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RESEARCH ARTICLE

# Logarithmic corrections to $\mathcal{N} = 2$ black hole entropy: an infrared window into the microstates

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**Abstract** Logarithmic corrections to the extremal black hole entropy can be computed purely in terms of the low energy data—the spectrum of massless fields and their interaction. The demand of reproducing these corrections provides a strong constraint on any microscopic theory of quantum gravity that attempts to explain the black hole entropy. Using quantum entropy function formalism we compute logarithmic corrections to the entropy of half BPS black holes in $\mathcal{N} = 2$ supersymmetric string theories. Our results allow us to test various proposals for the measure in the OSV formula, and we find agreement with the measure proposed by Denef and Moore if we assume their result to be valid at weak topological string coupling. Our analysis also gives the logarithmic corrections to the entropy of extremal Reissner–Nordstrom black holes in ordinary Einstein–Maxwell theory.

**Keywords** String theory · Black holes · $\mathcal{N} = 2$ supersymmetry

## Contents



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## 1 Introduction and summary

Recent years have seen considerable progress towards an understanding of the black hole entropy beyond the original formula due to Bekenstein and Hawking relating the entropy to the area of the event horizon. In particular Wald's formula gives a prescription for computing the black hole entropy in a classical theory of gravity with higher derivative terms, possibly coupled to other matter fields [1–4]. In the extremal limit this leads to a simple algebraic procedure for determining the near horizon field configurations and the entropy [5,6]. A proposal for computing quantum corrections to this formula was suggested in [7,8] by exploiting the presence of $AdS_2$ factors in the near horizon geometry of extremal black holes. In this formulation, called the quantum entropy function formalism, the degeneracy associated with the black hole horizon is given by the string theory partition function $Z_{AdS_2}$ in the near horizon geometry of the black hole. Such a partition function is divergent due to the infinite volume of $AdS_2$, but the rules of $AdS_2/CFT_1$ correspondence gives a precise procedure for removing this divergence. While in the classical limit this prescription gives us back the exponential of the Wald entropy, it can in principle be used to systematically calculate the quantum corrections to the entropy of an extremal black hole.

In this paper our main focus will be on logarithmic corrections to the black hole entropy. These arise from one loop quantum corrections to $Z_{AdS_2}$ involving massless fields and are insensitive to the details of the ultraviolet properties of the theory. On the other hand, being corrections to the black hole entropy, they give us non-trivial information about the microstates of the black hole. For this reason they can be regarded as an infrared window into the microphysics of black holes. In two previous papers [9,10] we used the quantum entropy function to compute logarithmic corrections to the entropy of 1/8 BPS and 1/4 BPS black holes in $\mathcal{N}=8$ and $\mathcal{N}=4$ supersymmetric string theories respectively and found results in perfect agreement with the microscopic results of [11–26]. In this paper we use this formalism to compute logarithmic correction to the entropy of half BPS black holes in $\mathcal{N}=2$ supersymmetric string theories. As in [9,10] we consider the limit in which all components of the charge become large at the same rate. In this limit we find that for a theory with $n_V$ massless vector multiplets and $n_H$ massless hypermultiplets, the entropy including logarithmic correction is given by

$$\frac{A_H}{4G_N}+\frac{1}{12}(23+n_H-n_V)\ln\frac{A_H}{G_N}+\mathcal{O}(1), \tag{1.1}$$

where $A_H$ is the area of the event horizon and $G_N$ is the Newton's constant. The $\mathcal{O}(1)$ terms include functions of ratios of charges, and also contains terms carrying inverse powers of charges.[1] Note that while the result depends on the number of vector

[1] Thus if we take another limit in which some ratios of charges become large then we may get additional logarithmic corrections.

and hypermultiplet fields, it does not depend on the details of the interaction involving these fields through the prepotential and the metric on the hypermultiplet moduli space. Eq. (1.1) is consistent with the version of the OSV formula [27] given in [28] if we take their result to be valid at weak topological string coupling. However (1.1) is in apparent disagreement with the measure proposed in [29,30]. A detailed discussion on this can be found in Sect. 9. For STU model [31,32] we have $n_H = 4$ and $n_V = 3$, leading to a logarithmic correction of $2\ln(A_H/G_N)$ to the entropy. This agrees with the result of [10].

We also give for comparison the result of [9,10] for supersymmetric black hole entropies in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supersymmetric theories:

$$\begin{aligned} \mathcal{N} = 4 &: \frac{A_H}{4G_N} + \mathcal{O}(1) \\ \mathcal{N} = 8 &: \frac{A_H}{4G_N} - 4\ln\frac{A_H}{G_N} + \mathcal{O}(1). \end{aligned} \tag{1.2}$$

Note that the coefficient given in (1.1) is proportional to the gravitational $\beta$-function in $\mathcal{N} = 2$ supergravity/string theory given in [33–36]. However this relation does not hold universally. For example in the $\mathcal{N} = 8$ supersymmetric theory the gravitational $\beta$-function vanishes [33] but the logarithmic correction to the entropy given in (1.2) does not vanish. The precise relation will be discussed in Sect. 7.

Our analysis also gives the result for the logarithmic correction to the entropy of an extremal Reissner–Nordstrom black hole in ordinary non-supersymmetric Einstein–Maxwell theory. The result is $-\frac{241}{45}\ln\frac{A_H}{G_N}$. If the theory in addition contains $n_S$ massless scalars, $n_F$ massless Dirac fermions and $n_V$ additional Maxwell fields, all minimally coupled to background gravitational field but not to the background electromagnetic flux, then the net entropy is given by

$$\frac{A_H}{4G_N} - \frac{1}{180}(964 + n_S + 62n_V + 11n_F)\ \ln\frac{A_H}{G_N} + \mathcal{O}(1). \tag{1.3}$$

We emphasize that in this formula $n_V$ is the number of *additional* minimally coupled Maxwell field. Thus if we just had an extremal Reissner–Nordstrom black hole in Einstein gravity coupled to a single Maxwell field then $n_V = 0$ in our convention.

Various other earlier approaches to computing logarithmic corrections to black hole entropy can be found in [37–53]. Of these the method advocated in [41], and subsequently developed further in [53] and reviewed in [54], is closest to the one we are following; so we have given a detailed comparison between the two methods below Eq. (3.31). For now we would like to mention that the method of [41,53,54] would correctly reproduce the dependence on $n_S$, $n_V$ and $n_F$ for extremal Reissner–Nordstrom black holes in (1.3) but will fail to give the constant term 964 correctly. This is due to the fact that the constant term comes from fluctuations of the metric and the gauge field under which the black hole is charged, and for these fields the analysis of [41,53,54] would be insufficient on two counts: first it does not take into account correctly the mixing between these fields due to the presence of the gauge field flux in the near horizon geometry of the black hole, and second it fails to take into account correctly the effect of integration over the zero modes of these fields.

The naive application of the analysis of [41,53,54] would also fail to get the result (1.1) or (1.2) for the supersymmetric black holes in $\mathcal{N} = 2, 4, 8$ supergravity for which both the mixing between the fields and the integration over the zero modes play a crucial role. As we discuss in Sect. 7, the effect of mixing between the fields can be incorporated by augmenting the analysis of [41,53,54] by supersymmetry—a fact already anticipated in [33]. However the effect of zero mode integration still needs to be taken into account separately.

References [55,56] attempted an exact evaluation of $Z_{AdS_2}$ using localization methods. The general formula for the logarithmic correction to the half BPS black hole entropy in these theories, described in (1.1), shows that $Z_{AdS_2}$ receives non-trivial contribution from not only the vector multiplets but also the gravity multiplet and the hypermultiplets. This makes the evaluation of this partition function a much more challenging problem, but also a more interesting one.

Before concluding the introduction we would like to discuss the region of validity of our formulæ. There are two independent questions: for which range of charges is our analysis valid and in which region of the moduli space is our analysis valid? As we have already mentioned, our analysis will be valid in the limit when all components of the charge are scaled uniformly, so that the four dimensional near horizon geometry becomes weakly curved and the internal space remains at a fixed shape and size as we scale the charges. The precise limit may be taken as follows. First we take a black hole solution in the $\mathcal{N} = 2$ supergravity with finite area event horizon and regular attractor values of the vector multiplet moduli, but do not require the charges to be quantized. We then scale all the charges carried by this black hole by some large number $\Lambda$ and at the end shift the charges by finite amounts to nearby integers in such a way that the final charge vector is primitive. In this limit the area of the horizon and hence the entropy scale as $\Lambda^2$ and the vector multiplet moduli remain fixed at regular values. To determine the chamber in the moduli space where our results are valid, note that our result is based on the analysis of the near horizon geometry of a single black hole. Thus we need to work in the attractor domain where the enigmatic configurations discussed in [28] are absent. Even in this case the total index receives contribution from multicentered scaling solutions besides the single centered black hole. In order that our result for single centered black hole entropy gives the dominant contribution we need to ensure that the contribution to the index from the scaling solutions are either absent or subleading. We discuss this point in detail in Sect. 8.

A related issue arises for extremal Reissner–Nordstrom black holes whose entropy is given by (1.3). Due to the existence of multicentered black holes with each center carrying a fraction of the total charge, the index receives contribution not only from single centered black holes but also from multi-centered configurations. This can be avoided by considering a dyonic configuration carrying a primitive charge vector instead of a purely electrically charged configuration. Since the Einstein-Maxwell theory is duality invariant, our result (1.3) will continue to be valid in such a situation, but the multicentered configurations are avoided since the total charge vector, being primitive, can no longer be split into multiple charge vectors which are proportional to each other. (A complete proof of this is still lacking however; see the discussion in Sect. 8.)

A final point about notation: while in the macroscopic description we compute the entropy of the black hole, on the microscopic side we always compute an appropriate index. It was argued in [57,58] that the entropy of the single centered black hole also represents the logarithm of the index carried by the same black hole. For this reason we shall use the word entropy and logarithm of the index interchangeably throughout our discussion.

The rest of the paper is organized as follows. Sections 2 and 3 contains mostly review of known material. In Sect. 2 we describe the general strategy for computing logarithmic corrections to the entropy of an extremal black hole. In Sect. 3 we illustrate this by calculating the logarithmic corrections to the entropy due to masless scalar, fermion and vector fields, *assuming that they only couple minimally to the background metric and is not affected by the any other background field if present.* In particular for the extremal Reissner–Nordstrom black hole this analysis does not apply to the gauge field which has non-zero background field strength since due to the Maxwell term in the action such gauge fields will be affected by the background flux. In Sect. 4 we apply the method reviewed in Sect. 2 to compute the logarithmic correction to the entropy of an extremal Reissner–Nordstrom black hole. This is important for our analysis since the bosonic sector of pure $\mathcal{N} = 2$ supergravity is described by ordinary Einstein-Maxwell theory and consequently the results of this section describe the bosonic contribution to the logarithmic correction to BPS black hole entropy in pure $\mathcal{N} = 2$ supergravity. In Sect. 5 we augment this result by computing the logarithmic correction to BPS black hole entropy due to the fermionic fields of $\mathcal{N} = 2$ supergravity. Adding the results of Sects. 4 and 5 we arrive at the result given in (1.1) for $n_H = n_V = 0$. In Sect. 6 we apply the same method to compute the logarithmic correction to the entropy of a BPS black hole in a general supergravity theory with arbitrary number of vector and hypermultiplets. This leads to (1.1). In Sect. 7 we discuss an alternative but equivalent method for deriving the same results, making use of the supersymmetry of the theory. This method is simpler, but requires certain assumption about possible supersymmetric one loop counterterms in $\mathcal{N} = 2$ supergravity theory. One could in principle elevate this into a rigorous analysis—at the same level as that in Sects. 4–6—by classifying all possible four derivative supersymmetric terms in the action that could be generated as one loop correction in $\mathcal{N} = 2$ supergravity. In Sect. 8 we explore if multi-centered scaling solutions could invalidate our result by generating new configurations whose entropy is of the same order or larger than the single center black hole entropy we analyze. Although we do not have any rigorous result we argue that it is extremely unlikely. In Sect. 9 we carry out a detailed comparison of our results with different versions of the OSV formula for black hole entropy which have been proposed in the literature. While our result agrees with that of [28] assuming its validity in the scaling limit we are studying, it disagrees with the proposal of [29]. We argue however this disagreement can be rectified by certain changes in the proposed formulæ of [29] without violating any basic principle used in arriving at these results. In Appendix A we collect the results for eigenfunctions and eigenvalues of the laplacian on $AdS_2 \times S^2$ for various fields. In Appendix B we collect some useful mathematical identities used in our analysis. Finally in Appendix C we demonstrate that in a general $\mathcal{N} = 2$ supergravity theory coupled to a set of vector and hypermultiplet fields, the action that describes the fluctuations of various fields around the BPS black hole background to quadratic order has a universal form that depends only on the number of vector and hypermultiplet fields but not on the details of their interaction e.g. the prepotential for the vector multiplet and the moduli space metric for the hypermultiplet. This has been used in the analysis of Sect. 6 and is responsible for the universal form of (1.1) that does not depend on the details of the interaction.

## 2 General strategy

In this section we shall review the general strategy for computing logarithmic corrections to the entropy of extremal black holes. We shall focus on spherically symmetric extremal black holes in four dimensions, but the method we describe is easily generalizable to non-spherical (rotating) black holes.

Suppose we have an extremal black hole with near horizon geometry $AdS_2 \times S^2$, with equal radius of curvature $a$ of $AdS_2$ and $S^2$. Then the Euclidean near horizon metric takes the form

$$ds^2 = a^2(d\eta^2 + \sinh^2\eta\, d\theta^2) + a^2(d\psi^2 + \sin^2\psi d\phi^2). \tag{2.1}$$

We shall denote by $x^\mu$ all four coordinates on $AdS_2 \times S^2$, by $x^m$ the coordinates $(\eta, \theta)$ on $AdS_2$ and by $x^\alpha$ the coordinates $(\psi, \phi)$ on $S^2$ and introduce the invariant antisymmetric tensors $\varepsilon_{\alpha\beta}$ on $S^2$ and $\varepsilon_{mn}$ on $AdS_2$ respectively, computed with the background metric (2.1):

$$\varepsilon_{\psi\phi} = a^2 \sin\psi, \quad \varepsilon_{\eta\theta} = a^2 \sinh\eta. \tag{2.2}$$

All indices will be raised and lowered with the background metric $g_{\mu\nu}$ defined in (2.1).

Let $Z_{AdS_2}$ denote the partition function of string theory in the near horizon geometry, evaluated by carrying out functional integral over all the string fields weighted by the exponential of the Euclidean action $\mathcal{S}$, with boundary conditions such that asymptotically the field configuration approaches the near horizon geometry of the black hole.[2] Since in $AdS_2$ the asymptotic boundary conditions fix the electric fields, or equivalently the charges carried by the black hole [7], and allow the constant modes of the gauge fields to fluctuate, we need to include in the path integral a boundary term $\exp(-i\oint \sum_k q_k A^{(k)}_\mu dx^\mu)$ where $A^{(k)}_\mu$ are the gauge fields and $q_k$ are the corresponding electric charges carried by the black hole [7]. Thus we have

$$Z_{AdS_2} = \int d\Psi \exp\left(\mathcal{S} - i\oint \sum_k q_k A^{(k)}_\mu dx^\mu\right), \tag{2.3}$$

[2] Our definition of the Euclidean action includes a minus sign so that the path integral is weighted by $e^{\mathcal{S}}$ instead of $e^{-\mathcal{S}}$.

where $\Psi$ stands for all the string fields. $AdS_2/CFT_1$ correspondence tells us that the full quantum corrected entropy $S_{BH}$ is related to $Z_{AdS_2}$ via [7]:

$$e^{S_{BH}-E_0 L} = Z_{AdS_2}, \tag{2.4}$$

where $E_0$ is the energy of the ground state of the black hole carrying a given set of charges, and $L$ denotes the length of the boundary of $AdS_2$ in a regularization scheme that renders the volume of $AdS_2$ finite by putting an infrared cut-off $\eta \leq \eta_0$. Equation (2.4) is valid in the limit of large $L$ and allows us to compute $S_{BH}$ from the knowledge of $Z_{AdS_2}$.

Let $\Delta\mathcal{L}_{eff}$ denote the one loop correction to the four dimensional effective lagrangian density evaluated in the background geometry (2.1). Then the one loop correction to $Z_{AdS_2}$ is given by

$$\exp\left[\int_0^{\eta_0} d\eta \int_0^{2\pi} d\theta \int_0^{\pi} d\psi \int_0^{2\pi} d\phi \sqrt{\det g}\, \Delta\mathcal{L}_{eff}\right] = \exp\left[8\pi^2 a^4 (\cosh\eta_0 - 1)\, \Delta\mathcal{L}_{eff}\right]. \tag{2.5}$$

Here we have used the fact that due to the $SO(2,1) \times SO(3)$ isometry of $AdS_2 \times S^2$, $\Delta\mathcal{L}_{eff}$ is independent of the coordinates of $AdS_2$ and $S^2$. Since the length of the boundary, situated at $\eta = \eta_0$, is given by $L = 2\pi a \sinh\eta_0$, the term proportional to $\cosh\eta_0$ in the exponent of (2.5) can be written as $-L\Delta E_0 + \mathcal{O}\left(L^{-1}\right)$ where $\Delta E_0 = -4\pi a^3 \Delta\mathcal{L}_{eff}$ has the interpretation of the shift in the ground state energy. The $L$-independent contribution in the exponent can be interpreted as the one loop correction to the black hole entropy [7]. Thus we have

$$\Delta S_{BH} = -8\pi^2 a^4\, \Delta\, \mathcal{L}_{eff}. \tag{2.6}$$

While the term in the exponent proportional to $L$ and hence $\Delta E_0$ can get further corrections from boundary terms in the action, the $L$-independent part $\Delta S_{BH}$ is defined unambiguously. This reduces the problem of computing one loop correction to the black hole entropy to that of computing one loop correction to $\mathcal{L}_{eff}$. We shall now describe the general procedure for calculating $\Delta\mathcal{L}_{eff}$.

Suppose we have a set of massless fields[3] $\{\phi^i\}$ where the index $i$ could run over several scalar fields, or the space-time indices of tensor fields. Let $\{f_n^{(i)}(x)\}$ denote an orthonormal basis of eigenfunctions of the kinetic operator expanded around the near horizon geometry, with eigenvalues $\{\kappa_n\}$:

$$\int d^4x \sqrt{\det g}\, G_{ij}\, f_n^{(i)}(x)\, f_m^{(j)}(x) = \delta_{mn}, \tag{2.7}$$

[3] Here by massless field we mean any field whose mass is of order $a^{-1}$ or less.

where $g_{\mu\nu}$ is the $AdS_2 \times S^2$ metric and $G_{ij}$ is a metric in the space of fields induced by the metric on $AdS_2 \times S^2$, e.g. for a vector field $A_\mu$, $G^{\mu\nu} = g^{\mu\nu}$. Then the heat kernel $K^{ij}(x, x')$ is defined as

$$K^{ij}(x, x'; s) = \sum_n e^{-\kappa_n s} f_n^{(i)}(x) f_n^{(j)}(x'). \tag{2.8}$$

In (2.7), (2.8) we have assumed that we are working in a basis in which the eigenfunctions are real; if this is not the case then we need to replace one of the $f_n^{(i)}$'s by $f_n^{(i)*}$. Among the $f_n^{(i)}$'s there may be a special set of modes for which $\kappa_n$ vanishes. We shall denote these zero modes by the special symbol $g_\ell^{(i)}(x)$, and define

$$\bar{K}^{ij}(x, x') = \sum_\ell g_\ell^{(i)}(x) g_\ell^{(j)}(x'). \tag{2.9}$$

Defining

$$K(0; s) = G_{ij} K^{ij}(x, x; s), \quad \bar{K}(0) = G_{ij} \bar{K}^{ij}(x, x; s), \tag{2.10}$$

and using orthonormality of the wave-functions, we get

$$\int d^4x \sqrt{\det g} \left(K(0; s) - \bar{K}(0)\right) = {\sum_n}' e^{-\kappa_n s}, \tag{2.11}$$

where $\sum_n'$ denotes sum over the non-zero modes only. Note that due to homogeneity of $AdS_2 \times S^2$ the right hand sides of (2.10) do not depend on $x$. The contribution of the non-zero modes of the massless fields to the one loop effective action can now be expressed as

$$\begin{aligned}\Delta\mathcal{S} &= -\frac{1}{2}{\sum_n}' \ln \kappa_n = \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s} {\sum_n}' e^{-\kappa_n s} \\ &= \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s} \int d^4x \sqrt{\det g} \left(K(0; s) - \bar{K}(0)\right),\end{aligned} \tag{2.12}$$

where $\epsilon$ is an ultraviolet cut-off which we shall take to be of order one, i.e. string scale.[4] Identifying this as the contribution to $\int d^4x \sqrt{\det g}\, \Delta\mathcal{L}_{eff}$ we get the contribution to $\Delta\mathcal{L}_{eff}$ from the non-zero modes:

$$\Delta\mathcal{L}_{eff}^{(nz)} = \frac{1}{2}\int_\epsilon^\infty \frac{ds}{s} \left(K(0; s) - \bar{K}(0)\right). \tag{2.13}$$

[4] Throughout this paper we shall assume that the horizon values of all the moduli fields are of order unity so that string scale and Planck scale are of the same order. This sets $G_N \sim 1$.

The logarithmic contribution to the entropy—term proportional to $\ln a$—arises from the $1 \ll s \ll a^2$ region in the $s$ integral. If we expand $K(0; s)$ in a Laurent series expansion in $\bar{s} = s/a^2$ around $\bar{s} = 0$, and if $K_0$ denotes the coefficient of the constant mode in this expansion, then using (2.6) and (2.13) we see that the net logarithmic correction to the entropy from the non-zero modes will be given by

$$-8\pi^2 a^4 \left(K_0 - \bar{K}(0)\right) \ln a = -4\pi^2 a^4 \left(K_0 - \bar{K}(0)\right) \ln A_H, \tag{2.14}$$

where $A_H = 4\pi a^2$ is the area of the event horizon.

