RESEARCH ARTICLE

Effects of electromagnetic field on the dynamical instability of expansionfree gravitational collapse

M. Sharif · M. Azam

Abstract   In this paper, we discuss the effects of electromagnetic field on the dynamical instability of a spherically symmetric expansionfree gravitational collapse. Darmois junction conditions are formulated by matching interior spherically symmetric spacetime to exterior Reissner–Nordström spacetime. We investigate the role of different terms in the dynamical equation at Newtonian and post Newtonian regimes by using perturbation scheme. It is concluded that instability range depends upon pressure anisotropy, radial profile of energy density and electromagnetic field, but not on the adiabatic index Γ. In particular, the electromagnetic field reduces the unstable region.

Keywords   Local anisotropy of pressure · Instability · Electromagnetic field

1 Introduction

The stability/instability of self-gravitating objects has great importance in general relativity. It is well-known that different ranges of stability would imply different kinds of evolution in the collapse as well as structure formation of self-gravitating objects. The adiabatic index Γ defines the range of instability which is less than $\frac{4}{3}$ for a spherically symmetric distribution of isotropic perfect fluid [1]. Also, it is obvious that a stellar model can exist only if it is stable against fluctuations. A stable stationary black hole solution under perturbations tells the final state of dynamical evolution of a gravitating system.

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The expansion scalar, Θ, measures the rate at which small volumes of the fluid may change. In the expanding sphere, the increase in volume due to increasing area of external surface must be reimbursed with the increasing area of internal boundary surface. A similar behavior of surface area can be observed in the case of contraction. Thus we have to keep Θ vanishing in each case. Skripkin [2] explored the central explosion of a spherically symmetric fluid distribution with expansionfree scalar. This leads to the formation of Minkowskian cavity at the center of the fluid. Eardley and Smarr [3] investigated that the collapse of self-gravitating fluids would lead to formation of naked singularity for inhomogeneous energy density but to black hole for homogenous case. It was found that expansionfree model requires locally anisotropic fluid and inhomogeneous energy density [4–6]. Herrera et al. [7] found that inhomogeneous expansionfree dust models with negative energy density has no physical significance. The same authors [8] discussed cavity evolution in relativistic self-gravitating fluid.

Rosseland [9] was the first to study self-gravitating spherically symmetric charged fluid distribution. Since then many people have considered the effect of electromagnetic charge on the structure and evolution of self-gravitating systems [10–14]. Di Prisco et al. [15] explored the effect of charge on the relation between the Weyl tensor and the inhomogeneity of energy density and concluded that Coulomb repulsion might prevent the gravitational collapse of the sphere. Thirukkanesh and Maharaj [16] investigated that gravitational attraction is compensated by the Coulomb's repulsive force along with gradient pressure in a gravitational collapse. Sharif and Abbas [17] discussed the effect of electromagnetic field on spherically symmetric gravitational collapse with cosmological constant. Sharif and Sundas [18] used Misner–Sharp formalism to discuss charged cylindrical collapse of anisotropic fluid and found that electric charge increases the active gravitational mass.

It is evident that anisotropy, free streaming radiation, thermal conduction and shearing viscosity affect the evolution of self-gravitating systems. In literature [19,20], it is shown that the thermal effects reduce the range of instability. Chan et al. [21] explored that the instability range depends upon the local anisotropy of the unperturbed fluid. The same authors [22] found the effects of shearing viscous fluid on the instability range. Chan [23] studied collapsing radiating star with shearing viscosity and concluded that it would increase anisotropy of pressure as well as the value of effective adiabatic index. Horvat et al. [24] explored that instability of anisotropic star occurs at higher surface compactness when the anisotropy of the pressure is present. Herrera et al. [25] discussed the dynamical instability of expansionfree fluid at Newtonian and post Newtonian order and found that the range of instability is determined by the anisotropic pressure and radial profile of the energy density. In a recent paper [26], this problem has been explored in $f(R)$ gravity.

In this paper, we take spherically symmetric distribution of collapsing fluid along with electromagnetic field and investigate how electromagnetic field would affect the range of instability. Darmois Junction conditions [27] are used to match the interior spherically symmetric spacetime to exterior Reissner–Nordström (RN) spacetime on the external hypersurface and on the internal hypersurface Minkowski spacetime within the cavity to the fluid distribution. We find that electromagnetic field, energy density and anisotropic pressure affect the stability of the system.

The paper has the following format. In Sect. 2, we discuss Einstein–Maxwell equations and some basic properties of anisotropic fluid. Section 3 provides the formulation of junction conditions. In Sect. 4, the perturbation scheme is applied on the field as well as dynamical equations. We discuss the Newtonian and post Newtonian regimes and obtain the dynamical equation in Sect. 5. Results are summarized in the last section.

