all* vectors at e that are tangent to light rays. This statement shows that although the tensor g_{ab} has been constructed by means of a particular particle P and parameter t on P it is, except for a non-vanishing factor, an intrinsic object of the light geometry. In order to characterize it relatively to any coordinate system by a unique system of numbers, we choose the signature of the quadratic form in eqn (2) to be $(+++-)$ always, and we normalize the common factor in the g_{ab} so that

$$\det(\mathscr{g}_{ab}) = -1; \tag{4}$$

we switched to the notation $\mathscr{g}_{ab}$ to stress that $\mathscr{g}_{ab}$ then is a tensor density of weight $-1/2$, not a tensor. This description of a conformal structure has also been used by Bergmann.[15] Having obtained $\mathscr{g}_{ab}$, we can and shall henceforth distinguish between time-like, space-like, and null vectors, and classify curves accordingly.

Relative to a coordinate system, one can find the numbers $\mathscr{g}_{ab}$ at e uniquely by taking nine null vectors $\mathbf{T}_A$ at e with linearly independent 'squares' $\mathbf{T}_A \otimes \mathbf{T}_A$—such a set $\{\mathbf{T}_A\}$ exists under the assumptions made so far—and then solving the system of linear equations

$$\mathscr{g}_{ab} T^a_A T^b_A = 0, \tag{5}$$

using also the normalization conditions imposed on $\mathscr{g}_{ab}$. This remark is useful for the following reason. If $\mathbf{T}$ is a null vector at e, we can draw a light ray L through e with tangent vector $\mathbf{T}$. By choosing a particle P through e and another one, Q, through some event $p \in L$ $(p \neq e)$,† one can construct a message from P to Q of which L is an element. According to Axiom D_4, this construction can be used to obtain a set $\{\mathbf{T}_p\}$ of null vectors at events $p \in P$, p near e, such that the function $p \rightarrow \mathbf{T}_p$ is smooth and $\mathbf{T}_e = \mathbf{T}$, the vector originally given at e. Doing this for each vector $\mathbf{T}_A$ of the set used in eqn (5), one obtains smooth null vector fields on P near e that uniquely determine the $\mathscr{g}_{ab}$ at all events p on P near e, from eqn (5). That implies that the $\mathscr{g}_{ab}$ are smooth functions of p on P. Since this holds for any particle through e and, as we shall see in the next section, the tangent vectors of particles generate M_e, the $\mathscr{g}_{ab}$ are smooth functions on M.

We have shown then that light propagation determines a conformal structure $\mathscr{C}$ such that the null vectors are precisely the tangent vectors of light rays, whence light rays are $\mathscr{C}$-null curves. Next, we wish to show that light rays are $\mathscr{C}$-null geodesics. To prove that, let us again consider the function g introduced in Axiom L_1. We have shown already that, at e, $g = 0$ and $g_{,a} = 0$. Therefore, if q is an event in U connected to e by a light ray L with tangent vector $\mathbf{T}$, we have

$$g_{,a}(q) = \int_e^q g_{,ab}\{x^c(u)\}T^b(u)\mathrm{d}u,$$

† The possibility of doing so is implied by Axioms P_1 and P_2. No circularity will result from using it already here.

where $L = \{x^a(u)\}$ and $T^a = \mathrm{d}x^a/\mathrm{d}u$. Since T is a non-vanishing $\mathscr{C}$-null vector, we know that, at e, $g_{ab}T^b \neq 0$. Since, moreover, $g_{,ab}\{x^c(u)\}T^b(u)$ can be considered as a continuous function of u and the initial direction of L at e, and since these directions form a compact set, it follows from the above formula that there exists a neighbourhood W of e such that if $e \neq q \in W$ and $g(q) = 0$, then $g_{,a}(q) \neq 0$. This result implies (by the implicit-function theorem) that if we denote by ν_e the set of events contained in light rays passing through e then this *light cone* is a smooth hypersurface near e, except at e itself.