The contribution to $Z_{AdS_2}$ from integration over the zero modes can be evaluated as follows.[5] First note that we can use (2.9), (2.10) to define the number of zero modes $N_{zm}$:

$$\int d^4x \sqrt{\det g}\, \bar{K}(0) = \sum_\ell 1 = N_{zm}. \tag{2.15}$$

In fact often the matrix $\bar{K}^{ij}$ takes a block diagonal form in the field space, with different blocks representing zero modes of different sets of fields. In that case we can use the analog of (2.15) to define the number of zero modes of each block. If these different blocks are labelled by different sets $\{A_r\}$ then the number of zero modes belonging to the set $A_r$ will be given by

$$\begin{aligned} N_{zm}^{(r)} &= \int d^4x \sqrt{\det g}\, \bar{K}^r(0) = 8\pi^2 a^4\, \bar{K}^r(0)\ (\cosh \eta_0 - 1), \\ \bar{K}^r(0) &\equiv \sum_{\ell \in A_r} G_{ij}\, g_\ell^{(i)}(x)\, g_\ell^{(j)}(x). \end{aligned} \tag{2.16}$$

Typically these zero modes are associated with certain asymptotic symmetries—gauge transformation with parameters which do not vanish at infinity. In this case we can evaluate the integration over the zero modes by making a change of variables from the coefficients of the zero modes to the parameters labelling the (super-)group of asymptotic symmetries. Suppose for the zero modes in the $r$'th block the Jacobian for the change of variables from the fields to supergroup parameters gives a factor of $a^{\beta_r}$ for each zero mode. Then the net $a$ dependent contribution to $Z_{AdS_2}$ from the zero mode integration will be given by

$$a^{\sum_r \beta_r N_{zm}^{(r)}} = \exp\left[8\pi^2 a^4\ (\cosh \eta_0 - 1) \ln a \sum_r \beta_r \bar{K}^r(0)\right]. \tag{2.17}$$

Again the coefficient of $\cosh \eta_0$ can be interpreted as due to a shift in the energy $E_0$, whereas the $\eta_0$ independent term has the interpretation of a contribution to the black

[5] Some discussion on the effect of zero modes on the ultraviolet divergent contribution to the black hole entropy can be found in [59,60].

hole entropy. This gives the following expression for the logarithmic correction to the entropy from the zero modes:

$$-8\pi^2 a^4 \ln a \sum_r \beta_r \bar{K}^r(0). \tag{2.18}$$

Adding this to (2.14) we get

$$\Delta S_{BH} = -4\pi^2 a^4 \ln A_H \left( K_0 + \sum_r (\beta_r - 1)\bar{K}^r(0) \right). \tag{2.19}$$

We shall refer to the term proportional to $\sum_r (\beta_r - 1)\bar{K}^r(0)$ as the zero mode contribution although it should be kept in mind that only the term proportional to $\beta_r$ arises from integration over the zero modes, and the $-1$ term is the result of subtracting the zero mode contribution from the heat kernel to correctly compute the result of integration over the non-zero modes.

The contribution from the fermionic fields can be included in the above analysis as follows. Let $\{\psi^i\}$ denote the set of fermion fields in the theory. Here $i$ labels the internal indices or space-time vector index (for the gravitino fields) but the spinor indices are suppressed. Without any loss of generality we can take the $\psi^i$'s to be Majorana spinors satisfying $\bar{\psi}^i = (\psi^i)^T \widetilde{C}$ where $\widetilde{C}$ is the charge conjugation operator. Then the kinetic term for the fermions have the form

$$-\frac{1}{2}\bar{\psi}^i \mathcal{D}_{ij}\psi^j = -\frac{1}{2}(\psi^i)^T \widetilde{C}\mathcal{D}_{ij}\psi^j, \tag{2.20}$$

for some appropriate operator $\mathcal{D}$. We can now proceed to define the heat kernel of the fermions in terms of eigenvalues of $\mathcal{D}$ in the usual manner, but with the following simple changes. Since the integration over the fermions produce $(\det \mathcal{D})^{1/2}$ instead of $(\det \mathcal{D})^{-1/2}$, we need to include an extra minus sign in the definition of the heat kernel. Also since the fermionic kinetic operator is linear in derivative, it will be convenient to first compute the determinant of $\mathcal{D}^2$ and then take an additional square root of the determinant. This is implemented by including an extra factor of $1/2$ in the definition of the heat kernel.[6] We shall denote by $K_0^f$ the constant part of the fermionic heat kernel in the small $s$ expansion after taking into account this factor of $-1/2$. For analysis of the zero modes however we need to work with the kinetic operator and not its square since the zero mode structure may get modified upon taking the square e.g., the kinetic operator may have blocks in the Jordan canonical form which squares to zero, but the matrix itself may be non-zero.[7] Let us denote by $\bar{K}^f(0)$ the total fermion zero mode contribution to the heat kernel. This must be subtracted from the total heat

[6] For this it is important to work with Majorana or Dirac fermions but not Weyl fermions since the action of $\mathcal{D}$ changes the chirality of the state. Thus $\det(\mathcal{D}^2) \neq (\det \mathcal{D})^2$ acting on a Weyl fermion if the action of $\mathcal{D}$ on the left and the right moving fermions are different.

[7] This problem would not arise if we work with $\widetilde{C}\mathcal{D}$ instead of $\mathcal{D}$ since $\widetilde{C}\mathcal{D}$ is represented by an anti-symmetric matrix. However for other reasons it is convenient to work with $\mathcal{D}$ instead of $\widetilde{C}\mathcal{D}$.

kernel. Thus we arrive at an expression similar to (2.14) for the fermionic non-zero mode contribution to the entropy:

$$-4\pi^2 a^4 \left(K_0^f - \bar{K}^f(0)\right) \ln A_H, \tag{2.21}$$

Next we need to carry out the integration over the zero modes. Taking into account the extra factor of $-1/2$ in the definition of the fermionic heat kernel we see that the analog of (2.16) for the total number of fermion zero modes $N_{zm}^{(f)}$ now takes the form

$$N_{zm}^{(f)} = -16\pi^2 a^4 \, \bar{K}^f(0) \, (\cosh\eta_0 - 1) \,. \tag{2.22}$$

Let us further assume that integration over each fermion zero modes gives a factor of $a^{-\beta_f/2}$ for some constant $\beta_f$. Then the total $a$-dependent contribution from integration over the fermion zero modes is given by

$$\exp\left[8\pi^2 a^4 \, (\cosh\eta_0 - 1) \, \beta_f \bar{K}^f(0) \, \ln a\right]. \tag{2.23}$$

As usual the coefficient of $\cosh\eta_0$ can be interpreted as due to a shift in the energy $E_0$, whereas the $\eta_0$ independent term has the interpretation of a contribution to the black hole entropy. Combining this with the contribution (2.21) from the non-zero modes we arive at the following expression for the logarithmic correction to the entropy from the fermion zero modes:

$$\Delta S_{BH} = -4\pi^2 a^4 \, \ln A_H \, \left(K_0^f + (\beta_f - 1)\bar{K}^f(0)\right). \tag{2.24}$$

In later sections we shall describe the computation of $K(0;s)$ and $\bar{K}^r(0)$ for various fields, as well as of the coefficients $\beta_r$ for gauge fields, metric and the gravitinos.

## 3 Simple examples with minimally coupled massless fields

We shall now review some simple applications of the results of the previous section by computing logarithmic corrections to the black hole entropy due to minimally coupled scalar, vector and fermion fields.[8] First consider the example of a massless scalar whose only interaction with other fields is a coupling to gravity via minimal coupling. Let us denote by $K^s(x, x'; s)$ the heat kernel associated with such a scalar. It follows from (2.8) and the fact that $\Box_{AdS_2\times S^2} = \Box_{AdS_2} + \Box_{S^2}$ that the heat kernel of a massless scalar field on $AdS_2 \times S^2$ is given by the product of the heat kernels on $AdS_2$ and $S^2$, and in the $x' \to x$ limit takes the form [62]

$$K^s(0; s) = K^s_{AdS_2}(0; s) K^s_{S^2}(0; s). \tag{3.1}$$

[8] Analysis of logarithmic correction to the black hole entropy due to massless scalars with non-minimal coupling to background gravity can be found in [61]. However for our analysis we also need to deal with the case where the fluctuations in various fields are coupled to background fluxes. These will be discussed in later sections.

$K^s_{S^2}$ and $K^s_{AdS_2}$ in turn can be calculated using (2.8) since we know the eigenfunctions and the eigenvalues of the Laplace operator on these respective spaces. The eigenfunctions $f_{\lambda,\ell}$ on $AdS_2$ are described in (A.1). Since $f_{\lambda,\ell}$ vanishes at $\eta = 0$ for $\ell \neq 0$, only the $\ell = 0$ eigenfunctions will contribute to $K^s_{AdS_2}(0; s)$. At $\eta = 0$ $f_{\lambda,0}$ has the value $\sqrt{\lambda \tanh(\pi\lambda)}/\sqrt{2\pi a^2}$. The corresponding eigenvalue of $-\Box_{AdS_2}$ is $(\lambda^2 + \frac{1}{4})/a^2$. Thus (2.8) gives

$$K^s_{AdS_2}(0; s) = \frac{1}{2\pi\, a^2} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, \exp\left[-s\left(\lambda^2 + \frac{1}{4}\right)/a^2\right]. \tag{3.2}$$

On $S^2$ the eigenfunctions are $Y_{lm}(\psi, \phi)/a$ and the corresponding eigenvalues are $-l(l+1)/a^2$. Since $Y_{lm}$ vanishes at $\psi = 0$ for $m \neq 0$, and $Y_{l0} = \sqrt{2l+1}/\sqrt{4\pi}$ at $\psi = 0$ we have

$$K^s_{S^2}(0; s) = \frac{1}{4\pi a^2} \sum_l e^{-sl(l+1)/a^2}(2l+1). \tag{3.3}$$

We can bring this to a form similar to (3.2) by expressing it as

$$\frac{1}{4\pi i\, a^2}\, e^{s/4a^2} \oint d\widetilde{\lambda}\, \widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, e^{-s\widetilde{\lambda}^2/a^2}, \tag{3.4}$$

where $\oint$ denotes integration along a contour that travels from $\infty$ to 0 staying below the real axis and returns to $\infty$ staying above the real axis. By deforming the integration contour to a pair of straight lines through the origin—one at an angle $\kappa$ below the positive real axis and the other at an angle $\kappa$ above the positive real axis—we get

$$K^s_{S^2}(0; s) = \frac{1}{2\pi a^2} e^{s/4a^2}\, \mathrm{Im} \int_0^{e^{i\kappa}\times\infty} \widetilde{\lambda}\, d\widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, e^{-s\widetilde{\lambda}^2/a^2}, \quad 0 < \kappa \ll 1. \tag{3.5}$$

Combining (3.3) and (3.2) we get the heat kernel of a scalar field on $AdS_2 \times S^2$:

$$\begin{aligned} K^s(0; s) &= \frac{1}{8\pi^2 a^4} \sum_{l=0}^\infty (2l+1) \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, \exp\left[-\bar{s}\lambda^2 - \bar{s}\left(l + \frac{1}{2}\right)^2\right] \\ &= \frac{1}{4\pi^2 a^4} \int_0^\infty d\lambda\, \lambda \tanh(\pi\lambda)\, \mathrm{Im} \int_0^{e^{i\kappa}\times\infty} \widetilde{\lambda}\, d\widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, \exp\left[-\bar{s}\lambda^2 - \bar{s}\widetilde{\lambda}^2\right], \end{aligned} \tag{3.6}$$

where

$$\bar{s} = s/a^2. \tag{3.7}$$

In order to find the logarithmic correction to the entropy we need to expand $K^s(0;s)$ in a power series expansion in $\bar{s}$ and pick the coefficient $K_0^s$ of the constant term in this expansion. With the help of (B.1), (B.2) we get:

$$\begin{aligned}
K^s_{AdS_2}(0;s) &= \frac{1}{4\pi a^2\,\bar{s}}\, e^{-\bar{s}/4} \\
&\quad \times \left[1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(2n+1)!\frac{\bar{s}^{n+1}}{\pi^{2n+2}}\frac{1}{2^{2n}}\left(2^{-2n-1}-1\right)\zeta(2n+2)\right] \\
&= \frac{1}{4\pi a^2\,\bar{s}}\, e^{-\bar{s}/4}\left(1 - \frac{1}{12}\bar{s} + \frac{7}{480}\bar{s}^2 + \mathcal{O}(\bar{s}^3)\right), \qquad (3.8) \\
K^s_{S^2}(0;s) &= \frac{1}{4\pi a^2\,\bar{s}}\, e^{\bar{s}/4} \\
&\quad \times \left[1 - \sum_{n=0}^{\infty} \frac{1}{n!}(2n+1)!\frac{\bar{s}^{n+1}}{\pi^{2n+2}}\frac{1}{2^{2n}}\left(2^{-2n-1}-1\right)\zeta(2n+2)\right] \\
&= \frac{1}{4\pi a^2\,\bar{s}}\, e^{\bar{s}/4}\left(1 + \frac{1}{12}\bar{s} + \frac{7}{480}\bar{s}^2 + \mathcal{O}(\bar{s}^3)\right). \qquad (3.9)
\end{aligned}$$

Substituting (3.8) and (3.9) into (3.1) we get

$$K^s(0;s) = \frac{1}{16\pi^2 a^4\,\bar{s}^2}\left(1 + \frac{1}{45}\bar{s}^2 + \mathcal{O}(s^4)\right). \qquad (3.10)$$

This gives $K_0^s = 1/720\pi^2 a^4$. Equation (A.12) shows that for the scalar all the eigenvalues of the kinetic operator $-\Box$ are positive and hence there are no zero modes. Hence, using (2.19) we get the logarithmic contribution to the entropy from a minimally coupled scalar to be

$$\Delta S_{BH} = -\frac{1}{180}\ln A_H. \qquad (3.11)$$

Next we consider the case of a Maxwell field $A_\mu$ whose only coupling is via the minimal coupling to the background metric. The action of such a field is given by

$$\mathcal{S}_A = -\frac{1}{4}\int d^4x\sqrt{\det g}\; F_{\mu\nu}F^{\mu\nu}, \qquad (3.12)$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength. Adding a gauge fixing term

$$S_{gf} = -\frac{1}{2}\int d^4x\sqrt{\det g}\;(D_\mu A^\mu)^2, \qquad (3.13)$$

we can express the action as

$$\mathcal{S}_A + \mathcal{S}_{gf} = -\frac{1}{2}\int d^4x\sqrt{\det g}\,A_\mu(\Delta A)^\mu, \qquad (3.14)$$

where

$$(\Delta A)_\mu \equiv -\Box A_\mu + R_{\mu\nu}A^\nu, \quad \Box A_\mu \equiv g^{\rho\sigma} D_\rho D_\sigma A_\mu. \tag{3.15}$$

A vector in $AdS_2 \times S^2$ decomposes into a (vector, scalar) plus a (scalar, vector), with the first and the second factors representing tensorial properties in $AdS_2$ and $S^2$ respectively. Furthermore, on any of these components the action of the kinetic operator can be expressed as $\Delta_{AdS_2} + \Delta_{S^2}$, with $\Delta$ defined as in (3.15) for vectors and as $-\Box$ for scalars. Thus we can construct the eigenfunctions of $\Delta$ by taking the product of appropriate eigenfunctions of $\Delta_{AdS_2}$ and $\Delta_{S^2}$, and the corresponding eigenvalue of $\Delta$ on $AdS_2 \times S^2$ will be given by the sum of the eigenvalues of $\Delta_{AdS_2}$ and $\Delta_{S^2}$. This gives

$$K^v(0; s) = K^v_{AdS_2}(0, s) K^s_{S^2}(0; s) + K^s_{AdS_2}(0, s) K^v_{S^2}(0; s). \tag{3.16}$$

Thus we need to compute $K^v_{AdS_2}(0, s)$ and $K^v_{S^2}(0; s)$. Finally, quantization of gauge fields also requires us to introduce two anticommuting scalar ghosts whose kinetic operator is given by the standard laplacian $-\Box$ in the harmonic gauge. They give a net contribution of $-2K^s(0; s)$ to the heat kernel.

To find $K^v_{S^2}$ we use the basis functions given in (A.2). These have $\Delta$ eigenvalue $\kappa_1^{(k)}$ and hence the contribution from any of these two eigenfunctions to the vector heat kernel $K^v_{S^2}(x, x; s)$ is given by $(\kappa_1^{(k)})^{-1} e^{-\kappa_1^{(k)} s} g^{\mu\nu} \partial_\mu U_k(x) \partial_\nu U_k(x)$. Now since $K^v_{S^2}(x, x; s)$ is independent of $x$ after summing over the contribution from all the states, we could compute it by taking the volume average of each term. Taking a volume average allows us to integrate by parts and gives the same result as the volume average of $(\kappa_1^{(k)})^{-1} e^{-\kappa_1^{(k)} s} U_k(x)(-\Box) U_k(x) = e^{-\kappa_1^{(k)} s} U_k(x)^2$. This is the same as the contribution from $U_k$ to the scalar heat kernel. Thus the net contribution to $K^v_{S^2}(0, s)$ from the pair of basis states given in (A.2) is given by $2K^s_{S^2}(0; s) - 1/2\pi a^2$, where the subtraction term $-1/2\pi a^2$ accounts for the absence of the contribution from the $l = 0$ modes. Similarly the contribution from the basis states (A.3) to $K^v_{AdS_2}(0; s)$ is given by $2K^s_{AdS_2}(0; s)$. We must add to this the contribution from the discrete modes given in (A.4). Using (A.5) we see that this contribution is given by $1/2\pi a^2$, leading to $K^v_{AdS_2}(0; s) = 2K^s_{AdS_2}(0; s) + 1/2\pi a^2$. Thus we get the net contribution to the $K(0; s)$ from the vector field, including the ghosts, to be:

$$K^v(0, s) = \left(2K^s_{S^2}(0; s) - \frac{1}{2\pi a^2}\right) K^s_{AdS_2}(0; s) + \left(2K^s_{AdS_2}(0; s) + \frac{1}{2\pi a^2}\right) K^s_{S^2}(0; s)$$
$$-2K^s_{S^2}(0; s) K^s_{AdS_2}(0; s). \tag{3.17}$$

Using (3.8), (3.9) we get

$$K^v(0, s) = \frac{1}{8\pi^2 a^4}\left(\frac{1}{\bar{s}^2} + \frac{31}{45} + \mathcal{O}(\bar{s}^4)\right), \tag{3.18}$$

leading to $K^v_0 = 31/360\pi^2 a^4$.

Gauge fields also have zero modes arising from the product of $a^{-1}Y_{00}(\psi,\phi)$ with the discrete modes $\partial_m\Phi^{(\ell)}$ given in (A.4). Using (2.10) and (A.5) we get the contribution to $\bar{K}$ from these zero modes to be

$$\bar{K}^v(0) = a^{-2}\sum_\ell (Y_{00}(\psi,\phi))^2\, g^{mn}\partial_m\Phi^{(\ell)}(x)\partial_n\Phi^{(\ell)}(x) = \frac{1}{8\pi^2a^4}. \tag{3.19}$$

We could also derive the expression for as follows. It follows from (2.16) that $8\pi^2a^4\bar{K}^v(0)(\cosh\eta_0-1)$ has the interpretation of the total number of gauge field zero modes. This in turn is given by the number of discrete modes $N_1$ on $AdS_2$ given in (A.6) since the gauge field zero modes are obtained by taking the product of the unique $l=0$ mode of a scalar in $S^2$ and the discrete modes of the vector field in $AdS_2$. Thus we have $8\pi^2a^4\bar{K}^v(0)=1$.

We now need to compute the coefficient $\beta_v$ appearing in (2.19) for the zero modes of the vector fields. This computation proceeds as follows. First we express the metric $g_{\mu\nu}$ on $AdS_2\times S^2$ as $a^2g^{(0)}_{\mu\nu}$ where $g^{(0)}_{\mu\nu}$ is independent of $a$. The path integral over $A_\mu$ is normalized such that

$$\int [DA_\mu]\exp\left[-\int d^4x\,\sqrt{\det g}\,g^{\mu\nu}A_\mu A_\nu\right] = 1, \tag{3.20}$$

i.e.

$$\int [DA_\mu]\exp\left[-a^2\int d^4x\,\sqrt{\det g^{(0)}}\,g^{(0)\mu\nu}A_\mu A_\nu\right] = 1. \tag{3.21}$$

From this we see that up to an $a$ independent normalization constant, $[DA_\mu]$ actually corresponds to integration with measure $\prod_{\mu,x}d(aA_\mu(x))$. On the other hand the gauge field zero modes are associated with deformations produced by the gauge transformations with non-normalizable parameters: $\delta A_\mu\propto\partial_\mu\Lambda(x)$ for some functions $\Lambda(x)$ with $a$-independent integration range. Thus the result of integration over the gauge field zero modes can be found by first changing the integration over the zero modes of $(aA_\mu)$ to integration over $\Lambda$ and then picking up the contribution from the Jacobian in this change of variables. This gives a factor of $a$ from integration over each zero mode of $A_\mu$. It now follows from the definition of $\beta_r$ given in the paragraph below (2.16) that we have

$$\beta_v = 1. \tag{3.22}$$

Equation (2.19) now gives the net logarithmic contribution to $S_{BH}$ from the minimally coupled vector field to be

$$-4\pi^2a^4\,\ln A_H\,\left(K_0^v+(\beta_v-1)\bar{K}^v(0)\right) = -\frac{31}{90}\ln A_H. \tag{3.23}$$

Note that the term proportional to $\bar{K}^v(0)$ does not contribute since $\beta_v=1$.

Next we consider the case of a massless Dirac fermion, again with only interaction being minimal coupling to the metric on $AdS_2 \times S^2$. The eigenfunctions and eigenvalues of the square of the Dirac operator are given by the direct product of $(\chi^\pm_{l.m}, \eta^\pm_{l,m})$ given in (A.18) with $(\chi^\pm_k(\lambda), \eta^\pm_k(\lambda))$ given in (A.22). We can compute the heat kernel for the fermion using the relations:

$$\sum_m \left((\chi^+_{l,m})^\dagger \chi^+_{l,m} + (\chi^-_{l,m})^\dagger \chi^-_{l,m} + (\eta^+_{l,m})^\dagger \eta^+_{l,m} + (\eta^-_{l,m})^\dagger \eta^-_{l,m}\right) = \frac{1}{\pi a^2}(l+1),$$

$$\sum_k \left((\chi^+_k(\lambda))^\dagger \chi^+_k(\lambda) + (\chi^-_k(\lambda))^\dagger \chi^-_k(\lambda) + (\eta^+_k(\lambda))^\dagger \eta^+_k(\lambda) + (\eta^-_k(\lambda))^\dagger \eta^-_k(\lambda)\right)$$
$$= \frac{1}{\pi a^2}\lambda \coth(\pi\lambda). \tag{3.24}$$

The first of these relations is derived by evaluating it at $\psi = 0$ where only the $m = 0$ terms contribute whereas the second relation is derived by evaluating it at $\eta = 0$ where only the $k = 0$ terms contribute. Using this we get the contribution to $K(0; s)$ from the fermion fields to be

$$K^f(0; s) = -\frac{1}{\pi^2 a^4}\int_0^\infty d\lambda e^{-\bar{s}\lambda^2}\, \lambda\, \coth(\pi\lambda) \sum_{l=0}^\infty (l+1)\, e^{-\bar{s}(l+1)^2}$$
$$= -\frac{1}{\pi^2 a^4}\int_0^\infty d\lambda\, \lambda\, \coth(\pi\lambda) \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda}\, \cot(\pi\widetilde{\lambda})\, e^{-\bar{s}\widetilde{\lambda}^2 - \bar{s}\lambda^2}. \tag{3.25}$$

Note that we have included a minus sign in the heat kernel to account for the fermionic nature of the fields. Since we are squaring the kinetic operator we should have also gotten a factor of 1/2, but this is compensated for by a factor of 2 arising out of the complex nature of the fields. In other words when we expand a Dirac fermion in the basis $(\chi^\pm_{l,m}, \eta^\pm_{l,m}) \otimes (\chi^\pm_k(\lambda), \eta^\pm_k(\lambda))$, the coefficients of expansion are arbitrary complex numbers, and hence we double the number of integration variables. Using (B.3), (B.4) we now get

$$K^f(0; s) = -\frac{1}{4\pi^2 a^4 \bar{s}^2}\left(1 - \frac{11}{180}\bar{s}^2 + \mathcal{O}(\bar{s}^4)\right), \tag{3.26}$$

leading to $K^f_0 = 11/720\pi^2 a^4$. Since there are no zero modes for the fermions, (2.19) leads to the following contribution to the black hole entropy due to a minimally coupled massless Dirac fermion:

$$-\frac{11}{180}\ln A_H. \tag{3.27}$$

If instead we choose to work with Majorana fermions then (3.27) is replaced by $-\frac{11}{360}\ln A_H$.