2 Fluid distribution and the field equations

Consider a spherically symmetric distribution of charged collapsing fluid bounded by a spherical surface Σ. The line element for the interior region is the most general spherically symmetric metric given by

$$ds_-^2 = -A^2(t, r)dt^2 + B^2(t, r)dr^2 + R^2(t, r)(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where we assume comoving coordinates inside the hypersurface Σ. The interior coordinates are taken as $x^{-0} = t, \ x^{-1} = r, \ x^{-2} = \theta, \ x^{-3} = \phi$. It is assumed that the fluid is locally anisotropic and the energy-momentum tensor for such a fluid is given by

$$T_{\alpha\beta}^- = (\mu + p_\perp)u_\alpha u_\beta + p_\perp g_{\alpha\beta} + (p_r - p_\perp)\chi_\alpha \chi_\beta, \tag{2}$$

where μ is the energy density, $p_\perp$ the tangential pressure, p_r the radial pressure, u_α the four-velocity of the fluid and χ_α is the unit four-vector along the radial direction. Using the following definitions in comoving coordinates

$$u^\alpha = A^{-1}\delta_0^\alpha, \quad \chi^\alpha = B^{-1}\delta_1^\alpha, \tag{3}$$

we can write

$$u^\alpha u_\alpha = -1, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha u_\alpha = 0.$$

The expansion scalar is defined as

$$\Theta = u_{;\alpha}^\alpha = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right). \tag{4}$$

Here dot and prime represent derivatives with respect to t and r respectively.

The Maxwell equations can be written as

$$F_{\alpha\beta} = \phi_{\beta,\alpha} - \phi_{\alpha,\beta}, \quad F_{;\beta}^{\alpha\beta} = 4\pi J^\alpha, \tag{5}$$

where ϕ_α is the four potential and J^α is the four current. The electromagnetic energy-momentum tensor is given by

$$E_{\alpha\beta} = \frac{1}{4\pi}\left(F_\alpha^\gamma F_{\beta\gamma} - \frac{1}{4}F^{\gamma\delta}F_{\gamma\delta}g_{\alpha\beta}\right), \tag{6}$$

where $F_{\alpha\beta}$ is the Maxwell field tensor. Since the charge is at rest with respect to comoving coordinates, the magnetic field will be zero. Thus we can write

$$\phi_\alpha = (\phi(t, r), 0, 0, 0), \quad J^\alpha = \xi u^\alpha, \tag{7}$$

where ξ is the charge density. The conservation of charge requires

$$q(r) = 4\pi \int_0^r \xi B R^2 dr \tag{8}$$

which is the electric charge interior to radius R. Using Eq. (1), the Maxwell equations (5) yield

$$\phi'' - \left(\frac{A'}{A} + \frac{B'}{B} - 2\frac{R'}{R}\right)\phi' = 4\pi \xi A B^2, \tag{9}$$

$$\dot{\phi}' - \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - 2\frac{\dot{R}}{R}\right)\phi' = 0. \tag{10}$$

Integration of Eq. (9) implies

$$\phi' = \frac{qAB}{R^2}. \tag{11}$$

The electric field intensity is defined as

$$E(t, r) = \frac{q}{4\pi R^2}. \tag{12}$$

The Einstein field equations

$$G_{\alpha\beta}^- = 8\pi \left(T_{\alpha\beta}^- + E_{\alpha\beta}^-\right), \tag{13}$$

for the interior metric gives the following set of equations

$$8\pi A^2(\mu + 2\pi E^2) = \left(\frac{2\dot{B}}{B} + \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}$$
$$- \left(\frac{A}{B}\right)^2 \left[\frac{2R''}{R} + \left(\frac{R'}{R}\right)^2 - \frac{2B'R'}{BR} - \left(\frac{B}{R}\right)^2\right], \tag{14}$$

$$0 = -2\left(\frac{\dot{R}'}{R} - \frac{\dot{R}A'}{RA} - \frac{\dot{B}R'}{BR}\right), \tag{15}$$

$$8\pi B^2(p_r - 2\pi E^2) = -\left(\frac{B}{A}\right)^2\left[\frac{2\ddot{R}}{R} - \left(\frac{2\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right]$$
$$+ \left(\frac{2A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \left(\frac{B}{R}\right)^2, \tag{16}$$

$$8\pi R^2(p_\perp + 2\pi E^2) = 8\pi R^2(p_\perp + 2\pi E^2)\sin^{-2}\theta$$
$$= -\left(\frac{R}{A}\right)^2\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right) + \frac{\dot{B}\dot{R}}{BR}\right]$$
$$+ \left(\frac{R}{B}\right)^2\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'B'}{AB} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{R'}{R}\right]. \tag{17}$$

The mass function is defined as follows [28]

$$m(t,r) = \frac{R}{2}(1 - g^{\alpha\beta}R_{,\alpha}R_{,\beta}) = \frac{R}{2}\left(1 + \frac{\dot{R}^2}{A^2} - \frac{R'^2}{B^2}\right) + \frac{q^2}{2R}. \tag{18}$$