Consider now a neighbourhood of e in which the last statement about ν_e holds, which is time-oriented, and which is a V-type neighbourhood as

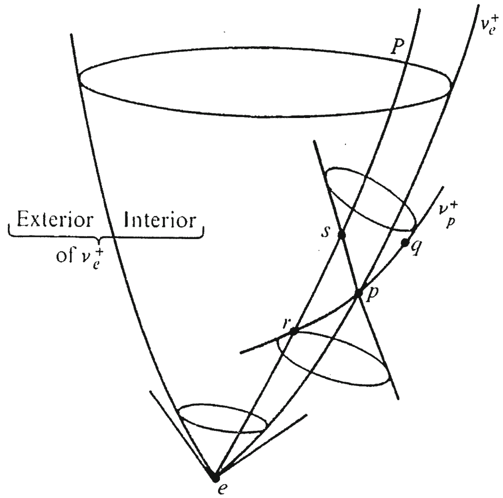

FIG. 7. Stacking of light cones.

postulated in Axiom L_1; such a neighbourhood exists. Let ν_e^+ denote the future light cone of e, i.e. that part of ν_e which is, at e, in contact with the future $\mathscr{C}$-null cone, and let p be an event on ν_e^+, $p \neq e$, contained in the specified neighbourhood. Since ν_e^+ is generated by light rays, its tangent space at p cannot be space-like. Suppose it were time-like. Then we should have the situation illustrated in Fig. 7. The light cone ν_p^+ would intersect, rather than touch, ν_e^+ at p. Hence, one could find a light ray (r, p, q) such that r is interior to ν_e^+.† Choosing r sufficiently close to p and anticipating Axiom P_2 (p. 77) that the interior of ν_e^+ is covered locally by particles passing through e,

† In a normal hyperbolic Riemannian manifold, the distinction between the interior and exterior of the null cone N_p of a point p can locally be defined by means of the sign of the world function $\Omega(p, q)$, for q near p. This extends immediately to a conformal space. Since ν_p is contained in the closure of the interior of N_p—for any smooth null curve through e is contained in that closure—the exterior of ν_e in a neighbourhood U of e can be defined as that part of $U - \nu_e$ which is connected with the exterior of N_p.

one could conclude that there would be a particle P through e and r which could be connected to p by *three* light rays, two on ν_p^- and one on ν_p^+. (The existence of the last one follows if Axiom L_1 is applied to a neighbourhood of p not containing e.) This result contradicts the assumption that we are working in a V-neighbourhood in the sense of Axiom L_1. Hence ν_e^+ cannot be time-like near e; it must be a $\mathscr{C}$-null hypersurface there. That, however, implies that light rays are $\mathscr{C}$-null geodesics, as we claimed, and that in turn implies that ν_e is identical with the $\mathscr{C}$-null cone at e.

The results obtained so far could also be expressed by saying that space–time has, in consequence of the Axioms D_1 to D_4, L_1, L_2, a *causal structure*, with an underlying C^3 manifold M and a C^2 conformal 'metric' $\mathscr{g}_{ab}$. One may wonder whether this differential and causal structure is uniquely determined by these axioms, given the underlying sets M, $\mathscr{L}$, $\mathscr{P}$ of events, light rays, and particles respectively. That is indeed the case, as R. P. Geroch pointed out to us. It follows immediately if one applies locally to the identity map $M \rightarrow M$ between two structures $(M, \mathscr{D}, \mathscr{C})$, $(M, \bar{\mathscr{D}}, \bar{\mathscr{C}})$, based on the same $(M, \mathscr{L}, \mathscr{P})$ (with $\mathscr{D}$, $\bar{\mathscr{D}}$ standing for differential structures), the following theorem due to Hawking.[16] *If M, $\bar{M}$ are four-dimensional C^3 manifolds with C^2 conformal structures $\mathscr{C}$, $\bar{\mathscr{C}}$ such that the strong-causality assumption holds on M, and if $\phi: M \rightarrow \bar{M}$ is a bijection such that ϕ and ϕ^{-1} preserve causal relationships, then ϕ is a C^3 diffeomorphism.* (The terms used are defined in Hawking's paper. Strong causality always holds locally, i.e. in sufficiently small, open neighbourhoods, considered as manifolds in their own right.)

Free fall and projective structure

A simple and empirically well-motivated assumption is the axiom that follows.

Axiom P_1. *Given an event e and a $\mathscr{C}$-time-like direction D at e, then there exists one and only one particle P passing through e with direction D.*

Unfortunately, this requirement alone is far too weak to characterize the 'inertial field' of general relativity. Consider, for example, a fixed electromagnetic field $\mathbf{F}$ in Minkowski space–time, and take particles with a fixed specific charge moving in that field according to Lorentz's equation of motion. The corresponding family of time-like world lines satisfies the law stated above, but no affine connection exists for which these curves are the geodesics. A great many similar examples can be constructed.