Our analysis shows that if we have a set of $n_S$ minimally coupled massless scalar fields, $n_V$ minimally coupled Maxwell fields and $n_F$ minimally coupled massless Dirac fields, then they lead to a net logarithmic contribution of

$$\Delta S_{BH} = -\frac{1}{180}\ln\, A_H(n_S + 62n_V + 11n_F) \tag{3.28}$$

to the black hole entropy. We shall now describe an alternative method for arriving at this result. First note that in all the cases discussed above only the $K_0$ term in (2.19) is responsible for the logarithmic correction; the contribution proportional to $(\beta_r - 1)\bar{K}^r$ vanishes either due to the vanishing of $\bar{K}^r$ due to absence of zero modes (as in the case of scalars and fermions) or due to the vanishing of $\beta_r - 1$ (as in the case of gauge fields). On the other hand one can show that [63–69] the contribution to $K_0$—the constant term in the small $\bar{s}$ expansion of the heat kernel—is given by

$$K_0 = -\frac{1}{90\pi^2}(n_S + 62n_V + 11n_F)E - \frac{1}{30\pi^2}(n_S + 12n_V + 6n_F)I, \tag{3.29}$$

where

$$\begin{aligned} E &= \frac{1}{64}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \\ I &= -\frac{1}{64}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right). \end{aligned} \tag{3.30}$$

For the metric (2.1) we have $I = 0$ and $E = -1/8a^4$. Thus we get

$$K_0 = \frac{1}{720\pi^2 a^4}(n_S + 62n_V + 11n_F). \tag{3.31}$$

Substituting this into (2.19) we recover (3.28).

The result (3.28) agrees with earlier results on logarithmic corrections to the extremal black hole entropy computed e.g. in [41,53,54]. This will not be the case for the results derived in later sections, so it is important to understand the relation between the two computations. First [41,53,54] do not use the quantum entropy function for their computation, but use the relation between the entanglement entropy and the partition function in the presence of a conical defect. But as argued in [37,70] the entropy computed by this method gives the same result computed using the $K_0$ given in (3.29)—so this is not a coincidence. Second, as we have seen in the analysis described above the zero modes conspire in such a way that the result is controlled completely by the coefficient $K_0$ arising in the small $\bar{s}$ expansion of the heat kernel. If this had not been the case then we would have to account for the extra contribution proportional to $\bar{K}^r(0)(\beta_r - 1)$ which is absent in the analysis of [41,53,54]. As we shall see in the next few sections, $\bar{K}^r(0)(\beta_r - 1)$ will be non-vanishing when we are considering fluctuations of the metric or gravitino degrees of freedom. Third, in arriving at (3.28) we have analyzed fields which couple to gravity minimally without any coupling to any background flux. This however is not always the case, e.g. whenever there is any background flux, e.g. for Reissner–Nordstrom black holes, the kinetic term of the metric and some gauge fields get additional contribution due to the background flux which is not captured in the simple formula given in (3.29). A similar effect occurs the the fermionic sector. It may be possible to generalize (3.29) and hence the analysis of [41,53,54] to such cases, but the results currently available in [41,53,54] are not sufficient to compute correctly the logarithmic correction to the extremal black hole entropy due to metric and gravitino fluctuations, and other fields with non-trivial coupling to the background flux. It will be interesting to generalize the earlier analysis of [41,53,54] to incorporate the effect of the zero modes and the background flux, and see if the results for logarithmic correction to the entropy agree with those given in Sects. 4–6.

## 4 Extremal Reissner–Nordstrom black holes

We now consider the Einstein-Maxwell theory with the action

$$\mathcal{S} = \int \sqrt{\det g}\, \mathcal{L}_b, \quad \mathcal{L}_b = \left[R - F_{\mu\nu}F^{\mu\nu}\right], \tag{4.1}$$

where $R$ is the scalar curvature computed with the metric $g_{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge field strength. Note that we have set $G_N = 1/16\pi$. The near horizon geometry of an extremal electrically charged Reissner–Nordstrom solution in this theory is given by (see e.g. [23])

$$\begin{aligned} ds^2 &\equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(d\eta^2 + \sinh^2\eta d\theta^2) + a^2(d\psi^2 + \sin^2\psi d\phi^2), \\ \bar{F}_{mn} &= i\, a^{-1}\, \varepsilon_{mn}. \end{aligned} \tag{4.2}$$

The parameter $a$ is related to the electric charge $q$ via the relation $q = a$. The classical Bekenstein–Hawking entropy of this black hole is given by

$$S_{BH} = 4\pi A_H = 16\pi^2 a^2 = 16\pi^2 q^2. \tag{4.3}$$

Since this theory possesses an electric-magnetic duality symmetry, the result for the entropy of a dyonic black hole carrying electric charge $q$ and magnetic charge $p$ can be found from that of an electrically charged black hole by replacing $q$ by $\sqrt{q^2 + p^2}$. This holds for the classical entropy given in (4.3) as well as the logarithmic correction that will be discussed below.

To compute logarithmic corrections to the entropy of this black hole we consider fluctuations of the metric and gauge fields of form

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad A_\mu = \bar{A}_\mu + \frac{1}{2}\mathcal{A}_\mu, \\ F_{\mu\nu} &= \bar{F}_{\mu\nu} + \frac{1}{2}(\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu) \equiv \bar{F}_{\mu\nu} + \frac{1}{2} f_{\mu\nu}. \end{aligned} \tag{4.4}$$

In subsequent discussions all indices will be raised and lowered by the background metric $\bar{g}$. Substituting (4.4) into (4.1), adding to this a gauge fixing term

$$\mathcal{L}_{gf} = -\frac{1}{2}g^{\rho\sigma}\left(D^{\mu}h_{\mu\rho} - \frac{1}{2}D_{\rho}\,h^{\mu}_{\ \mu}\right)\left(D^{\nu}\,h_{\nu\sigma} - \frac{1}{2}D_{\sigma}h^{\nu}_{\ \nu}\right) - \frac{1}{2}D^{\mu}\mathcal{A}_{\mu}D^{\nu}\mathcal{A}_{\nu}, \tag{4.5}$$

and throwing away total derivative terms, we get the total Lagrangian density for the fluctuating fields:

$$\begin{aligned}\mathcal{L}_{b} + \mathcal{L}_{gf} = \text{constant} &- \frac{1}{4}h_{\mu\nu}\left(\widetilde{\Delta}h\right)^{\mu\nu} + \frac{1}{2}\mathcal{A}_{\mu}(\bar{g}^{\mu\nu}\Box - R^{\mu\nu})\mathcal{A}_{\nu} \\ &+a^{-2}\left(\frac{1}{2}h^{mn}h_{mn} - \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta} + h^{m\alpha}h_{m\alpha} + \frac{1}{4}(h^{\alpha}_{\ \alpha} - h^{m}_{\ m})^{2}\right) \\ &-2ia^{-1}\varepsilon^{mn}\,f_{\alpha m}h^{\alpha}_{\ n} - \frac{i}{2}\,a^{-1}\,\varepsilon^{mn}f_{mn}\left(h^{\gamma}_{\ \gamma} - h^{p}_{\ p}\right),\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}\left(\widetilde{\Delta}h\right)_{\mu\nu} = -\Box h_{\mu\nu} - R_{\mu\tau}h^{\tau}_{\ \nu} - R_{\nu\tau}h^{\ \tau}_{\mu} - 2R_{\mu\rho\nu\tau}h^{\rho\tau} + \frac{1}{2}\,\bar{g}_{\mu\nu}\,\bar{g}^{\rho\sigma}\,\Box\,h_{\rho\sigma} \\ +R\,h_{\mu\nu} + \left(\bar{g}_{\mu\nu}R^{\rho\sigma} + R_{\mu\nu}\bar{g}^{\rho\sigma}\right)h_{\rho\sigma} - \frac{1}{2}\,R\,\bar{g}_{\mu\nu}\,\bar{g}^{\rho\sigma}\,h_{\rho\sigma}.\end{aligned} \tag{4.7}$$

In this formula all components of the Riemann and Ricci tensor and the curvature scalar are computed with the background metric $\bar{g}_{\mu\nu}$. To this we must also add the Lagrangian density for the ghost fields [9]:

$$\mathcal{L}_{ghost} = \left[b^{\mu}\left(\bar{g}_{\mu\nu}\Box + R_{\mu\nu}\right)c^{\nu} + b\Box c - 2\,b\bar{F}_{\mu\nu}\,D^{\mu}c^{\nu}\right]. \tag{4.8}$$

We now need to find the eigenmodes and eigenvalues of the kinetic operator and then calculate the determinant. We follow the same strategy as in [9,10], i.e. first expand the various fields as linear combinations of the eigenmodes described in Appendix A, substitute them into the action (4.6), (4.8), and then find the eigenvalues of the kinetic operator. For this we can work at fixed $l$ and $\lambda$ values since at the quadratic level the modes carrying different $l$ and $\lambda$ values do not mix. This simplifies the problem enormously since at fixed values of $l$ and $\lambda$ the kinetic operator reduces to a finite dimensional matrix $\mathcal{M}(l+\frac{1}{2},\lambda)$. The net contribution to $K(0;s)$ can then be computed using the formula

$$\begin{aligned}K(0;s) &= \frac{1}{8\pi^{2}a^{4}}\sum_{l=0}^{\infty}(2l+1)\int_{0}^{\infty}d\lambda\,\lambda\,\tanh(\pi\lambda)\,Tre^{s\mathcal{M}(l+\frac{1}{2},\lambda)} \\ &= \frac{1}{4\pi^{2}a^{4}}\,\text{Im}\int_{0}^{e^{i\kappa}\times\infty}d\widetilde{\lambda}\,\widetilde{\lambda}\,\tan(\pi\widetilde{\lambda})\int_{0}^{\infty}d\lambda\,\lambda\,\tanh(\pi\lambda)\,Tre^{s\mathcal{M}(\widetilde{\lambda},\lambda)}.\end{aligned} \tag{4.9}$$

It will be convenient to introduce a new matrix $M$ via the relation:

$$\mathcal{M} = \left\{-(\kappa_{1}+\kappa_{2})\,I + a^{-2}\,M\right\}, \tag{4.10}$$

where $I$ is the identity matrix and

$$\kappa_1 = a^{-2} l(l+1) = a^{-2}\left(\widetilde{\lambda}^2 - \frac{1}{4}\right), \quad \kappa_2 = a^{-2}\left(\lambda^2 + \frac{1}{4}\right). \tag{4.11}$$

Substituting this into (4.9) we get the first contribution to $K(0;s)$ which we denote by $\widetilde{K}^B_{(1)}(0;s)$:[9]

$$\widetilde{K}^B_{(1)}(0;s) = \frac{1}{4\pi^2 a^4} \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\tan(\pi\widetilde{\lambda}) \int_0^\infty d\lambda\,\lambda\,\tanh(\pi\lambda)\, e^{-\bar{s}(\lambda^2+\widetilde{\lambda}^2)}\, Tr(e^{\bar{s}M}). \tag{4.12}$$

We can now carry out the small $\bar{s}$ expansion by expanding the last term as

$$Tr(e^{\bar{s}M}) = \sum_{n=0}^\infty \frac{1}{n!}\bar{s}^n\, Tr(M^n) \tag{4.13}$$

and using (B.1), (B.2) to evaluate the integrals. (4.12) is not the complete contribution however, since for $l=0$ and 1 some modes will be absent due to the constraints on the modes mentioned below (A.2), (A.7). This requires a subtraction term which we shall call $\widetilde{K}^B_{(2)}$. Finally we also have to include the contribution from the discrete modes given in (A.4), (A.9) which we shall denote by $K^B_{(3)}$.

Our first task will be to find the matrix $M$. For this we expand the various fields as

$$\begin{aligned}
\mathcal{A}_\alpha &= \frac{1}{\sqrt{\kappa_1}}\left(C_1\partial_\alpha u + C_2\varepsilon_{\alpha\beta}\partial^\beta u\right), \quad \mathcal{A}_m = \frac{1}{\sqrt{\kappa_2}}\left(C_3\partial_m u + C_4\varepsilon_{mn}\partial^n u\right), \\
h_{m\alpha} &= \frac{1}{\sqrt{\kappa_1\kappa_2}}\left(B_1\,\partial_\alpha\partial_m u + B_2\,\varepsilon_{mn}\partial_\alpha\partial^n u + B_3\,\varepsilon_{\alpha\beta}\,\partial^\beta\partial_m u + B_4\,\varepsilon_{\alpha\beta}\,\varepsilon_{mn}\,\partial^\beta\partial^n u\right), \\
h_{\alpha\beta} &= \frac{1}{\sqrt{2}}(i\,B_5 + B_6)\,g_{\alpha\beta}\,u + \frac{1}{\sqrt{\kappa_1 - 2a^{-2}}}\left(D_\alpha\xi_\beta + D_\beta\xi_\alpha - g_{\alpha\beta}\,D^\gamma\xi_\gamma\right), \\
h_{mn} &= \frac{1}{\sqrt{2}}(i\,B_5 - B_6)\,g_{mn}\,u + \frac{1}{\sqrt{\kappa_2 + 2a^{-2}}}\left(D_m\widehat{\xi}_n + D_n\widehat{\xi}_m - g_{mn}\,D^p\widehat{\xi}_p\right), \\
\xi_\alpha &= \frac{1}{\sqrt{\kappa_1}}\left(B_7\partial_\alpha u + B_8\,\varepsilon_{\alpha\beta}\partial^\beta u\right), \quad \widehat{\xi}_m = \frac{1}{\sqrt{\kappa_2}}\left(B_9\partial_m u + B_0\,\varepsilon_{mn}\partial^n u\right).
\end{aligned} \tag{4.14}$$

[9] The superscript $B$ stands for bosonic fields. Of course in the Einstein–Maxwell theory all physical fields are bosonic and hence this symbol is redundant, but eventually we shall regard this as the bosonic sector of $\mathcal{N}=2$ supergravity. The 'tilde' on $K$ stands for the fact that we have overcounted the contribution from the $l=0$ and $l=1$ sectors by ignoring the constraints mentioned below (A.2), (A.7). Again this notation has been used keeping in mind a similar notation to be used in Sect. 5 for the fermionic sector of $\mathcal{N}=2$ supergravity.

Here $u$ denotes the product of $Y_{lm}(\psi, \phi)/a$ and a basis vector $f_{\lambda,\ell}(\eta, \theta)$ given in (A.1) for some fixed $(l, \lambda)$. $B_i$'s and $C_i$'s are constants labelling the fluctuations. Substituting this into the action we can compute the matrix $\mathcal{M}$ of the kinetic operator. The result is

$$\begin{aligned}
&\frac{1}{2}\left(\vec{B}\ \vec{C}\right)\mathcal{M}\begin{pmatrix}\vec{B}\\ \vec{C}\end{pmatrix}\\
&= -\frac{1}{2}(\kappa_1+\kappa_2)\left[\sum_{i=1}^{4} C_i^2 + \sum_{i=1}^{6} B_i^2\right] - \frac{1}{2}(\kappa_1+\kappa_2-4\,a^{-2})(B_7^2+B_8^2)\\
&\quad -\frac{1}{2}(\kappa_1+\kappa_2+4a^{-2})(B_9^2+B_0^2)\\
&\quad +a^{-2}\sum_{i=1}^{4} B_i^2 - 2ia^{-2}B_5B_6 - a^{-2}(B_7^2+B_8^2) + a^{-2}(B_9^2+B_0^2) + 2a^{-2}B_6^2\\
&\quad -2ia^{-1}\left[-\sqrt{\kappa_1}C_3B_2 + \sqrt{\kappa_1}C_4B_1 + \sqrt{\kappa_2}C_1B_2 + \sqrt{\kappa_2}C_2B_4 + \sqrt{2\kappa_2}B_6C_4\right].
\end{aligned} \tag{4.15}$$

The matrix $\mathcal{M}$ and hence the matrix $M$ defined via (4.10), (4.15) has block diagonal form and is easy to diagonalize. First of all we note the the modes labelled by $B_3$, $B_7$, $B_8$, $B_9$ and $B_0$ do not mix with any other mode and the modes $B_4$ and $C_2$ only mix with each other but not with any other mode. The modes $B_2$, $C_1$ and $C_3$ mix with each other but not with any other mode. Finally the modes $B_5$, $B_6$, $C_4$ and $B_1$ mix with each other but not with any other mode. The eigenvalues of $M$ in these different sectors are given by

$$\begin{aligned}
&B_3: 2,\quad B_7: 2,\quad B_8: 2,\quad B_9: -2,\quad B_0: -2,\quad B_4, C_2: 1 \pm i\sqrt{4\kappa_2 a^2 - 1},\\
&B_2, C_1, C_3: 0,\ 1 \pm i\sqrt{4a^2(\kappa_1+\kappa_2) - 1}\\
&B_5, B_6, C_4, B_1: \text{Eigenvalues of}\begin{pmatrix} 0 & -2i & 0 & 0\\ -2i & 4 & -2ia\sqrt{2\kappa_2} & 0\\ 0 & -2ia\sqrt{2\kappa_2} & 0 & -2ia\sqrt{\kappa_1}\\ 0 & 0 & -2ia\sqrt{\kappa_1} & 2\end{pmatrix}.
\end{aligned} \tag{4.16}$$

From this we get

$$\begin{aligned}
Tr(M) &= 12,\\
Tr(M^2) &= 36 - 32\lambda^2 - 16\widetilde{\lambda}^2,\\
Tr(M^3) &= 24 - 144\lambda^2 - 48\widetilde{\lambda}^2,\\
Tr(M^4) &= 68 - 464\lambda^2 + 192\lambda^4 - 112\widetilde{\lambda}^2 + 192\lambda^2\widetilde{\lambda}^2 + 64\widetilde{\lambda}^4.
\end{aligned} \tag{4.17}$$

Substituting this into (4.12) and carrying our the $\lambda$, $\widetilde{\lambda}$ integrals using (B.1), (B.2) we get the constant term in the small $\bar{s}$ expansion of $\widetilde{K}^B_{(1)}(0; s)$ to be

$$\widetilde{K}^B_{(1)}(0;s) : \frac{337}{360\pi^2 a^4}. \tag{4.18}$$

We now need to remove the contribution due to the modes which are absent for $l=0$ and $l=1$. For $l=1$ the modes $B_7$ and $B_8$ are absent due to the constraint mentioned below (A.7). The removed eigenvalues of $M$ are 2 and 2, and so those of $\mathcal{M}$ are $-a^{-2}(\lambda^2+\frac{1}{4})$ and $-a^{-2}(\lambda^2+\frac{1}{4})$. For $l=0$ the modes $C_1$, $C_2$, $B_1$, $B_2$, $B_3$, $B_4$, $B_7$, $B_8$ are absent due to the constraint mentioned below (A.2). The removed eigenvalues of $M$ are:

$$B_1:2,\quad B_3:2,\quad B_7:2,\quad B_8:2,\quad B_4,C_2:1\pm 2i\lambda,\quad B_2,C_1:1\pm 2i\lambda. \tag{4.19}$$

This gives a net subtraction term

$$\widetilde{K}^B_{(2)}(0;s) = -\frac{1}{8\pi^2 a^4}\int_0^\infty d\lambda\,\lambda\,\tanh\pi\lambda\, e^{-\bar{s}\lambda^2}\, e^{-\bar{s}/4}\Big[6+2e^{\bar{s}(1+2i\lambda)} + 2e^{\bar{s}(1-2i\lambda)}+4e^{2\bar{s}}\Big]. \tag{4.20}$$

The first term inside the square bracket is the contribution from the $l=1$ modes while the other terms represent contribution from the $l=0$ modes. Again by expanding the term inside the square bracket in a power series expansion in $\bar{s}$ and using (B.1) we get the $\bar{s}$ independent contribution to $\widetilde{K}^B_{(2)}$ in the small $\bar{s}$ expansion to be

$$\widetilde{K}^B_{(2)}(0;s) : \frac{1}{24\pi^2 a^4}. \tag{4.21}$$

Next we need to include the contribution due to the discrete modes. For this we expand the fields as

$$\begin{aligned}
\mathcal{A}_m &= E_1 v_m + E_2\varepsilon_{mn}v^n,\\
h_{m\alpha} &= \frac{1}{\sqrt{\kappa_1}}\left(E_3\partial_\alpha v_m + \widetilde{E}_3\varepsilon_{mn}\partial_\alpha v^n + E_4\varepsilon_{\alpha\beta}\partial^\beta v_m + \widetilde{E}_4\varepsilon_{\alpha\beta}\varepsilon_{mn}\partial^\beta v^n\right)\\
h_{mn} &= \frac{a}{\sqrt{2}}\left(D_m\widehat{\xi}_n + D_n\widehat{\xi}_m - g_{mn}D^p\widehat{\xi}_p\right),\quad \widehat{\xi}_m = E_5 v_m + \widetilde{E}_5\varepsilon_{mn}v^n,
\end{aligned} \tag{4.22}$$

and

$$h_{mn} = E_6 w_{mn}. \tag{4.23}$$

Here $v_m$ is the product of a spherical harmonic with one of the vectors in (A.3) and $w_{mn}$ is the product of a spherical harmonic with one of the basis vectors given in (A.9). Following the strategy of [9,10], we have taken $v_m$ to be a real basis vector, and regarded $v_m$ and $\varepsilon^{mn}v_n$ as independent. This effectively doubles the number of modes and hence we need to halve the contribution from each mode. Thus for example the contribution to the heat kernel on $AdS_2$ from each of these basis vectors is now given by a half of (A.5), i.e. $1/4\pi a^2$ since the net contribution is shared between $v_m$ and $\varepsilon^{mn} v_n$. There is no mixing between the modes described in (4.22) and (4.23); hence we can compute their contributions separately. Substituting (4.22) into the action we get the kinetic term to be

$$
\begin{aligned}
&-\frac{1}{2}\kappa_1 (E_1^2+E_2^2) - \frac{1}{2}\sum_{i=3}^{4}(\kappa_1 - 2a^{-2})(E_i^2+\widetilde{E}_i^2) - \frac{1}{2}(\kappa_1+2a^{-2})(E_5^2+\widetilde{E}_5^2) \\
&\quad +2ia^{-1}\sqrt{\kappa_1}\left(E_1\widetilde{E}_3 - E_2E_3\right) \\
&\equiv -\frac{1}{2}\kappa_1\left(E_1^2+E_2^2+\sum_{i=3}^{5}(E_i^2+\widetilde{E}_i^2)\right) + \frac{1}{2}a^{-2}\left(\vec{E}\ \vec{\widetilde{E}}\right)\widehat{M}\begin{pmatrix}\vec{E}\\ \vec{\widetilde{E}}\end{pmatrix} \qquad (4.24)
\end{aligned}
$$