One would like to express the fact that there is one class of test particles (neutral, spherically symmetrical ones) whose law of motion does not contain any 'non-geometrical' fields (like $\mathbf{F}$ in the example mentioned above) that would define preferred directions or field strengths at one event; any quantities occurring in the expression of the sought-for equation of motion relative to some coordinate system should be such that their components can be transformed away at an event on going over to some new coordinate system. We formalize these somewhat vague, negative statements, which are based on the so-called equality of inertial and (passive) gravitational mass, in the axiom that follows.

AXIOM P_2. *For each event $e \in M$, there exists a coordinate system $(\bar{x}^a)$, defined in a neighbourhood of e and permitted by the differential structure introduced in Axiom D_3, such that any particle P through e has a parameter representation $\bar{x}^a(\bar{u})$ with*

$$\left.\frac{d^2\bar{x}^a}{d\bar{u}^2}\right|_e = 0; \tag{6}$$

such a coordinate system is said to be projective at e.

This axiom might be called an infinitesimal version of the law of inertia, that takes into account the indistinguishability of gravitational and inertial forces. It is similar to Weyl's characterization of symmetric, affine connections on a manifold.[17]

If eqn (6) is transformed to an arbitrary coordinate system (x^a) and arbitrary parameter u of the particle and a dot is used to denote differentiation with respect to u, there results, at e,

$$\ddot{x}^a + \Pi^a_{bc}\dot{x}^b\dot{x}^c = \lambda\dot{x}^a, \tag{7}$$

where λ depends on the choice of u. We can and shall require

$$\Pi^a_{[bc]} = 0, \qquad \Pi^a_{ba} = 0. \tag{8}$$

For a given event e and coordinate system (x^a) around e, the coefficients Π^a_{bc} are then *uniquely* determined, for the difference Δ between two such systems Π, $\bar{\Pi}$ would, in view of eqn (7) and Axiom P_1, satisfy $T^{[a}\Delta^{b]}{}_{cd}T^cT^d = 0$ for all time-like vectors $\mathbf{T}$ at e, and a simple algebraic argument shows that this together with the relation $\Delta^a_{[bc]} = \Delta^a_{ba} = 0$ implies that $\Delta^a_{bc} = 0$.

The *projective coefficients* Π^a_{bc} associated with (x^a) can be determined by solving the linear equations

$$\dot{x}^{[a}(\ddot{x}^{b]} + \Pi^{b]}_{cd}\dot{x}^c\dot{x}^d) = 0, \tag{9}$$

for a sufficient number of particles $\{x^a(u)\}$. We believe that the smoothness assumptions made so far about M and the particles imply smoothness of the Π^a_{bc}; at any rate, we shall assume them to be at least of class C^1.

Any curve that satisfies eqn (7) is said to be a *geodesic*, and the structure thus imposed on a manifold is called a *projective structure*.

We may say, then, that the collection of free-fall trajectories assigns a unique projective structure $\mathscr{P}$ to space–time such that every particle is a geodesic, and every geodesic which is time-like at some event is a particle. (The last assertion follows from the preceding one and Axiom P_1.)

Compatibility of free fall and light propagation; Weyl structure

So far we have not yet fully characterized the set of *all* particles in terms of the structures $\mathscr{C}$ and $\mathscr{P}$. To achieve that, we formalize the idea described in § 1, remark (c) on pp. 65, 67 in

AXIOM C. *Each event e has a neighbourhood U such that an event $p \in U$, $p \neq e$ lies on a particle P through e if and only if p is contained in the interior of the light cone v_e of e.*