Eigenvalues of $\widehat{M}$ defined through (4.24) are given by

$$
\begin{aligned}
&E_4 : 2, \quad \widetilde{E}_4 : 2, \quad E_5 : -2, \quad \widetilde{E}_5 : -2, \quad (E_1, \widetilde{E}_3) : 1 \pm i\sqrt{4l^2+4l-1}, \\
&(E_2, E_3) : 1 \pm i\sqrt{4l^2+4l-1}. \qquad (4.25)
\end{aligned}
$$

For $l = 0$ however the modes $E_3, \widetilde{E}_3, E_4, \widetilde{E}_4$ carrying $\widehat{M}$ eigenvalues $2, 2, 2, 2$ are absent due to the condition mentioned below (A.2). Finally the mode (4.23) gives a kinetic term

$$
-\frac{1}{2}\kappa_1 E_6^2. \qquad (4.26)
$$

Combining these results we get the net contribution to the heat kernel from the discrete modes to be

$$
\begin{aligned}
K^B_{(3)}(0;s) &= \frac{1}{16\pi^2 a^4}\left[\sum_{l=0}^{\infty}\left\{(2l+1)e^{-\bar{s}l(l+1)}\left(2e^{2\bar{s}} + 2e^{-2\bar{s}} + 2e^{\bar{s}(1+i\sqrt{4l^2+4l-1})}\right.\right.\right. \\
&\quad \left.\left. +2e^{\bar{s}(1-i\sqrt{4l^2+4l-1})}\right)\right\} - 4\,e^{2\bar{s}} + 6\sum_{l=0}^{\infty}(2l+1)e^{-\bar{s}l(l+1)}\Bigg] \\
&= \frac{1}{4\pi^2 a^4}\mathrm{Im}\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\tan\pi\widetilde{\lambda}\, e^{-\bar{s}\widetilde{\lambda}^2} e^{\bar{s}/4}\left(e^{2\bar{s}} + e^{-2\bar{s}} + e^{\bar{s}(1+i\sqrt{4\widetilde{\lambda}^2-2})}\right. \\
&\quad \left. +e^{\bar{s}(1-i\sqrt{4\widetilde{\lambda}^2-2})} + 3\right) - \frac{1}{4\pi^2 a^4}e^{2\bar{s}} \qquad (4.27)
\end{aligned}
$$

The first line represents the contribution from the eigenvalues (4.25) and the second line represents the effect of removing the four $l = 0$ modes. The third line represents the contribution from the mode $E_6$ with kinetic term given in (4.26). We can evaluate the integral by expanding the terms inside ( ) in the fourth and fifth lines in a power series expansion in $\bar{s}$ and using (B.2). The result for the constant term in the small $\bar{s}$ expansion of $K^B_{(3)}$ is:

$$K^B_{(3)} : -\frac{5}{24\pi^2 a^4}. \tag{4.28}$$

Next we turn to the ghost fields. The last term in (4.8) describes mixing between the fields $b$ and $c^\nu$, but this has no effect on the determinant since the mixing matrix has an upper triangular form. Thus we can separately evaluate the contribution from the $(b, c)$ fields and $(b^\mu, c^\nu)$ fields. The contribution from the $b$, $c$ ghosts associated with the $U(1)$ gauge field is negative of that of two scalars. This gives the first contribution from the ghosts:

$$K^{ghost}_{(1)} = -\frac{1}{2\pi^2 a^4}\,\mathrm{Im}\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\tan\pi\widetilde{\lambda}\int_0^\infty d\lambda\,\lambda\,\tanh\pi\lambda\,e^{-\bar{s}\,(\lambda^2+\widetilde{\lambda}^2)}. \tag{4.29}$$

For finding the contribution due to the $b_\mu$, $c_\mu$ ghosts associated with general coordinate invariance, we expand them in modes:

$$\begin{aligned}
b_\alpha &= A\frac{1}{\sqrt{\kappa_1}}\,\partial_\alpha u + B\frac{1}{\sqrt{\kappa_1}}\,\varepsilon_{\alpha\beta}\partial^\beta u,\\
b_m &= C\frac{1}{\sqrt{\kappa_2}}\,\partial_m u + D\frac{1}{\sqrt{\kappa_2}}\,\varepsilon_{mn}\partial^n u,\\
c_\alpha &= E\frac{1}{\sqrt{\kappa_1}}\,\partial_\alpha u + F\frac{1}{\sqrt{\kappa_1}}\,\varepsilon_{\alpha\beta}\partial^\beta u,\\
c_m &= G\frac{1}{\sqrt{\kappa_2}}\,\partial_m u + H\frac{1}{\sqrt{\kappa_2}}\,\varepsilon_{mn}\partial^n u.
\end{aligned} \tag{4.30}$$

Substituting this into the first term in (4.8) we get the ghost kinetic term:

$$(\kappa_1 + \kappa_2 - 2a^{-2})(AE + BF) + (\kappa_1 + \kappa_2 + 2a^{-2})(CG + DH). \tag{4.31}$$

This gives the second contribution to the heat kernel of the ghosts

$$\begin{aligned}
K^{ghost}_{(2)} &= -\frac{1}{8\pi^2 a^4}\sum_{l=0}^\infty (2l+1)\int_0^\infty d\lambda\,\lambda\,\tanh\pi\lambda\,e^{-\bar{s}\,\lambda^2-\frac{1}{4}\bar{s}-\bar{s}l(l+1))}\left[4e^{-2\bar{s}} + 4e^{2\bar{s}}\right]\\
&= -\frac{1}{4\pi^2 a^4}\,\mathrm{Im}\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\tan\pi\widetilde{\lambda}\int_0^\infty d\lambda\,\lambda\,\tanh\pi\lambda\,e^{-\bar{s}\,(\lambda^2+\widetilde{\lambda}^2)}\left[4e^{-2\bar{s}}+4e^{2\bar{s}}\right].
\end{aligned} \tag{4.32}$$

We need to subtract from this the contribution due to the absent modes $A$, $B$, $E$, $F$ for $l = 0$. This is given by

$$K^{ghost}_{(3)} = \frac{1}{2\pi^2 a^4} \int_0^\infty d\lambda\, \lambda\, \tanh \pi\lambda\, e^{-\bar{s}\lambda^2} e^{2\bar{s}-\frac{1}{4}\bar{s}}. \tag{4.33}$$

Finally we need to include the contribution due to the discrete modes where we take $b_m$ and $c_m$ to be proportional to $v_m$. This gives the final contribution to the ghost heat kernel:

$$K^{ghost}_{(4)} = -\frac{1}{2\pi^2 a^4} \,\mathrm{Im} \int_0^{e^{i\kappa}\times\infty} d\tilde{\lambda}\, \tilde{\lambda}\, \tan \pi\tilde{\lambda}\, e^{-\bar{s}\tilde{\lambda}^2} e^{-2\bar{s}+\frac{1}{4}\bar{s}} \tag{4.34}$$

The small $s$ expansion of (4.29), (4.32)–(4.34) can be found by standard method described above and we get the following constant terms in the small $s$ expansion:

$$\begin{aligned} K^{ghost}_{(1)} &: -\frac{1}{360\pi^2 a^4} \\ K^{ghost}_{(2)} &: -\frac{91}{90\pi^2 a^4} \\ K^{ghost}_{(3)} &: \frac{5}{12\pi^2 a^4} \\ K^{ghost}_{(4)} &: \frac{5}{12\pi^2 a^4}. \end{aligned} \tag{4.35}$$

Adding all the contributions in (4.18), (4.21), (4.28) and (4.35) we get the total contribution to the constant term in the small $\bar{s}$ expansion of the heat kernel

$$K^B_0 = \frac{53}{90\pi^2 a^4}. \tag{4.36}$$

Next we turn to the contribution due to the zero modes. We first need to remove from $K^B_0$ the contribution due to the zero modes and then compute the contribution to $Z_{AdS_2}$ from integration over the zero modes. The combined effect of these is encoded in the $\sum_r (\beta_r - 1)\bar{K}^r(0)$ term in (2.19). Thus we need to compute $\beta_r$ and $\bar{K}^r(0)$ due to various zero modes. The relevant zero modes come from the gauge field $A_\mu$ and the metric $h_{\mu\nu}$ which we shall label by $r = v$ and $r = m$ respectively. We can identify these zero modes by examining the discrete mode contribution (4.27) to $K(0; s)$. First of all note that for $l = 0$ the $(2l+1)e^{\bar{s}(-l(l+1)+1+i\sqrt{4l^2+4l-1})}$ term becomes a constant signalling the presence of a zero mode. Working backwards we can identify them as due to the modes $E_1$, $E_2$ of the gauge field $A_\mu$. Since this term gives a contribution of $1/8\pi^2 a^4$ to $K(0; s)$ we have $\bar{K}^v(0) = 1/8\pi^2 a^4$. But we have seen that $\beta_v = 1$ for the gauge fields and hence these zero modes do not contribute to $\sum_r (\beta_r - 1)\bar{K}^r(0)$. The other zero modes come from the $3(2l+1)e^{-l(l+1)\bar{s}}$ term in (4.27) in the $l = 0$

sector and the $(2l+1)e^{-l(l+1)\bar{s}+2\bar{s}}$ term in the $l=1$ sector. The former corresponds to the modes represented by $E_6$ while the latter correspond to the modes represented by $E_5, \widetilde{E}_5$. By examining (4.22), (4.23) we see that both are modes of the metric. Physically the former represent deformations associated with the asymptotic Virasoro symmetries of the $AdS_2$ metric, while the latter are the zero modes of the $SU(2)$ gauge fields obtained from the dimensional reduction on $S^2$. The total contribution from these modes to $K(0,s)$ is given by $6/8\pi^2a^4$ and hence we have $\bar{K}^m(0) = 3/4\pi^2a^4$.

To complete the analysis we need to compute $\beta_m$ associated with the metric deformation. For this we proceed as in (3.20), (3.21). The path integral over the metric fluctuation $h_{\mu\nu}$ is normalized as

$$\int [Dh_{\mu\nu}] \exp\left[-\int d^4x \sqrt{\det g}\, g^{\mu\nu} g^{\rho\sigma} h_{\mu\rho} h_{\nu\sigma}\right] = 1, \tag{4.37}$$

i.e.

$$\int [Dh_{\mu\nu}] \exp\left[-\int d^4x \sqrt{\det g^{(0)}}\, g^{(0)\mu\nu} g^{(0)\rho\sigma} h_{\mu\rho} h_{\nu\sigma}\right] = 1. \tag{4.38}$$

Thus the correctly normalized integration measure, up to an $a$ independent constant, is $\prod_{x,(\mu\nu)} dh_{\mu\nu}(x)$. We now note that the zero modes are associated with diffeomorphisms with non-normalizable parameters: $h_{\mu\nu} \propto D_\mu\xi_\nu + D_\nu\xi_\mu$, with the diffeomorphism parameter $\xi^\mu(x)$ having $a$ independent integration range. Thus the $a$ dependence of the integral over the metric zero modes can be found by finding the Jacobian from the change of variables from $h_{\mu\nu}$ to $\xi^\mu$. Lowering of the index of $\xi^\mu$ gives a factor of $a^2$, leading to a factor of $a^2$ per zero mode. Thus we have $\beta_m = 2$ and hence the contribution to $\sum_r (\beta_r - 1)\bar{K}^r(0)$ from the zero modes of the metric is given by

$$(2-1)\frac{3}{4\pi^2a^4} = \frac{3}{4\pi^2a^4}. \tag{4.39}$$

Adding (4.39) to (4.36) and substituting this into (2.19), we get the net contribution to the logarithmic correction to the entropy of an extremal Reissner–Nordstrom black hole:

$$\Delta S_{BH} = -\frac{241}{45} \ln A_H. \tag{4.40}$$

If in addition the theory contains $n_S$ minimally coupled massless scalar, $n_F$ minimally coupled massless Dirac fermion and $n_V$ minimally coupled Maxwell fields, then the total logarithmic correction to $S_{BH}$ is given by the sum of (3.28) and (4.40):

$$\Delta S_{BH} = -\frac{1}{180}(964 + n_S + 62n_V + 11n_F) \ln A_H. \tag{4.41}$$

## 5 Half BPS black holes in pure $\mathcal{N} = 2$ supergravity

We shall now consider half BPS black holes in pure $\mathcal{N} = 2$ supergravity [71]. This requires adding to the Einstein–Maxwell action described in the previous section the fermionic action of a pair of Majorana spinors $\psi_\mu$ and $\varphi_\mu$ satisfying

$$\bar{\psi}_\mu = \psi_\mu^T \widetilde{C}, \quad \bar{\varphi}_\mu = \varphi_\mu^T \widetilde{C}, \tag{5.1}$$

for each $\mu$. Here $\widetilde{C}$ is the charge conjugation operator defined in (A.27). The quadratic part of the fermionic action is given by

$$\begin{aligned} S_f &= \int d^4x \sqrt{\det g}\, \mathcal{L}_f, \\ \mathcal{L}_f &= -\frac{1}{2}\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{2}\bar{\varphi}_\mu \gamma^{\mu\nu\rho} D_\nu \varphi_\rho + \frac{1}{2} F^{\mu\nu} \bar{\psi}_\mu \varphi_\nu + \frac{1}{4} F_{\rho\sigma} \bar{\psi}_\mu \gamma^{\mu\nu\rho\sigma} \varphi_\nu \\ &\quad -\frac{1}{2} F^{\mu\nu} \bar{\varphi}_\mu \psi_\nu - \frac{1}{4} F_{\rho\sigma} \bar{\varphi}_\mu \gamma^{\mu\nu\rho\sigma} \psi_\nu. \end{aligned} \tag{5.2}$$

For quantization we need to add to this a gauge fixing term

$$\mathcal{L}_{gf} = \frac{1}{4}\bar{\psi}_\mu \gamma^\mu \gamma^\nu D_\nu \gamma^\rho \psi_\rho + \frac{1}{4}\bar{\varphi}_\mu \gamma^\mu \gamma^\nu D_\nu \gamma^\rho \varphi_\rho, \tag{5.3}$$

and a ghost action

$$\mathcal{L}_{ghost} = \sum_{r=1}^{2} \left[ \bar{\tilde{b}}_r\, \Gamma^\mu D_\mu \tilde{c}_r + \bar{\tilde{e}}_r\, \Gamma^\mu D_\mu \tilde{e}_r \right]. \tag{5.4}$$

Here for each $r$ $(r = 1, 2)$ $\tilde{b}_r$, $\tilde{c}_r$ and $\tilde{e}_r$ represent spin half bosonic ghosts. The two values of $r$ correspond to two local supersymmetries which the theory possesses, $\tilde{b}_r$ and $\tilde{c}_r$ are the standard Fadeev–Popov ghosts, and $\tilde{e}_r$ is a special ghost originating due to the unusual nature of the gauge fixing terms we have used [9].

The sum of $\mathcal{L}_f$ and $\mathcal{L}_{gf}$, evaluated in the background (4.2), can be expressed as

$$\mathcal{L}_f + \mathcal{L}_{gf} = -\frac{1}{2}\left[ \bar{\psi}^\alpha \mathcal{K}_\alpha^{(1)} + \bar{\psi}^m \mathcal{K}_m^{(2)} + \bar{\varphi}^\alpha \mathcal{K}_\alpha^{(3)} + \bar{\varphi}^m \mathcal{K}_m^{(4)} \right] \tag{5.5}$$

where

$$\begin{aligned} \mathcal{K}_\alpha^{(1)} &= -\frac{1}{2}\gamma^n (\not{D}_{S^2} + \sigma_3\ \not{D}_{AdS_2}) \gamma_\alpha \psi_n - \frac{1}{2}\gamma^\beta \left( \not{D}_{S^2} + \sigma_3\ \not{D}_{AdS_2} \right) \gamma_\alpha \psi_\beta \\ &\quad + i\, a^{-1} \varepsilon_\alpha^{\ \beta} \sigma_3 \tau_3 \varphi_\beta \\ \mathcal{K}_m^{(2)} &= -\frac{1}{2}\gamma^\beta (\not{D}_{S^2} + \sigma_3\ \not{D}_{AdS_2}) \gamma_m \psi_\beta - \frac{1}{2}\gamma^n \left( \not{D}_{S^2} + \sigma_3\ \not{D}_{AdS_2} \right) \gamma_m \psi_n - i\, a^{-1} \varepsilon_m^{\ n} \varphi_n \\ \mathcal{K}_\alpha^{(3)} &= -\frac{1}{2}\gamma^n (\not{D}_{S^2} + \sigma_3\ \not{D}_{AdS_2}) \gamma_\alpha \varphi_n - \frac{1}{2}\gamma^\beta \left( \not{D}_{S^2} + \sigma_3\ \not{D}_{AdS_2} \right) \gamma_\alpha \varphi_\beta \\ &\quad - i\, a^{-1} \varepsilon_\alpha^{\ \beta} \sigma_3 \tau_3 \psi_\beta \end{aligned}$$

$$\mathcal{K}^{(4)}_m = -\frac{1}{2}\gamma^\beta (\not{D}_{S^2} + \sigma_3 \ \not{D}_{AdS_2})\gamma_m\varphi_\beta - \frac{1}{2}\gamma^n \left(\not{D}_{S^2} + \sigma_3 \ \not{D}_{AdS_2}\right)\gamma_m\varphi_n + i\, a^{-1}\varepsilon_m^{\ n}\psi_n. \tag{5.6}$$

We now expand the fermion fields in the basis described in Appendix A. As in the case of the bosonic fields we can work at fixed values of $l$ and $\lambda$. Let $\chi$ denote the product of $\chi^+_{l,m}$ or $\eta^+_{l,m}$ defined in (A.18) and $\chi^+_k(\lambda)$ or $\eta^+_k(\lambda)$ defined in (A.22). Then $\chi$ satisfies

$$\not{D}_{S^2}\chi = i\zeta_1\,\chi, \quad \not{D}_{AdS_2}\chi = i\zeta_2\,\chi, \quad \zeta_1 = (l+1)/a \geq 1/a, \quad \zeta_2 = \lambda/a \geq 0. \tag{5.7}$$

Furthermore, using Eqs. (5.7) and the representation of the $\gamma$-matrices given in (A.14) we get

$$\begin{aligned} \varepsilon_\alpha^{\ \beta}\gamma_\beta &= i\sigma_3\gamma_\alpha, \quad \varepsilon_{\alpha\beta}D^\beta\chi = -i\sigma_3 D_\alpha\chi - \zeta_1\sigma_3\gamma_\alpha\chi, \\ \varepsilon_m^{\ n}\gamma_n &= i\tau_3\gamma_m, \quad \varepsilon_{mn}D^n\chi = -i\tau_3 D_m\chi - \zeta_2\tau_3\sigma_3\gamma_m\chi. \end{aligned} \tag{5.8}$$

The basis functions involving $\chi^-_{l,m}$ and $\eta^-_{l,m}$ will be represented as $\sigma_3\chi^+_{l,m}$ and $\sigma_3\eta^+_{l,m}$ respectively; thus we shall not include them separately. Similarly the basis functions $\chi^-_k(\lambda)$ and $\eta^-_k(\lambda)$ will be represented as $\tau_3\chi^+_k(\lambda)$ and $\tau_3\eta^+_k(\lambda)$. We now introduce the modes of $\psi_\mu$ and $\varphi_\mu$ via the expansion

$$\begin{aligned} \psi_\alpha &= b_1\gamma_\alpha\chi + b_2\sigma_3\gamma_\alpha\chi + b_3 D_\alpha\chi + b_4\sigma_3 D_\alpha\chi \\ &\quad + b_1'\gamma_\alpha\tau_3\chi + b_2'\sigma_3\gamma_\alpha\tau_3\chi + b_3'\tau_3 D_\alpha\chi + b_4'\sigma_3\tau_3 D_\alpha\chi \\ \psi_m &= c_1\gamma_m\chi + c_2\sigma_3\gamma_m\chi + c_3\sigma_3 D_m\chi + c_4 D_m\chi \\ &\quad + c_1'\gamma_m\tau_3\chi + c_2'\sigma_3\gamma_m\tau_3\chi + c_3'\sigma_3\tau_3 D_m\chi + c_4'\tau_3 D_m\chi \\ \varphi_\alpha &= g_1\gamma_\alpha\chi + g_2\sigma_3\gamma_\alpha\chi + g_3 D_\alpha\chi + g_4\sigma_3 D_\alpha\chi \\ &\quad + g_1'\gamma_\alpha\tau_3\chi + g_2'\sigma_3\gamma_\alpha\tau_3\chi + g_3'\tau_3 D_\alpha\chi + g_4'\sigma_3\tau_3 D_\alpha\chi \\ \varphi_m &= h_1\gamma_m\chi + h_2\sigma_3\gamma_m\chi + h_3\sigma_3 D_m\chi + h_4 D_m\chi \\ &\quad + h_1'\gamma_m\tau_3\chi + h_2'\sigma_3\gamma_m\tau_3\chi + h_3'\sigma_3\tau_3 D_m\chi + h_4'\tau_3 D_m\chi \end{aligned} \tag{5.9}$$

where $b_i, b_i', c_i, c_i', g_i, g_i', h_i, h_i'$ are constants. Substituting this into (5.6) we get

$$\begin{aligned} \mathcal{K}^{(1)}_\alpha &= B_1\gamma_\alpha\chi + B_2\sigma_3\gamma_\alpha\chi + B_3 D_\alpha\chi + B_4\sigma_3 D_\alpha\chi \\ &\quad + B_1'\gamma_\alpha\tau_3\chi + B_2'\sigma_3\gamma_\alpha\tau_3\chi + B_3'\tau_3 D_\alpha\chi + B_4'\sigma_3\tau_3 D_\alpha\chi \\ \mathcal{K}^{(2)}_m &= C_1\gamma_m\chi + C_2\sigma_3\gamma_m\chi + C_3\sigma_3 D_m\chi + C_4 D_m\chi \\ &\quad + C_1'\gamma_m\tau_3\chi + C_2'\sigma_3\gamma_m\tau_3\chi + C_3'\sigma_3\tau_3 D_m\chi + C_4'\tau_3 D_m\chi \\ \mathcal{K}^{(3)}_\alpha &= G_1\gamma_\alpha\chi + G_2\sigma_3\gamma_\alpha\chi + G_3 D_\alpha\chi + G_4\sigma_3 D_\alpha\chi \\ &\quad + G_1'\gamma_\alpha\tau_3\chi + G_2'\sigma_3\gamma_\alpha\tau_3\chi + G_3'\tau_3 D_\alpha\chi + G_4'\sigma_3\tau_3 D_\alpha\chi \\ \mathcal{K}^{(4)}_m &= H_1\gamma_m\chi + H_2\sigma_3\gamma_m\chi + H_3\sigma_3 D_m\chi + H_4 D_m\chi \\ &\quad + H_1'\gamma_m\tau_3\chi + H_2'\sigma_3\gamma_m\tau_3\chi + H_3'\sigma_3\tau_3 D_m\chi + H_4'\tau_3 D_m\chi \end{aligned} \tag{5.10}$$