It has been shown on pp. 75–76 that v_e may be identified with the $\mathscr{C}$-null cone; this will be done henceforth. The results of the preceding section and Axiom C imply that every particle is a $\mathscr{P}$-geodesic which is nowhere $\mathscr{C}$-space-like, and *every $\mathscr{P}$-geodesic that is time-like at some event is nowhere space-like.* From this lemma, we shall deduce a relation between the quantities g_{ab} and Π^a_{bc} that characterize the conformal and projective structures of M. Let us introduce the auxiliary quantities g^{ab}, defined by

$$g^{ab} g_{bc} = \delta^a_c; \tag{10}$$

the components

$$K^a_{bc} := \tfrac{1}{2} g^{ad}(g_{db,c} + g_{dc,b} - g_{bc,d}) \tag{11}$$

of the conformal connection, and the differences

$$\Delta^a_{bc} := \Pi^a_{bc} - K^a_{bc} \tag{12}$$

between the components of the conformal and projective connections. Also, let us use the g_{ab}, g^{ab} to shift indices as, for example, in

$$\Delta_{abc} := g_{ad}\Delta^d_{bc}. \tag{13}$$

It is easily seen that

$$\Delta^a_{[bc]} = \Delta^a_{ba} = 0, \tag{14}$$

since the Ks and Πs have the corresponding properties. The algebraic properties (14) imply that Δ_{abc} can be represented in the form

$$\Delta_{abc} = \Delta_{(abc)} + \tfrac{1}{2}(p_a g_{bc} - g_{a(b}p_{c)}) + L_{a(bc)}, \tag{15}$$

where

$$L_{(ab)c} = L_{[abc]} = L^a_{ba} = 0. \tag{16}$$

(To prove this, define $p_a = -\frac{8}{9}\Delta_{[ac]}{}^c$ and $L_{abc} = \frac{4}{3}\Delta_{[ab]c} - p_{[a}g_{b]c}$, and work out $L_{a(bc)}$.)

Now, let $x^a(u)$ describe a $\mathscr{P}$-geodesic such that $\dot{x}^a(0)$ is a $\mathscr{C}$-null vector. We find from eqns (7), (11), (12), (13) at $u = 0$

$$\frac{\mathrm{d}}{\mathrm{d}u}(g_{ab}\dot{x}^a\dot{x}^b) = -\Delta_{abc}\dot{x}^a\dot{x}^b\dot{x}^c. \tag{17}$$

If this expression were different from zero, then $\mathscr{g}_{ab}\dot{x}^a\dot{x}^b$ would change sign along $x^a(u)$ at $u = 0$, i.e., the geodesic $x^a(u)$ would be time-like somewhere and space-like somewhere else. This is incompatible with the lemma stated above eqn (10), whence we conclude that the expression on the right-hand side of eqn (17) vanishes for all $\mathscr{C}$-null vectors. It follows that $\Delta_{abc}\xi^a\xi^b\xi^c = (\mathscr{g}_{ab}\xi^a\xi^b)(s_c\xi^c)$ identically in ξ^a, i.e.,

$$\Delta_{(abc)} = \mathscr{g}_{(ab}s_{c)}. \tag{18}$$

This relation permits us to simplify the representation (15) of Δ; taking into account eqns (14) and (16), we get

$$\Delta^a_{bc} = L^a_{(bc)} + 5q^a\mathscr{g}_{bc} - 2\delta^a{}_{(b}q_{c)} \tag{19}$$

with some functions q_c.

Equation (18) implies that on any $\mathscr{P}$-geodesic $x^a(u)$ a relation

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathscr{g}_{ab}\dot{x}^a\dot{x}^b) = (\mathscr{g}_{ab}\dot{x}^a\dot{x}^b)(2\lambda - s_c\dot{x}^c) \tag{20}$$

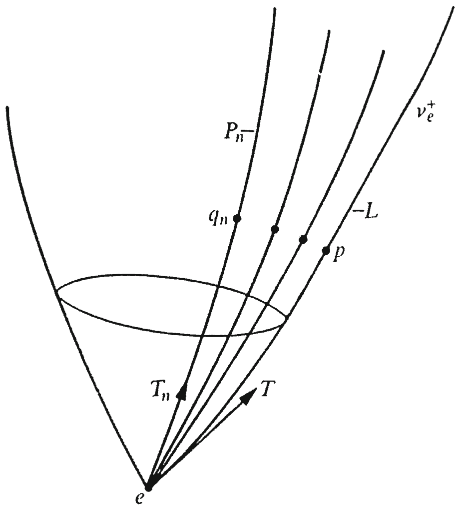

FIG. 8. Particles P_n converging to light ray L.

holds. Therefore, *a $\mathscr{P}$-geodesic that is time-like, space-like, or null respectively, with respect to $\mathscr{C}$ at one of its events, has this same orientation everywhere.* This is the first important link between $\mathscr{C}$ and $\mathscr{P}$ implied by our axioms, in particular, the compatibility Axiom C.