where

$$
\begin{aligned}
B_1 &= -i\zeta_1 b_1 + \frac{1}{2}\zeta_1^2 b_3 + \frac{1}{2}\zeta_1\zeta_2 b_4 + i\zeta_1 c_1 - \frac{1}{2}\zeta_1\zeta_2 c_3 + \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) c_4 \\
&\quad -a^{-1} g_1' - i\,\zeta_1 a^{-1} g_3' \\
B_2 &= i\zeta_1 b_2 + \frac{1}{2}\zeta_1\zeta_2 b_3 - \frac{1}{2}\zeta_1^2 b_4 + i\zeta_1 c_2 - \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) c_3 - \frac{1}{2}\zeta_1\zeta_2 c_4 \\
&\quad - a^{-1} g_2' - i\,\zeta_1 a^{-1} g_4' \\
B_3 &= i\zeta_2 b_4 - 2c_1 - i\zeta_2 c_3 + a^{-1} g_3' \\
B_4 &= i\zeta_2 b_3 - 2c_2 - i\zeta_2 c_4 + a^{-1} g_4' \\
C_1 &= -i\zeta_2 b_2 + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) b_3 + \frac{1}{2}\zeta_1\zeta_2 b_4 - i\zeta_2 c_2 - \frac{1}{2}\zeta_1\zeta_2 c_3 + \frac{1}{2}\zeta_2^2 c_4 \\
&\quad - a^{-1} h_1' + i\,\zeta_2 a^{-1} h_3' \\
C_2 &= i\zeta_2 b_1 - \frac{1}{2}\zeta_1\zeta_2 b_3 + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) b_4 - i\zeta_2 c_1 + \frac{1}{2}\zeta_2^2 c_3 + \frac{1}{2}\zeta_1\zeta_2 c_4 \\
&\quad - a^{-1} h_2' + i\,\zeta_2 a^{-1} h_4' \\
C_3 &= 2b_2 + i\zeta_1 b_4 - i\zeta_1 c_3 - a^{-1} h_3' \\
C_4 &= -2b_1 - i\zeta_1 b_3 + i\zeta_1 c_4 - a^{-1} h_4' \\
B_1' &= -i\zeta_1 b_1' + \frac{1}{2}\zeta_1^2 b_3' - \frac{1}{2}\zeta_1\zeta_2 b_4' + i\zeta_1 c_1' + \frac{1}{2}\zeta_1\zeta_2 c_3' + \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) c_4' \\
&\quad - a^{-1} g_1 - i\,\zeta_1 a^{-1} g_3 \\
B_2' &= i\zeta_1 b_2' - \frac{1}{2}\zeta_1\zeta_2 b_3' - \frac{1}{2}\zeta_1^2 b_4' + i\zeta_1 c_2' - \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) c_3' + \frac{1}{2}\zeta_1\zeta_2 c_4' \\
&\quad - a^{-1} g_2 - i\,\zeta_1 a^{-1} g_4 \\
B_3' &= -i\zeta_2 b_4' - 2c_1' + i\zeta_2 c_3' + a^{-1} g_3 \\
B_4' &= -i\zeta_2 b_3' - 2c_2' + i\zeta_2 c_4' + a^{-1} g_4 \\
C_1' &= i\zeta_2 b_2' + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) b_3' - \frac{1}{2}\zeta_1\zeta_2 b_4' + i\zeta_2 c_2' + \frac{1}{2}\zeta_1\zeta_2 c_3' + \frac{1}{2}\zeta_2^2 c_4' \\
&\quad - a^{-1} h_1 - i\,\zeta_2 a^{-1} h_3 \\
C_2' &= -i\zeta_2 b_1' + \frac{1}{2}\zeta_1\zeta_2 b_3' + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) b_4' + i\zeta_2 c_1' + \frac{1}{2}\zeta_2^2 c_3' - \frac{1}{2}\zeta_1\zeta_2 c_4' \\
&\quad - a^{-1} h_2 - i\zeta_2 a^{-1} h_4 \\
C_3' &= 2b_2' + i\zeta_1 b_4' - i\zeta_1 c_3' - a^{-1} h_3 \\
C_4' &= -2b_1' - i\zeta_1 b_3' + i\zeta_1 c_4' - a^{-1} h_4 \\
G_1 &= -i\zeta_1 g_1 + \frac{1}{2}\zeta_1^2 g_3 + \frac{1}{2}\zeta_1\zeta_2 g_4 + i\zeta_1 h_1 - \frac{1}{2}\zeta_1\zeta_2 h_3 + \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) h_4 \\
&\quad + a^{-1} b_1' + i\,\zeta_1 a^{-1} b_3'
\end{aligned}
$$

$$
\begin{aligned}
G_2 &= i\zeta_1 g_2 + \frac{1}{2}\zeta_1\zeta_2 g_3 - \frac{1}{2}\zeta_1^2 g_4 + i\zeta_1 h_2 - \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) h_3 - \frac{1}{2}\zeta_1\zeta_2 h_4 \\
&\quad + a^{-1}b_2' + i\,\zeta_1 a^{-1} b_4' \\
G_3 &= i\zeta_2 g_4 - 2h_1 - i\zeta_2 h_3 - a^{-1}b_3' \\
G_4 &= i\zeta_2 g_3 - 2h_2 - i\zeta_2 h_4 - a^{-1}b_4' \\
H_1 &= -i\zeta_2 g_2 + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) g_3 + \frac{1}{2}\zeta_1\zeta_2 g_4 - i\zeta_2 h_2 - \frac{1}{2}\zeta_1\zeta_2 h_3 + \frac{1}{2}\zeta_2^2 h_4 \\
&\quad + a^{-1}c_1' - i\,\zeta_2 a^{-1} c_3' \\
H_2 &= i\zeta_2 g_1 - \frac{1}{2}\zeta_1\zeta_2 g_3 + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) g_4 - i\zeta_2 h_1 + \frac{1}{2}\zeta_2^2 h_3 + \frac{1}{2}\zeta_1\zeta_2 h_4 \\
&\quad + a^{-1}c_2' - i\,\zeta_2 a^{-1} c_4' \\
H_3 &= 2g_2 + i\zeta_1 g_4 - i\zeta_1 h_3 + a^{-1}c_3' \\
H_4 &= -2g_1 - i\zeta_1 g_3 + i\zeta_1 h_4 + a^{-1}c_4' \\
G_1' &= -i\zeta_1 g_1' + \frac{1}{2}\zeta_1^2 g_3' - \frac{1}{2}\zeta_1\zeta_2 g_4' + i\zeta_1 h_1' + \frac{1}{2}\zeta_1\zeta_2 h_3' + \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) h_4' \\
&\quad + a^{-1}b_1 + i\,\zeta_1 a^{-1} b_3 \\
G_2' &= i\zeta_1 g_2' - \frac{1}{2}\zeta_1\zeta_2 g_3' - \frac{1}{2}\zeta_1^2 g_4' + i\zeta_1 h_2' - \frac{1}{2}\left(\zeta_2^2 + \frac{1}{a^2}\right) h_3' + \frac{1}{2}\zeta_1\zeta_2 h_4' \\
&\quad + a^{-1}b_2 + i\,\zeta_1 a^{-1} b_4 \\
G_3' &= -i\zeta_2 g_4' - 2h_1' + i\zeta_2 h_3' - a^{-1}b_3 \\
G_4' &= -i\zeta_2 g_3' - 2h_2' + i\zeta_2 h_4' - a^{-1}b_4 \\
H_1' &= i\zeta_2 g_2' + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) g_3' - \frac{1}{2}\zeta_1\zeta_2 g_4' + i\zeta_2 h_2' + \frac{1}{2}\zeta_1\zeta_2 h_3' + \frac{1}{2}\zeta_2^2 h_4' \\
&\quad + a^{-1}c_1 + i\,\zeta_2 a^{-1} c_3 \\
H_2' &= -i\zeta_2 g_1' + \frac{1}{2}\zeta_1\zeta_2 g_3' + \frac{1}{2}\left(\zeta_1^2 - \frac{1}{a^2}\right) g_4' + i\zeta_2 h_1' + \frac{1}{2}\zeta_2^2 h_3' - \frac{1}{2}\zeta_1\zeta_2 h_4' \\
&\quad + a^{-1}c_2 + i\,\zeta_2 a^{-1} c_4 \\
H_3' &= 2g_2' + i\zeta_1 g_4' - i\zeta_1 h_3' + a^{-1}c_3 \\
H_4' &= -2g_1' - i\zeta_1 g_3' + i\zeta_1 h_4' + a^{-1}c_4.
\end{aligned}
\tag{5.11}
$$

We can express this as

$$
\begin{pmatrix} \vec{B} \\ \vec{C} \\ \vec{G} \\ \vec{H} \\ \vec{B}' \\ \vec{C}' \\ \vec{G}' \\ \vec{H}' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \vec{b} \\ \vec{c} \\ \vec{g} \\ \vec{h} \\ \vec{b}' \\ \vec{c}' \\ \vec{g}' \\ \vec{h}' \end{pmatrix},
\tag{5.12}
$$

where $\mathcal{M}$ is a $32 \times 32$ matrix. If we introduce the matrix $\mathcal{M}_1$ through

$$\mathcal{M}^2 = -(\zeta_1^2 + \zeta_2^2) I_{32} + a^{-2}\mathcal{M}_1, \tag{5.13}$$

then the fermionic contribution to the heat kernel from the $l \geq 1$, i.e. $\zeta_1 \geq 2/a$ modes will be given by

$$K^f_{(1)}(0; s) = -\frac{1}{8\pi^2 a^4}, \sum_{l=1}^{\infty}(l+1)\int_0^{\infty} d\lambda\, \lambda\, \coth \pi\lambda\, e^{-\bar{s}\,((l+1)^2+\lambda^2)} \sum_{n=0}^{\infty} \frac{\bar{s}^n}{n!} Tr(\mathcal{M}_1^n). \tag{5.14}$$

Note the normalization factor 1/8 instead of 1 as in (3.25). A factor of 1/4 can be traced to the fact that in the analog of (3.24) we should no longer include the $\chi^-$'s or $\eta^-$'s in the sum since in the basis of expansion (5.9), (5.10) we have included, besides $\chi$, the states $\sigma_3\chi$, $\tau_3\chi$ and $\sigma_3\tau_3\chi$. Another factor of 1/2 arises from the fact that we are dealing with Majorana fermions instead of Dirac fermions.

In (5.14) we have not included the $l = 0$ contribution. This is due to the fact that for $l = 0$, i.e. $\zeta_1 = a^{-1}$ the modes $D_\alpha\chi$ and $\gamma_\alpha\chi$ are related by (A.30). Thus we can set $b_3 = b_4 = b_3' = b_4' = g_3 = g_4 = g_3' = g_4' = 0$ and replace the expressions for $B_1$, $B_2$, $B_1'$, $B_2'$, $G_1$, $G_2$, $G_1'$, $G_2'$ by those of $B_1 + iB_3/2a$, $B_2 + iB_4/2a$, $B_1' + iB_3'/2a$, $B_2' + iB_4'/2a$, $G_1 + iG_3/2a$, $G_2 + iG_4/2a$, $G_1' + iG_3'/2a$, $G_2' + iG_4'/2a$ respectively. This gives a $24 \times 24$ matrix $\widetilde{\mathcal{M}}$ relating $(B_1, B_2, B_1', B_2', C_1, \ldots, C_4, C_1', \ldots, C_4', G_1, G_2, G_1', G_2', H_1, \ldots, H_4, H_1', \ldots, H_4')$ to $(b_1, b_2, b_1', b_2', c_1, \ldots, c_4, c_1', \ldots, c_4', g_1, g_2, g_1', g_2', h_1, \ldots, h_4, h_1', \ldots, h_4')$. Let us introduce the matrix $\widetilde{\mathcal{M}}_1$ via:

$$\widetilde{\mathcal{M}}^2 = -(a^{-2} + \zeta_2^2) I_{24} + a^{-2}\widetilde{\mathcal{M}}_1. \tag{5.15}$$

Then the contribution from the $l = 0$ modes will be given by

$$K^f_{(2)}(0; s) = -\frac{1}{8\pi^2 a^4}\int_0^{\infty} d\lambda\, \lambda\, \coth \pi\lambda\, e^{-\bar{s}\,(1+\lambda^2)} \sum_{n=0}^{\infty} \frac{\bar{s}^n}{n!} Tr(\widetilde{\mathcal{M}}_1^n). \tag{5.16}$$

We can now write

$$K^f_{(1)}(0; s) + K^f_{(2)}(0; s) = \widetilde{K}^f_{(1)}(0; s) + \widetilde{K}^f_{(2)}(0; s), \tag{5.17}$$

where

$$\widetilde{K}^f_{(1)}(0;s) = -\frac{1}{8\pi^2 a^4} \sum_{l=0}^\infty (l+1) \int_0^\infty d\lambda\, \lambda\, \coth\pi\lambda\, e^{-\bar{s}\,((l+1)^2+\lambda^2)} \sum_{n=0}^\infty \frac{\bar{s}^n}{n!} Tr(\mathcal{M}_1^n)$$

$$= -\frac{1}{8\pi^2 a^4} \,\mathrm{Im} \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda}\, \cot\pi\widetilde{\lambda} \int_0^\infty d\lambda\, \lambda\, \coth\pi\lambda\, e^{-\bar{s}\,(\lambda^2+\widetilde{\lambda}^2)} \sum_{n=0}^\infty \frac{\bar{s}^n}{n!} Tr(\mathcal{M}_1^n), \tag{5.18}$$

and

$$\widetilde{K}^f_{(2)}(0;s) = -\frac{1}{8\pi^2 a^4} \int_0^\infty d\lambda\lambda \coth\pi\lambda e^{-\bar{s}(1+\lambda^2)} \sum_{n=0}^\infty \frac{\bar{s}^n}{n!} \left[ Tr(\widetilde{\mathcal{M}}_1^n) - Tr(\mathcal{M}_1^n)|_{l=0} \right]. \tag{5.19}$$

Finally we have to include the contribution from the discrete modes obtained by taking $\psi_m$ to be a linear combination of the product of the modes given in (A.31) and (A.18). The contribution from these modes may be analyzed by setting $\lambda = i$ i.e. $\zeta_2 = i/a, b_i = b'_i = g_i = g'_i = 0$ for $1 \le i \le 4$, and $c_{i+2} = 2ac_i, c'_{i+2} = 2ac'_i, h_{i+2} = 2ah_i, h'_{i+2} = 2ah'_i$ for $i = 1, 2$ in (5.11). Equation (5.11) now gives $B_i = B'_i = G_i = G'_i = 0$ for $1 \le i \le 4$, and $C_{i+2} = 2aC_i, C'_{i+2} = 2aC'_i, H_{i+2} = 2aH_i, H'_{i+2} = 2aH'_i$ for $i = 1, 2$, and we get a $8 \times 8$ matrix $\widehat{\mathcal{M}}$ that relates the constants $C_i, C'_i, H_i, H'_i$ to $c_i, c'_i, h_i, h'_i$ for $i = 1, 2$. We again introduce the matrix $\widehat{\mathcal{M}}_1$ via:

$$\widehat{\mathcal{M}}^2 = -(-a^{-2} + \zeta_1^2) I_8 + a^{-2} \widehat{\mathcal{M}}_1. \tag{5.20}$$

Then the contribution to the heat kernel from the fermionic discrete modes will be given by:

$$K^f_{(3)}(0;s) = -\frac{1}{8\pi^2 a^4} \sum_{l=0}^\infty (l+1)\, e^{-\bar{s}(l+1)^2+\bar{s}} \sum_{n=0}^\infty \frac{\bar{s}^n}{n!} Tr(\widehat{\mathcal{M}}_1^n)$$

$$= -\frac{1}{8\pi^2 a^4} \,\mathrm{Im} \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda}\, \cot\pi\widetilde{\lambda}\, e^{\bar{s}\,(1-\widetilde{\lambda}^2)} \sum_{n=0}^\infty \frac{\bar{s}^n}{n!} Tr(\widehat{\mathcal{M}}_1^n). \tag{5.21}$$

Explicit computation gives

$$\begin{aligned} Tr(\mathcal{M}_1) &= -32 \\ Tr(\mathcal{M}_1^2) &= 128 + 64(l+1)^2 + 64\lambda^2 \\ Tr(\mathcal{M}_1^3) &= -512 - 384(l+1)^2 - 384\lambda^2 \end{aligned} \tag{5.22}$$

$$
\begin{aligned}
Tr(\mathcal{M}_1^4) &= 2048 + 2048(l+1)^2 + 256(l+1)^4 \\
&\quad +2048\lambda^2 + 512(l+1)^2\lambda^2 + 256\lambda^4, \\
Tr(\widetilde{\mathcal{M}}_1) &= -32 \\
Tr(\widetilde{\mathcal{M}}_1^2) &= 192 + 64\lambda^2 \\
Tr(\widetilde{\mathcal{M}}_1^3) &= -896 - 384\lambda^2 \\
Tr(\widetilde{\mathcal{M}}_1^4) &= 4352 + 2560\lambda^2 + 256\lambda^4,
\end{aligned}
\tag{5.23}
$$

and

$$
\begin{aligned}
Tr(\widehat{\mathcal{M}}_1) &= -16 \\
Tr(\widehat{\mathcal{M}}_1^2) &= 32 + 32(l+1)^2 \\
Tr(\widehat{\mathcal{M}}_1^3) &= -64 - 192(l+1)^2 \\
Tr(\widehat{\mathcal{M}}_1^4) &= 128 + 768(l+1)^2 + 128(l+1)^4.
\end{aligned}
\tag{5.24}
$$

Substituting these into Eqs. (5.18), (5.19) and (5.21) and using Eqs. (B.3), (B.4) we get the following constant terms in the small $\bar{s}$ expansions of the heat kernels:

$$\widetilde{K}^f_{(1)} : \frac{11}{180\pi^2 a^4}, \tag{5.25}$$

$$\widetilde{K}^f_{(2)} : -\frac{5}{12\pi^2 a^4}, \tag{5.26}$$

and

$$K^f_{(3)} : -\frac{5}{12\pi^2 a^4}. \tag{5.27}$$

Finally the six Majorana ghost fields give a contribution equal to that of three minimally coupled Dirac fermions but with opposite sign. Thus using (3.26) we get the constant term in the heat kernel from the ghost fields to be:

$$K^f_{ghost} : -\frac{11}{240\pi^2 a^4} \tag{5.28}$$

Adding up the contributions (5.25)–(5.28) we get the total fermionic contribution to the constant term in $K(0; s)$:

$$K^f_0 = -\frac{589}{720\pi^2 a^4}. \tag{5.29}$$

To this we have to add the extra contribution due to the zero modes. These modes arise in the sector containing the discrete modes with $l = 0$. The kinetic operator in this sector is represented by the matrix $\widehat{\mathcal{M}}$ defined above (5.20). Explicit computation shows that for $l=0$, i.e. $\zeta_1=1$ this matrix has the form:

$$\begin{pmatrix} -i & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}. \tag{5.30}$$

This has four zero eigenvalues, representing four zero modes. Equation(5.21) now shows that the net contribution to $K(0;s)$ from these zero modes is given by $-1/2\pi^2a^4$. This is to be identified as the contribution $\bar{K}^f(0)$ in (2.21) that must be subtracted from the heat kernel.

It remains to calculate the constant $\beta_f$ that appears in (2.24). It was shown in [10] that the effect of fermion zero mode integration is to add back to $K_0^f$ three times the contribution that we subtract, i.e. we have $\beta_f=3$. For completeness we shall briefly recall the argument. First following an argument similar to the one given below (3.21) for the gauge fields, one can show that the path integral measure for the gravitino fields $\psi_\mu$ corresponds to $\prod_{\mu,x} d(a\psi_\mu(x))$. To evaluate the integral we note that the fermion zero mode deformations correspond to local supersymmetry transformation ($\delta\psi_\mu \propto D_\mu\epsilon$) with supersymmetry transformation parameters $\epsilon$ which do not vanish at infinity. Now since the anti-commutator of two supersymmetry transformations correspond to a general coordinate transformation with parameter $\xi^\mu=\bar{\epsilon}\gamma^\mu\epsilon$, and since $\gamma^\mu \sim a^{-1}$, we conclude that $\epsilon_0=a^{-1/2}\epsilon$ provides a parametrization of the asymptotic supergroup in an $a$-independent manner. Writing $\delta(a\psi_\mu) \propto a^{3/2}D_\mu\epsilon_0$, and using the fact that the integration over the supergroup parameter $\epsilon_0$ produces an $a$ independent result, we now see that each fermion zero mode integration produces a factor of $a^{-3/2}$. Comparing this with the definition of $\beta_f$ given below (2.22) we get $\beta_f=3$.

Using (2.24) we now see that the net logarithmic contribution to the entropy from the gravitino fields is given by

$$-4\pi^2a^4 \ln A_H \left(-\frac{589}{720\pi^2a^4} - \frac{1}{2\pi^2a^4}(3-1)\right) = (1309/180)\ln A_H. \tag{5.31}$$

Adding this to the bosonic contribution given in (4.40) we get a net contribution of

$$\frac{23}{12}\ln A_H, \tag{5.32}$$

to the black hole entropy.

## 6 Half BPS black holes in $\mathcal{N} = 2$ supergravity coupled to matter fields

We shall now consider a more general $\mathcal{N} = 2$ supergravity theory containing $n_V$ vector multiplets and $n_H$ hypermultiplets. Since at quadratic order in the expansion around the near horizon background the fluctuations in the vector multiplet fields do not mix with the fluctuations in the hypermultiplet fields, we can evaluate separately the logarithmic correction to the entropy due to the vector multiplets and the hypermultiplets. The action involving these fields can be found in [72].

Let us begin with the vector multiplet fields. Suppose we have an $\mathcal{N} = 2$ supergravity theory coupled to $n_V$ vector multiplets. The coupling of the vector multiplet fields to supergravity will be described by the prepotential $F(\vec{X})$ which is a homogeneous function of degree 2 in $n_V + 1$ complex variables $X^0, \ldots, X^{n_V}$, with $X^k/X^0$ having the interpretation of the $n_V$ complex scalars in the $n_V$ vector multiplets. Now it has been shown in Appendix C that with the help of a symplectic transformation we can introduce new special coordinates $Z^A$ $(0 \le A \le n_V)$ in the vector multiplet moduli space such that

1. In the near horizon geometry $Z^k = 0 \quad \text{for} \quad k = 1, \ldots, n_V$.
2. The prepotential in the new coordinate system has the form:

$$F = -\frac{i}{2}\left((Z^0)^2 - \sum_{k=1}^{n_V}(Z^k)^2\right) + \cdots, \tag{6.1}$$

   where $\cdots$ denotes terms which are cubic and higher order in the $Z^k$'s and hence do not effect the action up to quadratic order in the fluctuations around the near horizon geometry.
3. The only non-vanishing background electromagnetic field in the near horizon geometry is $F^0_{mn}$ of the form:

$$F^0_{mn} = -2ia^{-1}\varepsilon_{mn}, \quad m, n \in AdS_2, \tag{6.2}$$

   in the gauge $Z^0 = 1$. Here $a$ denotes the radii of the near horizon $AdS_2$ and $S^2$.