In the next step, we wish to exploit the postulate, implied by Axiom C, that each event p on ν_e sufficiently close to e can be approximated arbitrarily closely by events q situated on particles through e. Let $p \in \nu_e$, $p \neq e$, and let q_n be a sequence of events contained in particles P_n through e such that

$q_n \to p$ (see Fig. 8). The P_n are geodesics and thus satisfy eqn (7). By choosing their parameters suitably, we can arrange that $\lambda = 0$ in eqn (7), that the parameter value 0 corresponds to e, and that 1 corresponds to q_n. Standard theorems on differential equations then tell us that the sequence $\{\mathbf{T}_n\}$ of the tangent vectors of the P_n at $e = 0$ converges to the tangent vector $\mathbf{T}$ of that geodesic P which passes, for $u = 0$ and $u = 1$, respectively, through e and p, provided we work in a $\mathscr{P}$-convex coordinate neighbourhood of e, as we shall do. Since $\mathbf{T}_n \to \mathbf{T}$ and the $\mathbf{T}_n$, belonging to particles, are not space-like, it follows that $\mathbf{T}$ is time-like or null.

If $\mathbf{T}$ and, therefore, P were time-like, then P would intersect ν_e at p. That, however, is impossible in a time-oriented neighbourhood of e, since a future-directed time-like line through e cannot intersect ν_e^+, as is geometrically

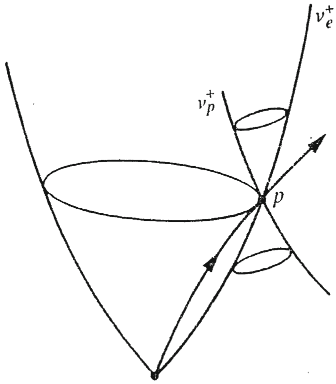

FIG. 9. A time-like line cannot leave 'its' light cone locally.

obvious (see Fig. 9). Since our considerations are local, we can exclude this possibility.

Let, then, $\mathbf{T}$ and hence P be null. Since, as we have pointed out in the footnote on p. 75, P cannot have any events exterior to ν_e, the events on P between e and p must be situated either on ν_e or in the interior of ν_e. Events of the latter kind, however, cannot occur in a time-oriented, sufficiently small neighbourhood, because if q were such an event the part of P after q could not escape from ν_q^+ and hence could not reach the event p. Thus P is contained, at least between e and p, in ν_e, and hence is a $\mathscr{C}$-null geodesic. Since this is true for each $p \in \nu_e$ near e and since being geodesic is a local property of a curve, we obtain our second main consequence of Axiom C, viz., *the projective null geodesics are identical with the conformal null geodesics.*

According to this result, the two equations

$$\mathscr{g}_{ab}\dot{x}^a\dot{x}^b = 0, \tag{21}$$

and

$$\ddot{x}^a + K^a_{bc}\dot{x}^b\dot{x}^c = \nu\dot{x}^a \tag{22}$$

(which characterize $\mathscr{C}$-null geodesics) imply eqn (7). Consequently, subtraction of eqn (22) from eqn (7) implies

$$\Delta^a_{bc}\dot{x}^b\dot{x}^c = (\lambda-\nu)\dot{x}^a. \tag{23}$$

From eqn (19), it follows that a relation of the form

$$L^a_{bc}T^bT^c = \mu T^a \tag{24}$$

holds for any $\mathscr{C}$-null vector T. If one takes into account the properties (16) of L^a_{bc}, one can derive from this property the fact that

$$L^a_{bc} = 0,$$

so that eqn (19) simplifies to

$$\Delta^a_{bc} = 5q^a\mathscr{g}_{bc}-2\delta^a_{(b}q_{c)}. \tag{25}$$

(For the case considered here—dim $M = 4$ and signature $(+++-)$—the proof is particularly simple and elegant if eqns (16) and (24) are translated into spinor language. The statement concerning L holds for any dimension and normal-hyperbolic signature.)

A second consequence of the last statement in italics, if combined with Axiom C, is that *the set of all particles is identical with the set of all $\mathscr{C}$-timelike geodesics of $\mathscr{P}$*. This is the characterization announced at the beginning of this subsection.