With this choice of the prepotential, the relevant part of the bosonic action can be computed using the general formulæ given e.g. in [72]. We work in the gauge $Z^0 = 1$ and define a set of complex scalar fields $\phi^k$ through the equation:

$$Z^k = \frac{1}{2}\phi^k = \frac{1}{2}(\phi^k_R + i\phi^k_I). \tag{6.3}$$

Up to quadratic order in the fluctuations in the near horizon geometry the action given in [72] takes the form:

$$
\begin{aligned}
&\int d^4x\sqrt{\det g}\Big[R-\frac{1}{2}\partial_\mu\phi_R^k\partial^\mu\phi_R^k-\frac{1}{2}\partial_\mu\phi_I^k\partial^\mu\phi_I^k\\
&\quad-\frac{1}{4}\left\{1+\frac{1}{2}\sum_k\left((\phi_R^k)^2-(\phi_I^k)^2\right)\right\}F^{0\mu\nu}F_{0\mu\nu}\\
&\quad-\frac{1}{4}F^{k\mu\nu}F^k_{\mu\nu}-\frac{1}{2}\phi_R^kF^{0\mu\nu}F^k_{\mu\nu}+\frac{1}{2}\phi_I^k\widetilde{F}^{0\mu\nu}F^k_{\mu\nu}+\cdots\Big],
\end{aligned}
\tag{6.4}
$$

where $\cdots$ denotes terms cubic and higher order in the fluctuations and

$$
\widetilde{F}^{0\mu\nu}=\frac{1}{2}i\epsilon^{\mu\nu\rho\sigma}F^0_{\rho\sigma},\quad \epsilon^{mn\alpha\beta}=\varepsilon^{mn}\varepsilon^{\alpha\beta}.
\tag{6.5}
$$

Comparing (6.2) with (4.2) (or (6.4) with (4.1)) we see that $F^0_{\mu\nu}$ can be identified as $-2F_{\mu\nu}$ where $F_{\mu\nu}$ is the graviphoton field strength appearing in §4. The bosonic fields in the vector multiplet are the real scalar fields $\phi^k_{R,I}$ and the vector fields $A^k_\mu$ whose field strengths are given by $F^k_{\mu\nu}$. In the background (6.2) the action involving these fields to quadratic order is given by:

$$
\begin{aligned}
&\int d^4x\sqrt{\det g}\Big[-\frac{1}{4}F^{k\mu\nu}F^k_{\mu\nu}-\frac{1}{2}\partial_\mu\phi_R^k\partial^\mu\phi_R^k-\frac{1}{2}\partial_\mu\phi_I^k\partial^\mu\phi_I^k+a^{-2}\sum_{k=1}^{n_V}((\phi_R^k)^2\\
&\quad-(\phi_I^k)^2)+i\,a^{-1}\phi_R^k\varepsilon^{mn}F^k_{mn}+a^{-1}\,\phi_I^k\varepsilon_{\alpha\beta}F^k_{\alpha\beta}\Big].
\end{aligned}
\tag{6.6}
$$

Note the mass terms for the scalars and the mixing between the vector and the scalar fields appearing in the last two terms. This has exactly the same structure as the one which appeared in the analysis of the matter multiplet fields in $\mathcal{N}=4$ supergravity in [9]. Thus we can borrow the result of [9], which shows that the net contribution to the heat kernel from these fields, after taking into account the effect of the ghost fields, is given by $4K^s(0;s)$ for each vector multiplet, with $K^s$ given in (3.10). Since $\beta_v=1$ we do not need to give any special treatment to the zero modes of the vector fields.

Let us now turn to the contribution from the fermions in the vector multiplet. Each vector multiplet contains two Majorana fermions or equivalently one Dirac fermion. It can be shown using the results of [72] that for quadratic prepotential of the type we have, the kinetic operator of the vector multiplet fermions is the standard Dirac operator in the $AdS_2\times S^2$ background metric. Thus the heat Kernel is given by $K^f(0;s)$ given in (3.25). As a result the net contribution to the heat Kernel from each vector multiplet field is given by

$$
4K^s(0;s)+K^f(0;s)=\frac{4}{720\pi^2a^4}+\frac{11}{720\pi^2a^4}+\cdots=\frac{1}{48\pi^2a^4}+\cdots,
\tag{6.7}
$$

where as usual $\cdots$ represent terms containing other powers of $s$. This corresponds to a contribution to the entropy of $-\frac{1}{12}\ln A_H$ per vector multiplet.

Let us now turn to the hypermultiplet fields consisting of four real scalars and a pair of Weyl fermions. The four scalars are minimally coupled to the background gravitational field without any coupling to the graviphoton flux, and give a contribution of $4K^s(0; s)$. Each hypermultiplet contains a pair of Weyl fermions $\zeta_a$ ($a = 1, 2$) whose action in the Lorentzian theory, to quadratic order, is given by [72]

$$-\frac{1}{2}\bar{\zeta}^a \not{D}\zeta_a + \frac{1}{4}\bar{\zeta}^a \varepsilon_{ab} \Sigma_{\mu\nu} F^{0\mu\nu} \zeta^b + \text{h.c.}, \tag{6.8}$$

where $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, and $\zeta_a$ and $\bar{\zeta}^a$ are related as

$$\bar{\zeta}^a = (\zeta_a)^\dagger \gamma^0 = (\zeta^a)^T \widetilde{C}, \tag{6.9}$$

$\widetilde{C}$ being the charge conjugation operator. In writing down (6.8) we have already used the fact that for the background we are considering $F^0_{\mu\nu}$ is the only non-vanishing field strength. (6.9) can be taken as the definition of $\zeta^a$ in terms of $\zeta_a$. Since $\zeta^a$ defined via (6.9) has opposite chirality of $\zeta_a$, we can define a Majorana spinor $\chi^a$ via

$$\chi^a = \zeta_a + \zeta^a, \tag{6.10}$$

and express the action as

$$-\frac{1}{2}\bar{\chi}^a \not{D}\chi^a + \frac{1}{4}\bar{\chi}^a \varepsilon_{ab} \Sigma_{\mu\nu} F^{0\mu\nu} \chi^b. \tag{6.11}$$

This can now be continued to Euclidean space with $\bar{\chi}^a \equiv (\chi^a)^T \widetilde{C}$. Using the explicit form of the $\gamma$ matrices given in (A.14) and the background value of $F^0_{\mu\nu}$ given in (6.2) we get

$$-\frac{1}{2}\bar{\chi}^a \not{D}\chi^a - \frac{1}{2} a^{-1} \bar{\chi}^a \varepsilon_{ab} \tau_3 \chi^b. \tag{6.12}$$

Thus the kinetic operator is given by

$$\begin{aligned} &\delta_{ab} \not{D} + a^{-1} \varepsilon_{ab} \tau_3 = \mathcal{D}_1 + \mathcal{D}_2, \quad \mathcal{D}_1 \equiv \delta_{ab} \not{D}_{S^2} + a^{-1} \varepsilon_{ab} \tau_3, \\ &\mathcal{D}_2 \equiv \delta_{ab} \sigma_3 \not{D}_{AdS_2}. \end{aligned} \tag{6.13}$$

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ anti-commute we have $(\mathcal{D}_1 + \mathcal{D}_2)^2 = (\mathcal{D}_1)^2 + (\mathcal{D}_2)^2$. The eigenvalues of $\mathcal{D}_2^2$ are given by $-\lambda^2/a^2$. On the other hand since $\not{D}_{S^2}$ has eigenvalues $\pm i(l+1)a^{-1}$, and $-a^{-1}\varepsilon_{ab}\tau_3$ has eigenvalues $\pm i\, a^{-1}$, and these operators act on different spaces, the eigenvalues of $\mathcal{D}_1$ are given by $\pm i(l+1\pm 1)a^{-1}$. Thus $(\mathcal{D}_1)^2 + (\mathcal{D}_2)^2$ has eigenvalues $-(l+1\pm 1)^2/a^2 - \lambda^2/a^2$ and the net contribution to the heat kernel from the two Majorana fermions in the hypermultiplet is given by

$$-\frac{1}{2\pi^2 a^4}\int_0^\infty d\lambda e^{-\bar{s}\lambda^2}\,\lambda\,\coth(\pi\lambda)\sum_{l=0}^{\infty}(l+1)\left[e^{-\bar{s}(l+2)^2}+e^{-\bar{s}l^2}\right]. \tag{6.14}$$

We can evaluate this in two different ways—either by shifting $l \to l \mp 1$ in the two terms as in [9], or by directly expressing this as a double integral and using Eqs. (B.3), (B.4). We shall follow the second approach and express (6.14) as

$$-\frac{1}{2\pi^2 a^4}\mathrm{Im}\int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\,\widetilde{\lambda}\,\cot(\pi\widetilde{\lambda})\int_0^\infty d\lambda\,\lambda\,\coth(\pi\lambda)e^{-\bar{s}\widetilde{\lambda}^2-\bar{s}\widetilde{\lambda}^2}\left[e^{-2\bar{s}\widetilde{\lambda}-\bar{s}}+e^{2\bar{s}\widetilde{\lambda}-\bar{s}}\right]. \tag{6.15}$$

The terms in the square bracket can now by expanded in a power series in $\bar{s}$ and we can evaluate the integrals using (B.3), (B.4). The resulting constant term in the small $\bar{s}$ expansion of the expression is given by $-19/720\pi^2 a^4$. Combining this with the contribution $4/720\pi^2 a^4$ from the bosonic contribution $4K^s(0;s)$, we get

$$K^{hyper}(0;s) = -\frac{1}{48\pi^2 a^4}+\cdots. \tag{6.16}$$

This corresponds to a contribution of $\frac{1}{12}\ln A_H$ per hypermultiplet. Combining (5.32) with the results of this section we see that an $\mathcal{N}=2$ supergravity theory with $n_V$ vector multiplets and $n_H$ hypermultiplets will have a logarithmic correction to the entropy given by

$$\frac{1}{12}(23+n_H-n_V)\,\ln\,A_H. \tag{6.17}$$

## 7 Local method, duality anomaly and ensemble choice

In this section we shall discuss an alternative derivation of the results for $\mathcal{N}=2$ supergravity using local methods. Indeed, with hindsight we could have read out these results from those in [33] which computed the trace anomalies due to various fields in gauged supergravity theories. For this we begin with the generalized version of (3.29) including the effect of $n_{3/2}$ Majorana spin 3/2 field and $n_2$ spin 2 fields. Then (3.29) takes the form [64–69] (for a recent review see [54])[10]

$$K_0 = -\frac{1}{90\pi^2}(n_S+62n_V+11n_F)E-\frac{1}{30\pi^2}(n_S+12n_V+6n_F \\ -\frac{233}{6}n_{3/2}+\frac{424}{3}n_2)I, \tag{7.1}$$

[10] For metric and spin 3/2 fields the individual coefficients multiplying $E$ and $I$ are gauge dependent [64] but the coefficient of $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is gauge independent. As we shall see, this will be the only relevant coefficient that enters our analysis.

$$E = \frac{1}{64}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right),$$
$$I = -\frac{1}{64}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right). \tag{7.2}$$

Now in the near horizon background we are interested in, we also have background gauge fields besides the background metric, and so we cannot apply (7.1) directly. But we can try to use supersymmetry to find the supersymmetric completion of these terms. Of these since $E$ is a topological term, it is supersymmetric by itself and does not require the addition of any other term. On the other hand supersymmetrization of $I$ has been carried out in [73–75]. Although the resulting action is quite complicated, it is known that supersymmetrization of $I$, evaluated in the near horizon background of the black hole [75–80], takes the same value as $-E$ [6,81] even though $I$ itself vanishes in the near horizon geometry and $E$ does not vanish.[11] Thus for our analysis we can replace the supersymmetrized $I$ by $-E$ on the right hand side of (7.1). This gives

$$K_0 = -\frac{1}{90\pi^2}(-2n_S + 26n_V - 7n_F + \frac{233}{2}n_{3/2} - 424n_2)E. \tag{7.3}$$

Using $E = -1/8a^4$ for the $AdS_2 \times S^2$ background, we get

$$K_0 = \frac{1}{720\pi^2 a^4}(-2n_S + 26n_V - 7n_F + \frac{233}{2}n_{3/2} - 424n_2). \tag{7.4}$$

These coefficients agree with those given in [33]. Using this result we can reproduce all the results of the previous sections for $\mathcal{N} \geq 2$ supergravity theories correctly. For example for the hypermultiplet we have $n_S = 4$, $n_F = 1$ leading to $K_0 = -1/48\pi^2 a^4$ in agreement with (6.16). On the other hand for vector multiplets we have $n_V = 1$, $n_S = 2$ and $n_F = 1$ leading to $K_0 = 1/48\pi^2 a^4$ in agreement with (6.7). For the $\mathcal{N} = 2$ supergravity multiplet we have $n_2 = 1, n_{3/2} = 2$ and $n_V = 1$ leading to $K_0 = -11/48\pi^2 a^4$. This agrees with the sum of (4.36) and (5.29). For $\mathcal{N} = 4$ supergravity multiplet we have $n_2 = 1, n_{3/2} = 4, n_V = 6, n_F = 2$ and $n_S = 2$ leading to $K_0 = 1/4\pi^2 a^4$ and for $\mathcal{N} = 8$ supergravity we have $n_2 = 1, n_{3/2} = 8, n_V = 28, n_F = 28$ and $n_S = 70$, leading to $K_0 = 5/4\pi^2 a^4$. These results agree with the corresponding results in [10]. In each of these cases however, the effect of zero modes needs to be accounted for separately.

Even though this analysis appears to be simpler than the one carried out in the previous sections, it requires us to assume that there are no other local four derivative supersymmetric terms that could contribute to $K_0$, or, if such terms are present, they must vanish when evaluated in the near horizon geometry of the black hole.[12] In contrast the analysis of the previous sections does not require any such assumption

[11] This could be due to the fact that supersymmetrization of $I$ and $-E$ are equivalent via a field redefinition since they have the same coefficient of the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term, but we shall not need this stronger result.

[12] For a recent discussion on possible higher derivative terms in $\mathcal{N} = 2$ supergravity, see [82].

since we compute the complete contribution to $K_0$ in the near horizon geometry of the black hole.

[66] found an ambiguity in computing the coefficient of $E$ in the trace anomaly: if we replace a field by its dual field—e.g., a scalar field by a 2-form field—the coefficient of $E$ changes. A recent discussion on this in the context of black hole entropy can be found in [83]. This has been understood as due to the contribution to the trace anomaly from the zero modes [33,84]. Using this ambiguity [33] suggested replacing the scalar field by the 2-form field since that is what appears naturally in string theory. The resulting contribution to $K_0$ agrees with the result of direct string computation in [34–36], and would also produce correctly the coefficient of the log term in (1.1) without having to give special treatment to the zero modes. This procedure of replacing a scalar by a 2-form field would also reproduce correctly the zero result given in (1.2) for $\mathcal{N} = 4$ supersymmetric theories. This however is a coincidence; it just so happens that the extra term we get by first removing the contribution from the metric and the gravitino zero modes to the heat kernel and then carrying out separately the integration over these zero modes is the same as the extra term we get in computation of the coefficient of $E$ if we replace the scalar field by a 2-form field. A similar replacement for type II string theory on a torus (where several scalars need to be replaced by 2-form fields and we also need to include the contribution from some non-dynamical 3-form fields) will give zero coefficient of the logarithmic correction [33] while the correct coefficient as given in (1.2) is $-4$. In contrast the procedure we suggest gives the correct answer matching the microscopic results in the $\mathcal{N} = 4$ and 8 supersymmetric theories where the microscopic results are known.

Also note that our procedure for computing the coefficient of the logarithmic correction does not suffer from the ambiguity described in the previous paragraph, since we remove the zero mode contribution from the heat kernel completely, and then integrate separately over the zero modes of the physical fields. Even in this case one might have expected an ambiguity depending on which duality frame we use, since the zero modes over which we integrate depend on this frame. This is however fixed by the physical problem at hand. Let us for example consider adding to the theory a non-dynamical 3-form field. In this case the non-zero mode contribution to the heat kernel vanishes, but integration over the zero modes could produce non-zero contribution. To be more specific, the dimensional reduction of the 3-form field on $S^2$ gives a gauge field on $AdS_2$ which has a set of zero modes. If we are to integrate over these zero modes then we would get some additional logarithmic correction to the entropy. However in this case the ensemble that it represents will have the charge associated with this gauge field fixed. This will correspond to membrane charge wrapped on $S^2$. This is not a physical gauge charge from the point of view of an asymptotic observer in the four dimensional Minkowski space-time and hence should not be fixed in the ensemble. This is turn shows that we should not be integrating over the zero modes of the gauge fields sourced by this membrane charge. Thus we see that the physical ensemble we want to calculate the entropy in automatically fixes the duality frame. This in turn fixes the relevant zero modes over which we need to integrate.

## 8 Multi-centered black hole solutions

Our analysis of logarithmic corrections refers to single centered black hole solutions only. However the microscopic counting formula does not distinguish between the contributions from single and multi-centered contributions—it simply counts the total index/degeneracy for a given total charge. Thus if we are to compare our results with the result of microscopic counting when such results become available, we need to either include the contribution from multi-centered black holes or argue that such contributions are small compared to that of single centered black holes.

There are two types of multi-centered black hole solutions we can consider. If the total charge carried by the black hole is non-primitive, i.e. can be written as an integral multiple of another charge vector, then the total charge can be distributed among multiple centers, carrying parallel charge vectors. These solutions exist for arbitrary values of the asymptotic values of the moduli scalar fields. Furthermore in this case the positions of the centers are arbitrary, and the centers can come arbitrarily close to each other producing an intermediate $AdS_2 \times S^2$ throat associated with the near horizon geometry of the single centered black hole carrying the same total charge. As we go down the throat, it splits into multiple $AdS_2 \times S^2$ throats each carrying a fraction of the total flux vector, and representing the near horizon geometry of individual centers. This phenomenon is known as the anti-de Sitter fragmentation [85] via Brill instantons [86]. This can however be avoided by taking the total charge vector to be primitive since in this case it is not possible for the total charge vector to split into a set of parallel charge vectors.

The second class of multi-centered solutions arise from the mechanism discussed in [28,87–89]. In this case the charges carried by the centers are not parallel and there are certain constraints among the relative distances between the centers. The (non-)existence of these solutions depends on the asymptotic values of the moduli scalar fields, and most of these solutions cease to exist if we set the asymptotic values of the moduli fields to be equal to their attractor values—the values they take in the near horizon geometry of a single centered black hole carrying the same total charge. Nevertheless [28] pointed out the existence of a class of solutions which exist even when the asymptotic values of the scalar fields are set equal to their attractor values. These solutions are known as scaling solutions since in one corner of the space of parameters labelling these solutions the distances between the centers go to zero. This leads to a phenomenon similar to anti-de Sitter fragmentation [90].

The existence of these scaling solutions could cause potential problem for comparing our macroscopic results with any microscopic result since we need to add the contribution from the scaling solutions to the single centered entropy before comparing it to the microscopic results. A general formula for computing the contribution to the index from these solutions was given in [91] generalizing the results of [92,93]. It takes the form

$$f(\{\vec{q}_{(i)}\}, \{\vec{p}_{(i)}\}) \prod_i d(\vec{q}_{(i)}, \vec{p}_{(i)}), \tag{8.1}$$

when the charges carried by the individual centers are not identical. Here $(\vec{q}_{(i)}, \vec{p}_{(i)})$ denote the electric and the magnetic charge vectors carried by the $i$th center, $d(\vec{q}, \vec{p})$ is the contribution to the index from a single centered black hole carrying charge $(\vec{q}, \vec{p})$ and $f(\{\vec{q}_{(i)}, \vec{p}_{(i)}\})$ is a function of the charges carried by all the centers, representing the contribution to the index from the quantum system describing the relative motion between the centers. When some of the centers carry identical charges the result gets modified [91], but not in a way that invalidates our discussion below. The contribution from these configurations could dominate the single centered contribution in two ways: the number of such multi-centered configurations could be exponentially large, giving a contribution to the entropy that is of the same order or larger that that of the single centered contribution to the entropy, or individual terms could dominate over the entropy of single centered black holes. For the special case of D6-$\bar{\text{D}}$6-D0 systems the number of configurations was estimated in [93], and although it grows exponentially with the charge, the power of the charge in the exponent was found to be smaller than 2. Given the rarity of scaling solutions to be discussed shortly, we believe that this is probably a generic features of these solutions. Furthermore there can also be cancelations between the contributions from different configurations if they contribute to the index with opposite signs. In order to estimate the contribution from the individual terms we use the result of [91] from which it follows that while the index of individual centers could grow exponentially with the charges, the function $f(\{\vec{q}_{(i)}, \vec{p}_{(i)}\})$ grows polynomially with the charges. Thus in order for (8.1) to dominate or be of the same order as the contribution from the single centered black hole, $\sum_i \ln |d(\vec{q}_{(i)}, \vec{p}_{(i)})|$ should either exceed or be of the same order as $\ln \left|d\left(\sum_i \vec{q}_{(i)}, \sum_i \vec{p}_{(i)}\right)\right|$—the latter representing the contribution to the entropy from a single centered black hole with total charge $\left(\sum_i \vec{q}_{(i)}, \sum_i \vec{p}_{(i)}\right)$. For this reason it is important to classify all the scaling solutions carrying a given total charge and examine if their contribution could dominate or be of the same order as the contribution from a single centered black hole.[13]

Let us now review the condition under which the scaling solutions exist. We shall describe the solution in the limit when all the centers come close to each other since the (non-)existence of the solution in this limit will imply (non-)existence of the whole family. If we define

$$\alpha_{ij} = \vec{q}_{(i)} \cdot \vec{p}_{(j)} - \vec{q}_{(j)} \cdot \vec{p}_{(i)}, \tag{8.2}$$

and $\vec{x}_{(i)}$ denotes the position of the $i$-th center, then these positions are constrained by the requirement [28]:

$$\sum_{j, j\neq i} \frac{\alpha_{ij}}{|\vec{x}_{(i)} - \vec{x}_{(j)}|} = 0 \quad \forall i. \tag{8.3}$$

For three centered black hole this translates to the condition that $\alpha_{12}$, $\alpha_{23}$ and $\alpha_{31}$ have the same sign and satisfy the triangle inequality so that they form three sides of a

[13] It has been suggested by Frederik Denef that the sum of the classical entropies of the individual centers could not possibly exceed that of the single centered black hole since this will violate the holographic bound. Although there is no direct proof of this, some special cases have been discussed in [94].

triangle. Another requirement comes from the regularity of the metric. Let the entropy of a single centered BPS black hole carrying charge $(\vec{q}, \vec{p})$ be denoted by $\pi\sqrt{D(\vec{q}, \vec{p})}$. Then the regularity condition takes the form

$$D(\vec{h}(\vec{x}), \vec{g}(\vec{x})) > 0 \quad \forall \vec{x}, \quad \vec{h}(\vec{x}) \equiv \sum_i \frac{\vec{q}_{(i)}}{|\vec{x} - \vec{x}_{(i)}|}, \quad \vec{g}(\vec{x}) \equiv \sum_i \frac{\vec{p}_{(i)}}{|\vec{x} - \vec{x}_{(i)}|}. \tag{8.4}$$

Note that while (8.3) is independent of the details of the theory e.g. the prepotential, (8.4) is sensitive to the details of the theory since the function $D(\vec{q}, \vec{p})$ depends on the prepotential. There are further requirements, e.g. the matrix multiplying the gauge kinetic term, which is a function of the vector multiplet scalars, must be positive definite everywhere in space. These conditions also depend on the prepotential.

For two centered black holes (8.3) requires $\alpha_{12}$ to vanish. In this case the function $f(\{\vec{q}_{(i)}, \vec{p}_{(i)}\})$ turns out to be proportional to $\alpha_{12}$ and as a result two centered scaling solutions do not contribute to the index. However there are plenty of solutions to (8.3) involving three or more centers, giving rise to potential contributors to the index. The condition (8.4) as well as the requirement of a positive definite gauge kinetic term has been less studied since this has to be done on a case by case basis as it depends on the details of the theory. Manschot et al. [91] considered a special example of a theory with a single vector multiplet with prepotential $-(X^1)^3/6X^0$ and found that a 3-centered solution to (8.3), with each center described by a regular event horizon, fails to satisfy (8.4). This leads us to suspect that the scaling solutions may be rare and may not be a potential competitor to the contribution to the index from a single centered black hole. We shall now describe the results for some simple systems.

First we consider pure supergravity, or more generally supergravity coupled to hypermultiplets but no vector multiplets. Such theories can arise from type IIB string theory on Calabi–Yau manifolds which do not admit any deformation of the complex structure. In this case we do not expect any non-singular multi-centered solutions with non-parallel charges since the only forces are due to gravity and electromagnetism, and for non-parallel charges the gravitational force wins over the electromagnetic force. This argument of course ignores the non-linear effects of gravity and in order to have a convincing result we need to analyze the possibility of simultaneous solutions to (8.3), (8.4). In this case the charge vectors are one dimensional and $D(q, p) \propto (q^2 + p^2)$. Thus the only way (8.4) can fail is if the functions $h(\vec{x})$ and $g(\vec{x})$ both vanish at the same point, i.e. the surfaces $f(\vec{x}) = 0$ and $g(\vec{x}) = 0$ intersect. It was shown in [94] that for three centered solutions these surfaces always intersect, showing the absence of scaling solutions. For larger number of centers a general proof of absence does not exist, but none have been found so far in numerical searches.

We have also examined the solution to (8.4) in the one vector multiplet model with prepotential $-(X^1)^3/6X^0$. Here we have [95]

$$D(p^0, p^1, q_1, q_0) = \frac{1}{9}\Big[3(q_1 p^1)^2 - 18 q_0\, p^0\, q_1\, p^1 - 9 q_0^2\, (p^0)^2 - 6 (p^1)^3\, q_0 + 8 p^0\, (q_1)^3\Big]. \tag{8.5}$$

In this case there are known examples of scaling solutions satisfying (8.4), e.g. the D6-$\bar{\text{D}}$6-D0 system discussed in [28,91–93]. These solutions by themselves have individual centers carrying zero entropy, but by adding sufficiently small amount of charges to each center we can ensure that the each center has non-zero (although small) entropy and yet the solution continues to satisfy the condition (8.4).[14] Nevertheless it is instructive to explore how pervasive these solutions are. For this we have randomly generated the charges carried by the three centers and picked among them those sets for which $\alpha_{12}, \alpha_{23}$ and $\alpha_{31}$ satisfy the triangle inequality and the discriminant $D$ is positive for each center as well as for the total charge carried by all the centers. For each of these sets we then test the positivity of $D(\vec{f}(\vec{x}), \vec{g}(\vec{x}))$ as a function of $\vec{x}$. We find that in each of the 30 examples generated this way, $D$ fails to be positive in some region of space.

While we do not have any rigorous result, the results reviewed in this section indicate that scaling solutions satisfying (8.3) and (8.4) simultaneously are rare. This in turn gives us reason to hope that at least in some of the theories the contribution from the single centered black holes dominate the index, and we can directly compare our results for logarithmic corrections to the microscopic results. It will clearly be useful to have a better analytic understanding of the problem.

## 9 Comparison with the OSV formula

In this section we shall compare our result with various versions of the OSV formula [27]. In a nutshell an OSV type formula is a proposal for the asymptotic expansion of the black hole entropy in the large charge limit, giving the expression for the entropy as a function of the charges to all orders in an expansion in inverse powers of charges. In particular any such formula will give a definite predictions for the logarithmic corrections to the entropy which are the first subleading corrections to the Bekenstein–Hawking entropy. Thus it can be compared with (1.1).

We begin with the version of the OSV formula proposed in [28]. Although this formula was derived for a limit of the charges different from the one we are considering, we shall go ahead with the assumption that it is valid also in the limit in which all the charges are scaled uniformly i.e. for 'weak topological string coupling' and at the attractor point in the moduli space where single centered black hole gives the dominant contribution to the entropy. If the theory has $n_V$ vector multiplets and is described by the prepotential $F(X^0, \ldots, X^{n_V})$, then the relevant part of the formula for the index of a single centered black hole carrying electric charges $\{q_I\}$ and magnetic charges $\{p^I\}$ is given by

$$e^{S_{BH}} = \text{constant} \times \int \prod_{I=0}^{n_V} d\phi^I\, e^{-\pi\phi^I q_I}\, |g_{top}|^{-2}\, e^{-K}\, |Z_{top}|^2, \tag{9.1}$$

where

$$e^{-K} = i(\bar{X}^I F_I - X^I \bar{F}_I), \quad X^I = \phi^I + ip^I, \tag{9.2}$$

$$Z_{top} = \left(\frac{g_{top}}{2\pi}\right)^{\chi/24} \exp[-i\frac{\pi}{2}F(X) + \cdots] \tag{9.3}$$

[14] I wish to thank Frederik Denef for suggesting this construction.

and

$$g_{top} = \frac{4\pi}{X^0}. \tag{9.4}$$

$\chi$ is the euler character of the Calabi–Yau threefold on which type IIA string theory is compactified to produce the $\mathcal{N} = 2$ supersymmetric string theory. It is related to $n_H$ and $n_V$ via:

$$\chi = 2(n_V - n_H + 1). \tag{9.5}$$

The $\left(g_{top}/2\pi\right)^{\chi/24}$ factor was not present explicitly in the original OSV definition of $Z_{top}$ but first made its appearance in [96]. $\cdots$ in (9.3) denotes additional terms containing non-negative powers of $g_{top}$ and non-trivial functions of $X^k/X^0$ and will not be relevant for our analysis. Finally it must be mentioned that the analysis of [28] was carried out for $p^0 = 0$ i.e., real $g_{top}$.

Let us now consider the limit in which all the charges are scaled by a large parameter $\Lambda$: $(q^I, p^I) \to (\Lambda q^I, \Lambda p^I)$. Under this rescaling $A_H \to \Lambda^2 A_H$. We now try to evaluate the integration over $\phi^I$ using saddle point method. To leading order the relevant saddle point lies at the extremum of

$$-\pi\phi^I q_I + \pi \,\mathrm{Im}\, F, \tag{9.6}$$

and sets $\phi^I$—the real parts of $X^I$—to be equal to the attractor values of the electric fields given in (C.9) in the $w = 8$ gauge. Since $F$ is a homogeneous function of degree 2 in the $X^I$'s and since $q^I$ and $\mathrm{Im}(X^I) = p^I$ scale as $\Lambda$, it follows that the saddle point values of $\phi^I$ also scale as $\Lambda$. Furthermore since the second derivatives of $\mathrm{Im} F$ with respect to $\phi^I$ scale as $\Lambda^0$, the determinant from the $\phi$ integral has no $\Lambda$ dependence. Finally $e^{-K}$ scales as $\Lambda^2$ and $g_{top}$ scales as $\Lambda^{-1}$. From (9.1) we now see that in the large $\Lambda$ limit

$$\begin{aligned} e^{S_{BH}} &= C(\vec{q}, \vec{p})\, e^{-\pi\phi^I q_I + \pi \mathrm{Im}\, F}\, \Lambda^{(4-\frac{\chi}{12})} \\ &= C(\vec{q}, \vec{p})\, \exp\left[-\pi\phi^I q_I + \pi \mathrm{Im}\, F + \frac{1}{12}(23 - n_V + n_H)\ln\Lambda^2\right], \end{aligned} \tag{9.7}$$

where $C(\vec{q}, \vec{p})$ represents sum of terms which scale as $\Lambda^n$ for $n \le 0$. The $-\pi\phi^I q_I + \pi \mathrm{Im}\, F$ term has to be evaluated at the saddle point and gives the classical Bekenstein–Hawking entropy $A_H/4G_N$. Since this scales as $\Lambda^2$, we can replace $\ln\Lambda^2$ by $\ln(A_H/G_N)$ at the cost of redefining the order one multiplicative factor $C(\vec{q}, \vec{p})$. This precisely agrees with (1.1).

There are other proposals for modifying the OSV formula by introducing an additional measure. For example at the order in which we are working, the measure used in [29,30] differs from that of [28] by a multiplicative factor of $\exp\left[\left(2 - \frac{\chi}{24}\right) K\right]$. This makes the measure a homogeneous function of degree zero in the $X^I$'s and predicts zero coefficient of the logarithmic correction in contradiction to (1.1).

Given that the OSV formula has played an important role in our search for an exact/approximate formula for the black hole entropy in $\mathcal{N} = 2$ supersymmetric string theories, it will be useful to explore in some detail the significance of possible agreement and disagreement between different formulæ. The original proposal of OSV [27] made use of the observation that the Wald entropy of a black hole in $\mathcal{N} = 2$ string theory, corrected by higher derivative terms [77,78], is given by the Legendre transform of $\ln |Z_{top}|^2$ where $Z_{top}$ is the topological string partition function. OSV then suggested that the exact index is given by the Laplace transform of $|Z_{top}|^2$—this reduces to the exponential of the Legendre transform of $\ln |Z_{top}|^2$ in the saddle point approximation. There were however indications that this cannot be completely correct (see e.g. [15,96–98]), one needs to include additional measure factor in the integral while performing the Laplace transform. If we are allowed to choose the measure freely then any correction to the leading entropy can be encoded in an appropriate factor in the measure, at least order by order in an expansion in inverse powers of charges. Thus in order to make OSV formula useful one must have an a priori description of the measure. Denef and Moore [28] derived the measure from an indirect microscopic analysis of the degeneracy of D4–D2–D0 system wrapped on appropriate cycles of a Calabi–Yau manifold.[15] Modular invariance of the partition function allowed them to use Rademacher expansion and express the partition function in terms of the index associated with polar states—states carrying special charge vectors—and they then identified the polar states which give dominant contribution to the entropy. However since their analysis only keeps a subset of the terms in the full Rademacher expansion, there are error terms. It was found that while the error terms are small for a certain range of charges (in particular when the D0-brane charge is large), in general there is no guarantee that they will be small when all the charges are scaled uniformly. Indeed it will require surprising cancellations for their formula to be valid for this range of charges. Thus while the agreement of our Eq. (1.1) with [28] indicates that such cancellations might be present, at present we should treat this agreement as accidental. It is however encouraging to note that there have been independent indications that such cancellations might take place [102].

In contrast [29,98] started from a different perspective, using symplectic invariance as the basic principle.[16] OSV formula treats electric and magnetic charges differently, and to generalize this to a symplectic invariant from refs. [29,98] had to begin with an integral that involves double the number of integration variables. They then recovered the OSV type integral by integrating out half of the variables using saddle point approximation. However symplectic invariance by itself does not completely fix the form of the original integrand—this has to be fixed using the knowledge of the effective action. Using the known local terms in the one loop effective action and their effect on the black holes entropy [29] suggested a specific measure that differs from the measure of [28] by a factor of $\exp\left[\left(2 - \frac{\chi}{24}\right) K\right]$ to the order at which we are ana-

[15] For other attempts to derive OSV conjecture see [99–101].

[16] Symplectic invariance does not necessarily refer to a symmetry of the OSV formula, but represents the fact that we could change the electric and magnetic charges by a symplectic transformation and at the same time change the prepotential according to the specified rules without changing the value of the integral. In special cases when the prepotential remains invariant under such a transformation, the transformation may be a genuine duality symmetry of the theory.

lyzing the entropy. However since $K$ is invariant under a symplectic transformation, we could multiply the original integrand of [29] by a factor of $\exp\left[-\left(2-\frac{\chi}{24}\right)K\right]$ without violating symplectic invariance. Then to this order the results of [29] and [28] would agree and will both be consistent with (1.1). Multiplying the integrand of [29] by $\exp\left[-\left(2-\frac{\chi}{24}\right)K\right]$ corresponds to adding to the effective action a non-local but symplectic invariant term beyond the local terms considered in [29].

In fact the quantum entropy function formalism that we are using for computing the entropy is designed to precisely take into account the contribution to the black hole entropy from both the local and the non-local terms in the 1PI effective action. The effect of local terms can also be taken into account using Wald's formula, and for these quantum entropy function will give the same result as Wald's formula. However Wald's formula is not directly applicable to the non-local terms in the effective action. Quantum entropy function takes such corrections into account by directly evaluating the path integral of string theory in the near horizon geometry which, by virtue of the intrinsic curvature of $AdS_2$, comes with an automatic infrared cut-off. This allows us to treat the non-local terms as corrections to the local effective Lagrangian density. This can be seen from Eq. (2.13)—it describes a correction to $\mathcal{L}_{eff}$ which has logarithmic dependence on the radius of curvature $a$ of $AdS_2$ but is otherwise infrared finite. The logarithmic dependence on $a$ shows that these terms are non-analytic in the $a\to\infty$ i.e., flat space limit. The other ingredient of [29]—symplectic invariance—is also implicitly built in our formalism since quantum entropy function is expressed as a functional integral over all the fields in the theory. Symplectic transformation can be implemented explicitly at the level of path integral, and using this we can formally transform the expression for the quantum entropy function written in one duality frame to the expression written in another duality frame.

Thus we conclude that while our result is in conflict with the explicit form for the OSV integral that appears in [29], there is no disagreement between the basic principles of [29] and the quantum entropy function formalism. The cause of the explicit disagreement can be traced to certain non-local terms in the one loop effective action which have been included in our analysis but were not present in [29]. On the other hand the agreement of our result with that of [28] seems somewhat accidental since the latter was derived for a different scaling limits of charges instead of the uniform scaling limit used here, and at a different point in the moduli space where multi-centered black holes could give dominant contribution to the index. It will be interesting to explore if due to some underlying miraculous cancellation the formula given in [28] could be an exact asymptotic expansion of the index of a single centered black hole in the large charge limit, giving the result to all orders in inverse powers of $\Lambda$. While order by order analysis is not suited for this study, localization methods discussed in [55,56] could help prove or disprove such a claim.

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## Appendix A: The basis functions in $AdS_2 \times S^2$

In this appendix we shall review the results on eigenfunctions and eigenvalues of the Laplacian operator $\Box \equiv g^{\mu\nu}D_\mu D_\nu$ on $AdS_2$ and $S^2$ for different tensor and spinor fields following [62,103–105]. First consider the Laplacian acting on the scalar fields. On $S^2$ the normalized eigenfunctions of $-\Box$ are just the usual spherical harmonics $Y_{lm}(\psi,\phi)/a$ with eigenvalues $l(l+1)/a^2$. On the other hand on $AdS_2$ the $\delta$-function normalized eigenfunctions of $-\Box$ are given by [103][17]

$$f_{\lambda,\ell}(\eta,\theta) = \frac{1}{\sqrt{2\pi\, a^2}}\,\frac{1}{2^{|\ell|}(|\ell|)!}\left|\frac{\Gamma\left(i\lambda+\frac{1}{2}+|\ell|\right)}{\Gamma(i\lambda)}\right| e^{i\ell\theta}\sinh^{|\ell|}\eta$$
$$F\left(i\lambda+\frac{1}{2}+|\ell|, -i\lambda+\frac{1}{2}+|\ell|; |\ell|+1; -\sinh^2\frac{\eta}{2}\right),$$
$$\ell\in\mathbb{Z},\quad 0<\lambda<\infty, \tag{A.1}$$

with eigenvalue $\left(\frac{1}{4}+\lambda^2\right)/a^2$. Here $F$ denotes hypergeometric function.

The normalized basis of vector fields on $S^2$ may be taken as

$$\frac{1}{\sqrt{\kappa_1^{(k)}}}\,\partial_\alpha U_k, \qquad \frac{1}{\sqrt{\kappa_1^{(k)}}}\,\varepsilon_{\alpha\beta}\partial^\beta U_k, \tag{A.2}$$

where $\{U_k\}$ denote normalized eigenfunctions of the scalar Laplacian with eigenvalue $\kappa_1^{(k)}$. The basis states given in (A.2) have eigenvalue of $-\Box$ equal to $\kappa_1^{(k)} - a^{-2}$. Note that for $\kappa_1^{(k)} = 0$, i.e. for $l=0$, $U_k$ is a constant and $\partial_\alpha U_k$ vanishes. Hence these modes do not exist for $l=0$.

Similarly a normalized basis of vector fields on $AdS_2$ may be taken as

$$\frac{1}{\sqrt{\kappa_2^{(k)}}}\,\partial_m W_k, \qquad \frac{1}{\sqrt{\kappa_2^{(k)}}}\,\varepsilon_{mn}\partial^n W_k, \tag{A.3}$$

where $W_k$ are the $\delta$-function normalized eigenfunctions of the scalar Laplacian with eigenvalue $\kappa_2^{(k)}$. The basis states given in (A.3) have eigenvalues of $-\Box$ equal to $\kappa_2^{(k)} + a^{-2}$. There are also additional square integrable modes of eigenvalue $a^{-2}$, given by [103]

$$A = d\Phi^{(\ell)}, \quad \Phi^{(\ell)} = \frac{1}{\sqrt{2\pi|\ell|}}\left[\frac{\sinh\eta}{1+\cosh\eta}\right]^{|\ell|} e^{i\ell\theta}, \quad \ell=\pm1,\pm2,\pm3,\ldots. \tag{A.4}$$

These are not included in (A.3) since the $\Phi^{(\ell)}$ given in (A.4) is not normalizable. $d\Phi$ given in (A.4) is self-dual or anti-self-dual depending on the sign of $\ell$. Thus we do not

[17] Although often we shall give the basis states in terms of complex functions, we can always work with a real basis by choosing the real and imaginary parts of the function.

get independent eigenfunctions from $*d\Phi^{(\ell)}$. However we can also work with a real basis in which we take $d\text{Re}(\Phi^{(\ell)})$ and $d\text{Im}(\Phi^{(\ell)}) \propto *d\text{Re}(\Phi^{(\ell)})$ as the independent basis states for $\ell > 0$. The basis states (A.4) satisfy

$$\sum_\ell g^{mn}\partial_m\Phi^{(\ell)*}(x)\partial_n\Phi^{(\ell)}(x) = \frac{1}{2\pi a^2}. \tag{A.5}$$

We have derived this using the fact that due to homogeneity of $AdS_2$ this sum is independent of $x$, and that at $\eta = 0$ only the $\ell = \pm 1$ terms contribute to the sum. Thus the total number of such discrete modes of spin 1 field on $AdS_2$ is given by

$$\begin{aligned} N_1 &= \int_{AdS_2} d^2x\,\sqrt{g_{AdS_2}}\sum_\ell g^{mn}\partial_m\Phi^{(\ell)*}(x)\partial_n\Phi^{(\ell)}(x) \\ &= \frac{1}{2\pi}\int_0^{\eta_0}\sinh\eta\,d\eta\int d\theta = \cosh\eta_0 - 1. \end{aligned} \tag{A.6}$$

A similar choice of basis can be made for a symmetric rank two tensor representing the graviton fluctuation. For example on $S^2$ we can choose a basis of these modes to be

$$\frac{1}{\sqrt{2}}g_{\alpha\beta}U_k, \quad \frac{1}{\sqrt{2(\kappa_1^{(k)} - 2a^{-2})}}\left[D_\alpha\xi_\beta + D_\beta\xi_\alpha - D^\gamma\xi_\gamma\, g_{\alpha\beta}\right], \tag{A.7}$$

where $\xi_\alpha$ denotes one of the two vectors given in (A.2). The first set of states have $-\Box$ eigenvalue $\kappa_1^{(k)}$ and the second set of states have $-\Box$ eigenvalue $\kappa_1^{(k)} - 4a^{-2}$. Note that for $\kappa_1^{(k)} = 2a^{-2}$, i.e., for $l = 1$, the second set of states given in (A.7) vanishes since the corresponding $\xi_\alpha$'s label the conformal Killing vectors of the sphere.

On $AdS_2$ the basis states for a symmetric rank two tensor may be chosen as

$$\frac{1}{\sqrt{2}}g_{mn}W_k, \quad \frac{1}{\sqrt{2(\kappa_2^{(k)} + 2a^{-2})}}\left[D_m\widehat{\xi}_n + D_n\widehat{\xi}_m - D^p\widehat{\xi}_p\, g_{mn}\right], \tag{A.8}$$

where $\widehat{\xi}_m$ denotes one of the two vectors given in (A.3), or the vector given in (A.4). The first set of states have $-\Box$ eigenvalue $\kappa_2^{(k)}$ and the second set of states have $-\Box$ eigenvalue $\kappa_2^{(k)} + 4a^{-2}$. Besides these there is another set of square integrable modes of eigenvalue $2a^{-2}$ of $-\Box$, given by [103]

$$\begin{aligned} h_{mn} &= w^{(\ell)}_{mn}, \\ w^{(\ell)}_{mn}dx^m dx^n &= \frac{a}{\sqrt{\pi}}\left[\frac{|\ell|(\ell^2-1)}{2}\right]^{1/2}\frac{(\sinh\eta)^{|\ell|-2}}{(1+\cosh\eta)^{|\ell|}}\,e^{i\ell\theta} \\ &\quad\times(d\eta^2 + 2i\,\sinh\eta\,d\eta d\theta - \sinh^2\eta\,d\theta^2)\quad \ell\in\mathbb{Z},\quad |\ell|\geq 2. \end{aligned} \tag{A.9}$$

Locally these can be regarded as deformations generated by a diffeomorphism on $AdS_2$, but these diffeomorphisms themselves are not square integrable. The basis states (A.9) satisfy

$$\sum_{\ell} g^{mn} g^{pq} w_{mp}^{(\ell)*}(x) w_{nq}^{(\ell)}(x) = \frac{3}{2\pi a^2}. \tag{A.10}$$

We have derived this using the fact that due to homogeneity of $AdS_2$ this sum is independent of $x$, and that at $\eta = 0$ only the $\ell = \pm 2$ terms contribute to the sum. Thus as in (A.6) the total number of such discrete modes is given by

$$N_2 = 3 \cosh \eta_0 - 3. \tag{A.11}$$

We can construct the basis states of various fields on $AdS_2 \times S^2$ by taking the product of the basis states on $S^2$ and $AdS_2$. For example for a scalar field the basis states will be given by the product of $Y_{lm}(\psi, \phi)$ with the states given in (A.1), and satisfy

$$\Box\, f_{\lambda,k}(\eta, \theta)\, Y_{lm}(\psi, \phi) = -\frac{1}{a^2} \left\{ l(l+1) + \lambda^2 + \frac{1}{4} \right\} f_{\lambda,k}(\eta, \theta)\, Y_{lm}(\psi, \phi). \tag{A.12}$$

For a vector field on $AdS_2 \times S^2$ the basis states will contain two sets. One set will be given by the product of $Y_{lm}(\psi, \phi)$ and (A.3) or (A.4). The other set will contain the product of the functions (A.1) on $AdS_2$ and the vector fields (A.2) on $S^2$. The basis states for a symmetric rank two tensor field on $AdS_2 \times S^2$ can be constructed in a similar manner.