We have now established that $\mathscr{C}$ and $\mathscr{P}$ define a *Weyl structure* on M and that eqn (25) is the formal expression of the compatibility of $\mathscr{C}$ and $\mathscr{P}$. (The verification of the sufficiency of (25) is trivial.) It is a straightforward matter to show on this basis that the functions

$$\Gamma^a_{bc} = K^a_{bc}+5q^a\mathscr{g}_{bc}-10\delta^a{}_{(b}q_{c)} \tag{26}$$

(with q_a from eqn (25)) define a *symmetric linear connection* with the properties stated in § 1, remark (c) on pp. 65, 67, and to establish the converse theorem mentioned there.

As to the Kundt–Hoffmann experiment, we wish to remark here only that the measurements conceived by these authors make use of only those properties of particles and light rays that are embodied in the Weyl geometry as outlined here. Moreover, the calculations that contain the theory of the experiment can be so rearranged that they contain only the quantities $(\mathscr{g}_{ab}, \Pi^a_{bc})$ of a Weyl space, together with Weyl proper time (defined in remark (d) on p. 67) along the observer's world line; it is the latter quantity that is measured.

Curvature and Riemannian space–time

Since parallel transport in a Weyl space preserves nullity of vectors, the linear transformation induced in a tangent space M_p by parallel displacement around a loop at p can be decomposed uniquely into a Lorentz transformation and a scalar multiplication. Therefore, the curvature tensor R^a_{bcd} associated with a Weyl connection Γ decomposes uniquely according to

$$R^a_{bcd} = \hat{R}^a_{bcd} + \tfrac{1}{2}\delta^a_b F_{cd}, \tag{27}$$

where

$$\mathscr{g}_{e(a}\hat{R}^e_{b)cd} = 0, \qquad F_{(ab)} = 0. \tag{28}$$

The bivector **F** is Weyl's *Streckenkrümmung.* Its vanishing is necessary and sufficient in order that the identity component of the holonomy group of Γ be a subgroup of the (restricted) Lorentz group. Therefore the vanishing of **F** is also necessary and sufficient for the existence (in any simply connected domain of M) of a Riemannian metric g_{ab}, compatible with the conformal structure $\mathscr{C}$, such that Γ is metric with respect to g_{ab}. This establishes the second criterion stated in remark (e), p. 69. To obtain the first criterion, we observe that in the equation

$$\ddot{V}^a = \hat{R}^a_{bcd}U^bU^cV^d + \tfrac{1}{2}U^aF_{bc}U^bV^c \tag{29}$$

of geodesic deviation the first vector on the right-hand side is orthogonal to and the second one parallel to the tangent vector **U** of the first of two adjacent, affinely parametrized geodesics P, P'. If equidistant events on P correspond, in the sense described in remark (e) on p. 69, to equidistant events on P', the connection vector **V** in eqn (29) can be chosen orthogonal to P; hence, the second term must vanish, and if this is true for arbitrary **U** and **V**, **F** must vanish. If, conversely, $\mathbf{F} = 0$, there exists a Riemannian metric compatible with $\mathscr{C}$ and $\mathscr{A}$, and in that case the desired property holds true.

If one of these criteria is used as an additional *Riemannian axiom*, one obtains a Riemannian metric g_{ab}, unique up to a constant, positive factor, which is compatible with the more primitive structures $\mathscr{C}$, $\mathscr{P}$, and $\mathscr{A}$, and thus one arrives finally at the usual, full space–time structure of general-relativity theory. This last step seems unavoidable on empirical grounds if equality of gravitational time (as given by the Weyl arc length and measurable, for example, by the method of Kundt and Hoffmann) and atomic time is assumed. This is because the latter is transported in an integrable fashion, which was pointed out by Einstein in his criticism of Weyl's theory and supported by (among other things) the consistency of the interpretations of observed red-shifts. (But how compelling is the time-equality postulate?)

In a Weyl space–time, the bivector **F** is closed, $d\mathbf{F} = 0$, and the vector density (of weight one)

$$\mathscr{J}^a := (\mathscr{g}^{ac}\mathscr{g}^{bd}F_{cd})_{,b}$$

is conserved, $\mathscr{J}^a_{\ ,a} = 0$. Weyl used these facts to interpret **F** as the electromagnetic field, and $\mathscr{J}$ as the electric-charge current density. Nowadays, this identification is no longer compelling; one may, however, ask whether other interpretations of **F**, and $\mathscr{J}$, relating the gravitational field to another universally conserved current, might contain some physical truth.

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