Finally we turn to the basis states for the fermion fields. Consider a Dirac spinor on $AdS_2 \times S^2$. It decomposes into a product of a Dirac spinor on $AdS_2$ and a Dirac spinor on $S^2$. We use the following conventions for the vierbeins and the gamma matrices

$$e^0 = a \sinh \eta\, d\theta, \quad e^1 = a\, d\eta, \quad e^2 = a \sin \psi\, d\phi, \quad e^3 = a\, d\psi, \tag{A.13}$$
$$\gamma^0 = -\sigma_3 \otimes \tau_2, \quad \gamma^1 = \sigma_3 \otimes \tau_1, \quad \gamma^2 = -\sigma_2 \otimes I_2, \quad \gamma^3 = \sigma_1 \otimes I_2, \tag{A.14}$$

where $\sigma_i$ and $\tau_i$ are two dimensional Pauli matrices acting on different spaces and $I_2$ is $2 \times 2$ identity matrix. In this convention the Dirac operator on $AdS_2 \times S^2$ can be written as

$$\not{D}_{AdS_2 \times S^2} = \not{D}_{S^2} + \sigma_3\, \not{D}_{AdS_2}, \tag{A.15}$$

where

$$\not{D}_{S^2} = a^{-1} \left[ -\sigma^2 \frac{1}{\sin \psi} \partial_\phi + \sigma^1 \partial_\psi + \frac{1}{2} \sigma^1 \cot \psi \right], \tag{A.16}$$

and

$$\not{D}_{AdS_2} = a^{-1}\left[-\tau^2 \frac{1}{\sinh\eta}\partial_\theta + \tau^1\,\partial_\eta + \frac{1}{2}\,\tau^1\,\coth\eta\right]. \tag{A.17}$$

The eigenstates of $\not{D}_{S^2}$ are given by [106]

$$\begin{aligned}
\chi^\pm_{l,m} &= \frac{1}{\sqrt{4\pi a^2}}\,\frac{\sqrt{(l-m)!(l+m+1)!}}{l!}\,e^{i\left(m+\frac{1}{2}\right)\phi} \\
&\quad\times\begin{pmatrix} i\,\sin^{m+1}\frac{\psi}{2}\cos^m\frac{\psi}{2}\,P^{(m+1,m)}_{l-m}(\cos\psi) \\ \pm\sin^m\frac{\psi}{2}\cos^{m+1}\frac{\psi}{2}\,P^{(m,m+1)}_{l-m}(\cos\psi)\end{pmatrix}, \\
\eta^\pm_{l,m} &= \frac{1}{\sqrt{4\pi a^2}}\,\frac{\sqrt{(l-m)!(l+m+1)!}}{l!}\,e^{-i\left(m+\frac{1}{2}\right)\phi} \\
&\quad\times\begin{pmatrix} \sin^m\frac{\psi}{2}\cos^{m+1}\frac{\psi}{2}\,P^{(m,m+1)}_{l-m}(\cos\psi) \\ \pm i\,\sin^{m+1}\frac{\psi}{2}\cos^m\frac{\psi}{2}\,P^{(m+1,m)}_{l-m}(\cos\psi)\end{pmatrix}, \\
&\quad l, m\in\mathbb{Z},\quad l\geq 0,\quad 0\leq m\leq l,
\end{aligned} \tag{A.18}$$

satisfying

$$\not{D}_{S^2}\chi^\pm_{l,m} = \pm i\,a^{-1}\,(l+1)\,\chi^\pm_{l,m}, \qquad \not{D}_{S^2}\eta^\pm_{l,m} = \pm i\,a^{-1}\,(l+1)\,\eta^\pm_{l,m}. \tag{A.19}$$

Here $P^{\alpha,\beta}_n(x)$ are the Jacobi Polynomials:

$$P^{(\alpha,\beta)}_n(x) = \frac{(-1)^n}{2^n\,n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]. \tag{A.20}$$

$\chi^\pm_{l,m}$ and $\eta^\pm_{l,m}$ provide an orthonormal set of basis functions, e.g.

$$a^2\int_{S^2}\left(\chi^\pm_{l,m}\right)^\dagger\,\chi^\pm_{l',m'}\,\sin\psi\,d\psi\,d\phi = \delta_{ll'}\delta_{mm'} \tag{A.21}$$

etc.

The eigenstates of $\not{D}_{AdS_2}$ are given by [106]

$$\chi_k^\pm(\lambda) = \frac{1}{\sqrt{4\pi a^2}} \left|\frac{\Gamma\left(1+k+i\lambda\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}+i\lambda\right)}\right| e^{i\left(k+\frac{1}{2}\right)\theta} \times \begin{pmatrix} i\,\frac{\lambda}{k+1}\,\cosh^k\frac{\eta}{2}\sinh^{k+1}\frac{\eta}{2}F\left(k+1+i\lambda,k+1-i\lambda;k+2;-\sinh^2\frac{\eta}{2}\right) \\ \pm\cosh^{k+1}\frac{\eta}{2}\sinh^k\frac{\eta}{2}F\left(k+1+i\lambda,k+1-i\lambda;k+1;-\sinh^2\frac{\eta}{2}\right) \end{pmatrix},$$

$$\eta_k^\pm(\lambda) = \frac{1}{\sqrt{4\pi a^2}} \left|\frac{\Gamma\left(1+k+i\lambda\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}+i\lambda\right)}\right| e^{-i\left(k+\frac{1}{2}\right)\theta} \times \begin{pmatrix} \cosh^{k+1}\frac{\eta}{2}\sinh^k\frac{\eta}{2}F\left(k+1+i\lambda,k+1-i\lambda;k+1;-\sinh^2\frac{\eta}{2}\right) \\ \pm i\,\frac{\lambda}{k+1}\,\cosh^k\frac{\eta}{2}\sinh^{k+1}\frac{\eta}{2}F\left(k+1+i\lambda,k+1-i\lambda;k+2;-\sinh^2\frac{\eta}{2}\right) \end{pmatrix},$$

$$k\in\mathbb{Z},\quad 0\le k<\infty,\quad 0<\lambda<\infty, \tag{A.22}$$

satisfying

$$\not{D}_{AdS_2}\chi_k^\pm(\lambda) = \pm i\,a^{-1}\,\lambda\,\chi_k^\pm(\lambda), \qquad \not{D}_{AdS_2}\eta_k^\pm(\lambda) = \pm i\,a^{-1}\,\lambda\,\eta_k^\pm(\lambda). \tag{A.23}$$

$\chi_k^\pm(\lambda)$ and $\eta_k^\pm(\lambda)$ provide an orthonormal set of basis functions on $AdS_2$, e.g.,

$$a^2\int \sinh\eta\,d\eta\,d\theta\,(\chi_k^\pm(\lambda))^\dagger\,\chi_{k'}^\pm(\lambda') = \delta_{kk'}\delta(\lambda-\lambda'), \tag{A.24}$$

etc.

The basis of spinors on $AdS_2\times S^2$ can be constructed by taking the direct product of the spinors given in (A.18) and (A.22). Let $\psi_1$ denotes an eigenstate of $\not{D}_{S^2}$ with eigenvalue $i\zeta_1=\pm ia^{-1}(l+1)$ and $\psi_2$ denotes an eigenstate of $\not{D}_{AdS_2}$ with eigenvalue $i\zeta_2=\pm ia^{-1}\lambda$. Since $\sigma_3$ anti-commutes with $\not{D}_{S^2}$ and commutes with $\not{D}_{AdS_2}$, we have, using (A.15),

$$\begin{aligned} \not{D}_{AdS_2\times S^2}\,\psi_1\otimes\psi_2 &= i\zeta_1\psi_1\otimes\psi_2 + i\zeta_2\sigma_3\,\psi_1\otimes\psi_2, \\ \not{D}_{AdS_2\times S^2}\,\sigma_3\,\psi_1\otimes\psi_2 &= i\zeta_2\,\psi_1\otimes\psi_2 - i\zeta_1\sigma_3\,\psi_1\otimes\psi_2. \end{aligned} \tag{A.25}$$

Diagonalizing the $2\times 2$ matrix we see that $\not{D}_{AdS_2\times S^2}$ has eigenvalues $\pm i\sqrt{\zeta_1^2+\zeta_2^2}$. Thus the square of the eigenvalue of $\not{D}_{AdS_2\times S^2}$ is given by the sum of squares of the eigenvalues of $\not{D}_{AdS_2}$ and $\not{D}_{S^2}$, and we have

$$\begin{aligned} (\not{D}_{AdS_2\times S^2})^2\psi_1\otimes\psi_2 &= -(\zeta_1^2+\zeta_2^2)\,\psi_1\otimes\psi_2, \\ (\not{D}_{AdS_2\times S^2})^2\sigma_3\psi_1\otimes\psi_2 &= -(\zeta_1^2+\zeta_2^2)\,\sigma_3\psi_1\otimes\psi_2. \end{aligned} \tag{A.26}$$

By introducing the 'charge conjugation operator'

$$\widetilde{C} = \sigma_2\otimes\tau_1 \tag{A.27}$$

and defining $\bar{\psi} = \psi^T \widetilde{C}$, we can express the orthonormality relations (A.21), (A.24) as

$$\int d^4x \sqrt{\det g}\, \overline{\left(\chi^+_{l,m} \otimes \chi^+_k(\lambda)\right)} \left(\eta^+_{l',m'} \otimes \eta^-_{k'}(\lambda')\right) = i\, \delta_{l,l'}\delta_{m,m'}\delta_{k,k'}\delta(\lambda-\lambda'), \tag{A.28}$$

etc. This is important since eventually we shall be dealing with fields satisfying appropriate reality conditions for which $\bar{\psi}$ will be defined as $\psi^T \widetilde{C}$.

In our analysis we shall also need to find a basis in which we can expand the Rarita–Schwinger field $\Psi_\mu$. Let us denote by $\chi$ the spinor $\psi_1 \otimes \psi_2$ where $\psi_1$ and $\psi_2$ are eigenstates of $\not{D}_{S^2}$ and $\not{D}_{AdS_2}$ with eigenvalues $i\zeta_1$ and $i\zeta_2$ respectively. Then a (non-orthonormal set of) basis states for expanding $\Psi_\mu$ on $AdS_2 \times S^2$ can be chosen as follows:

$$\begin{aligned}
&\Psi_\alpha = \gamma_\alpha \chi, \quad \Psi_m = 0,\\
&\Psi_\alpha = 0, \quad \Psi_m = \gamma_m \chi,\\
&\Psi_\alpha = D_\alpha \chi, \quad \Psi_m = 0,\\
&\Psi_\alpha = 0, \quad \Psi_m = D_m \chi.
\end{aligned} \tag{A.29}$$

By including all possible eigenstates $\chi$ of $\not{D}_{S^2}$ and $\not{D}_{AdS_2}$ we shall generate the complete set of basis states for expanding the Rarita–Schwinger field barring the subtleties mentioned below.

The first subtlety arises due to the relations

$$D_\alpha \chi^\pm_{0,0} = \pm\frac{i}{2}\, a^{-1}\, \gamma_\alpha \chi^\pm_{0,0}, \quad D_\alpha \eta^\pm_{0,0} = \pm\frac{i}{2}\, a^{-1}\, \gamma_\alpha \eta^\pm_{0,0}. \tag{A.30}$$

Thus if we take $\chi = \psi_1 \otimes \psi_2$ where $\psi_1$ corresponds to any of the states $\chi^\pm_{0,0}$ or $\eta^\pm_{0,0}$, and $\psi_2$ is any eigenstate of $\not{D}_{AdS_2}$, then the basis vectors appearing in (A.29) are not all independent—the modes in the third row of (A.29) are related to those in the first row. The second point is that the modes given in (A.29) do not exhaust all the modes of the Rarita Schwinger operator; there are some additional discrete modes of the form

$$\xi^{(k)\pm}_m \equiv \psi_1 \otimes \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m\right) \chi^\pm_k(i), \quad \widehat{\xi}^{(k)\pm}_m \equiv \psi_1 \otimes \left(D_m \pm \frac{1}{2a}\sigma_3\gamma_m\right) \eta^\pm_k(i), \quad k = 1, \ldots \infty, \tag{A.31}$$

where $\chi^\pm_k(\lambda)$ and $\eta^\pm_k(\lambda)$ have been defined in (A.22). Since $\chi^\pm_k(i)$ and $\eta^\pm_k(i)$ are not square integrable, these states are not included in the set given in (A.29). However the modes described in (A.31) are square integrable and hence they must be included among the eigenstates of the Rarita–Schwinger operator. These modes can be shown to satisfy the chirality projection condition

$$\tau_3 \left( D_m \pm \frac{1}{2a}\sigma_3 \gamma_m \right) \chi_k^{\pm}(i) = -\left( D_m \pm \frac{1}{2a}\sigma_3 \gamma_m \right) \chi_k^{\pm}(i),$$
$$\tau_3 \left( D_m \pm \frac{1}{2a}\sigma_3 \gamma_m \right) \eta_k^{\pm}(i) = \left( D_m \pm \frac{1}{2a}\sigma_3 \gamma_m \right) \eta_k^{\pm}(i). \tag{A.32}$$

## Appendix B: Some useful relations

In this appendix we shall collect the results of some useful integrals. Their derivation has been reviewed in [9,10].

$$\int_0^\infty d\lambda\, \lambda\, \tanh(\pi\lambda)\, e^{-\bar{s}\lambda^2}\, \lambda^{2n}$$
$$= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^\infty \bar{s}^m \frac{(2m+2n+1)!}{m!} (2\pi)^{-2(m+n+1)} (-1)^m$$
$$\times (2^{-2m-2n-1} - 1)\, \zeta(2(m+n+1)), \tag{B.1}$$

$$\text{Im} \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda}\, \tan(\pi\widetilde{\lambda})\, e^{-\bar{s}\widetilde{\lambda}^2}\, \widetilde{\lambda}^{2n}$$
$$= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^\infty \bar{s}^m \frac{(2m+2n+1)!}{m!} (2\pi)^{-2(m+n+1)} (-1)^{n+1}$$
$$\times (2^{-2m-2n-1} - 1)\, \zeta(2(m+n+1)), \tag{B.2}$$

$$\int_0^\infty d\lambda\, \lambda\, \coth(\pi\lambda)\, e^{-\bar{s}\lambda^2}\, \lambda^{2n}$$
$$= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^\infty \bar{s}^m \frac{(2m+2n+1)!}{m!} (2\pi)^{-2(m+n+1)} (-1)^m$$
$$\times \zeta(2(m+n+1)), \tag{B.3}$$

$$\text{Im} \int_0^{e^{i\kappa}\times\infty} d\widetilde{\lambda}\, \widetilde{\lambda}\, \cot(\pi\widetilde{\lambda})\, e^{-\bar{s}\widetilde{\lambda}^2}\, \widetilde{\lambda}^{2n}$$
$$= \frac{1}{2}\bar{s}^{-1-n}\Gamma(1+n) + 2\sum_{m=0}^\infty \bar{s}^m \frac{(2m+2n+1)!}{m!} (2\pi)^{-2(m+n+1)} (-1)^{n+1}$$
$$\times \zeta(2(m+n+1)). \tag{B.4}$$

## Appendix C: Symplectic transformation of the prepotential

In general the coupling of the vector multiplet fields to supergravity is determined by a prepotential $F(X^0, \ldots, X^{n_V})$ where $F$ is a homogeneous function of degree 2 and

$n_V$ is the number of vector multiplets. A general symplectic transformation takes the form

$$X^r \to M_{rs}X^s + N_{rs}F_s, \quad F_r = P_{rs}X^s + Q_{rs}F_s, \quad 0 \le r, s \le n_V, \tag{C.1}$$

where $F_s = \partial F/\partial X^s$ and $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is an $Sp(2n_V + 2)$ matrix satisfying

$$M^T P - P^T M = 0, \quad N^T Q - Q^T N = 0, \quad M^T Q - P^T N = I. \tag{C.2}$$

Our goal is to show that by a symplectic transformation we can introduce new coordinates $Z^0, \ldots, Z^{n_V}$ such that at the attractor geometry $Z^k = 0$ for $1 \le k \le n_V$ and the prepotential takes the form

$$\widehat{F} = -\frac{i}{2}\left((Z^0)^2 - \sum_{k=1}^{n_V}(Z^k)^2\right) + \cdots, \tag{C.3}$$

where $\cdots$ denote terms which are cubic or higher order in $Z^1, \ldots, Z^{n_V}$. These higher order terms contain information about the interactions of the theory and hence are important in the full theory. But the quadratic terms in the fluctuations about the black hole background are controlled by the terms up to quadratic order in $Z^1, \ldots, Z^{n_V}$, and hence for our analysis we can ignore the effects of the cubic and higher order terms.

Since $Sp(2n_V+2)$ has $2(n_V+1)^2+(n_V+1) = 2n_V^2+5n_V+3$ parameters, in the generic case we can use them to introduce new special coordinates $Y^0, Y^1, \ldots, Y^{n_V}$ such that at the attractor value $Y^k = 0$ for $k = 1, \ldots, n_V$. Since $Y^k$ are in general complex, this uses up $2n_V$ of the $2n_V^2 + 5n_V + 3$ parameters. We shall denote the new prepotential by $\check{F}$. If we expand $\check{F}$ around the point $Y^i = 0$ the expansion takes the form:

$$\check{F} = \frac{i}{2}A(Y^0)^2 + B_k Y^k Y^0 + \frac{i}{2}C_{kl}Y^k Y^l + \cdots, \tag{C.4}$$

for some complex constants $A, B_k, C_{kl}$. The $\cdots$ terms are cubic and higher order in $Y^1, \ldots, Y^{n_V}$ and as a result does not affect the terms in the action quadratic in the fluctuations. In order to arrive at the form (C.3) we need to make another set of symplectic transformations which sets $A = 1$, $B_k = 0$ and $C_{kl} = -\delta_{kl}$. This corresponds to $1 + n_V + n_V(n_V + 1)/2$ complex constraints, i.e. $n_V^2 + 3n_V + 2$ real constraints and, in the generic case, can be achieved by utilizing $n_V^2 + 3n_V + 2$ parameters of $Sp(2n_V + 2)$. Adding this to the $2n_V$ constraints which keep the attractor values of $Z^k$ to be fixed at 0, we see that we have $(n_V^2 + 5n_V + 2)$ conditions. This is less than the number of parameters $2n_V^2 + 5n_V + 3$ of $Sp(2n_V)$ and hence is achievable for a generic choice of the starting prepotential.

We shall now show how to find the required symplectic transformation explicitly in the case where the form of the prepotential given in (C.4) differs from the one in

(C.3) by an infinitesimal amount, i.e., when

$$A = -1 + \epsilon \tilde{A}, \quad B_k = \epsilon \tilde{B}_k, \quad C_{kl} = \delta_{kl} + \epsilon \tilde{C}_{kl}, \tag{C.5}$$

for an infinitesimal parameter $\epsilon$. Now a general symplectic transformation relating the variables $\vec{Z}$ and $\vec{Y}$ takes the form

$$Y^r = M_{rs} Z^s + N_{rs} \widehat{F}_s, \quad \check{F}_r = P_{rs} Z^s + Q_{rs} \widehat{F}_s, \quad 0 \le r, s \le n, \tag{C.6}$$

where $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ is an $Sp(2n_V+2)$ matrix satisfying (C.2). We choose the following infinitesimal $Sp(2n_V+2)$ matrices:

$$\begin{aligned} &M = I + \epsilon \widetilde{M}, \quad Q = I + \epsilon \widetilde{Q}, \quad P = \epsilon \widetilde{P}, \quad N = \epsilon \widetilde{N}, \quad \widetilde{Q} = -\widetilde{M}^T, \quad \widetilde{N} = \widetilde{N}^T, \\ &\widetilde{P} = \widetilde{P}^T, \widetilde{M}_{i0} = 0, \quad \widetilde{N}_{i0} = 0, \\ &2\widetilde{M}_{00} - i(\widetilde{N}_{00} + \widetilde{P}_{00}) = \widetilde{A}, \quad \widetilde{P}_{0i} + i\widetilde{M}_{0i} = \widetilde{B}_i, \quad -\widetilde{M}_{ij} - \widetilde{M}_{ji} - i\widetilde{N}_{ij} - i\widetilde{P}_{ij} = \widetilde{C}_{ij}. \end{aligned} \tag{C.7}$$

The first line ensures that the matrix $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ describes an $Sp(2n_V+2)$ matrix to order $\epsilon$. The second line ensures that the attractor point $Y^i = 0$ gets mapped to $Z^i = 0$ for $i = 1, \ldots, n_V$. Finally the last line ensures that $\check{F}$ computed from (C.6), (C.3) agrees with (C.4) to first order in $\epsilon$.

At the end of this process we are still left with $n_V^2 + 1$ parameters of $Sp(2n_V+2)$. These transformations do not change the prepotential but generate electric-magnetic duality rotation among the Maxwell fields. For example we can still make the symplectic transformation of the form

$$Z^0 \to \cos\alpha\, Z^0 + \sin\alpha\, F_0, \quad F_0 \to -\sin\alpha\, Z^0 + \cos\alpha\, F_0, \tag{C.8}$$

for some constant $\alpha$ without changing the form of the prepotential. This induces an electric-magnetic duality rotation among the electric and magnetic fields $F^0_{\mu\nu}$ and $\widetilde{F}^0_{\mu\nu}$.

It is instructive to find the electric and magnetic charges $\{q_I, p^I\}$ and the near horizon electric field $e^I$ $(0 \le I \le n_V)$ carried by the black hole when the near horizon background is described by $Z^k = 0$ for $k = 1, \ldots, n_V$. For this we use the attractor equations, derived for two derivative action in [107–109] and for higher derivative terms in [76–78]. In the convention of [80] we have

$$\begin{aligned} a^2 &= \frac{16}{w\bar{w}}, \\ q_I &= 4i\left(\bar{w}^{-1}\overline{\widehat{F}}_I - w^{-1}\widehat{F}_I\right) \\ p^I &= 4i\left(\bar{w}^{-1}\bar{Z}^I - w^{-1}Z^I\right) \\ e^I &= 4\left(\bar{w}^{-1}\bar{Z}^I + w^{-1}Z^I\right) \quad 0 \le I \le n_V, \end{aligned} \tag{C.9}$$

where $a$ is the radii of $S^2$ and $AdS_2$ and $w$ is the background value of an auxiliary anti-self-dual tensor field $T^-_{\mu\nu}$: $T^-_{mn} = -iw\varepsilon_{mn}$ for $m, n \in AdS_2$. We shall choose the gauge $Z^0 = 1$. Since $\widehat{F}_k = iZ_k$ for $1 \le k \le n_V$ it follows from (C.9) that for $Z^k = 0$, $p^k = q_k = e^k = 0$ for $1 \le k \le n_V$. If we further choose $p^0 = 0$ with the help of the duality rotation (C.8), then we see from (C.9) that in the $Z^0 = 1$ gauge $w$ must be real, and we have

$$w = 4a^{-1}, \quad q_0 = -8w^{-1} = -2a, \quad e^0 = 8w^{-1} = 2a. \tag{C.10}$$

The near horizon electromagnetic fields are now given by

$$F^k_{\mu\nu} = 0 \quad \text{for} \quad 1 \le k \le n_V, \quad F^0_{\alpha\beta} = 0, \quad F^0_{mn} = -ie^0\, a^{-2}\, \varepsilon_{mn} = -2ia^{-1}\varepsilon_{mn},$$
$$\mu, \nu \in AdS_2 \times S^2, \quad \alpha, \beta \in S^2, \quad m, n \in AdS_2. \tag{C.11}$$

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