Differentiating this equation with respect to r and using Eq. (14), we get

$$m' = 4\pi\mu R'R^2 + 16\pi^2 R^2 E(RE' + 2R'E). \tag{19}$$

The proper time and radial derivatives are given by

$$D_T = \frac{1}{A}\frac{\partial}{\partial t}, \quad D_R = \frac{1}{R'}\frac{\partial}{\partial r}, \tag{20}$$

where R is the areal radius of the spherical surface. The velocity of the collapsing fluid is defined by the proper time derivative of R, i.e.,

$$U = D_T R = \frac{\dot{R}}{A} \tag{21}$$

which is always negative in case of collapse. Using this expression, Eq. (18) can be written as

$$\tilde{E} \equiv \frac{R'}{B} = \left[1 + U^2 - \frac{2m}{R} + \left(\frac{q}{R}\right)^2\right]^{1/2}. \tag{22}$$

The conservation of energy-momentum tensor yields

$$(T^{-\alpha\beta} + E^{-\alpha\beta})_{;\beta}u_\alpha = -\frac{1}{A}\left[\dot{\mu} + (\mu + p_r)\frac{\dot{B}}{B} + 2(\mu + p_\perp)\frac{\dot{R}}{R}\right] = 0 \tag{23}$$

which can be rewritten as

$$\dot{\mu} + (\mu + p_r)A\Theta + 2(p_\perp - p_r)\frac{\dot{R}}{R} = 0, \tag{24}$$

and

$$
(T^{-\alpha\beta} + E^{-\alpha\beta})_{;\beta}\chi_\alpha = \frac{1}{B}\left[p'_r + (\mu + p_r)\frac{A'}{A} + 2(p_r - p_\perp)\frac{R'}{R} \right.
$$
$$
\left. - \frac{E}{R}(4\pi RE' + 8\pi R'E) \right] = 0.
\tag{25}
$$

3 Junction conditions

In this section, we formulate the Darmois junction conditions for the general spherically symmetric spacetime in the interior region and RN spacetime in the exterior region. The line element for RN spacetime in Eddington–Finkelstein coordinates is given as

$$
ds_+^2 = -\left(1 - \frac{2M}{\rho} + \frac{Q^2}{\rho^2}\right)dv^2 - 2d\rho dv + \rho^2(d\theta^2 + \sin^2\theta d\phi^2),
\tag{26}
$$

where M, Q and v are the total mass, charge and retarded time respectively. For smooth matching of the interior and exterior regions, Darmois conditions [27] can be stated as follows:

1. The continuity of the line elements over Σ

$$
\left(ds_-^2\right)_\Sigma = \left(ds_+^2\right)_\Sigma = \left(ds^2\right)_\Sigma.
\tag{27}
$$

2. The continuity of the extrinsic curvature over Σ

$$
\left[K_{ij}\right] = K_{ij}^+ - K_{ij}^- = 0, \quad (i, j = 0, 2, 3).
\tag{28}
$$

The boundary surface Σ in terms of interior and exterior coordinates can be defined as

$$
f_-(t, r) = r - r_\Sigma = 0,
\tag{29}
$$
$$
f_+(v, \rho) = \rho - \rho(v_\Sigma) = 0,
\tag{30}
$$

where r_Σ is a constant. Using Eqs. (29) and (30), the interior and exterior metrics take the following form over Σ

$$
(ds_-^2)_\Sigma = -A^2(t, r_\Sigma)dt^2 + R^2(t, r_\Sigma)(d\theta^2 + \sin^2\theta d\phi^2),
\tag{31}
$$
$$
(ds_+^2)_\Sigma = -\left(1 - \frac{2M}{\rho_\Sigma} + \frac{Q^2}{\rho_\Sigma^2} + 2\frac{d\rho_\Sigma}{dv}\right)dv^2 + \rho_\Sigma^2(d\theta^2 + \sin^2\theta d\phi^2).
\tag{32}
$$

The continuity of the first fundamental form implies

$$\frac{dt}{d\tau} = A(t, r_\Sigma)^{-1}, \quad R(t, r_\Sigma) = \rho_\Sigma(v), \tag{33}$$

$$\left(\frac{dv}{d\tau}\right)^{-2} = \left(1 - \frac{2M}{\rho_\Sigma} + \frac{Q^2}{\rho_\Sigma^2} + 2\frac{d\rho_\Sigma}{dv}\right). \tag{34}$$

For the second fundamental form, we evaluate outward unit normals to Σ by using Eqs. (29) and (30) as follows

$$n_\alpha^- = (0, B(t, r_\Sigma), 0, 0), \tag{35}$$

$$n_\alpha^+ = \left(1 - \frac{2M}{\rho_\Sigma} + \frac{Q^2}{\rho_\Sigma^2} + 2\frac{d\rho_\Sigma}{dv}\right)^{-\frac{1}{2}} \left(-\frac{d\rho_\Sigma}{dv}, 1, 0, 0\right). \tag{36}$$

The non-vanishing components of the extrinsic curvature in terms of interior and exterior coordinates are

$$K_{00}^- = -\left[\frac{A'}{AB}\right]_\Sigma, \quad K_{22}^- = \left[\frac{RR'}{B}\right]_\Sigma, \quad K_{33}^- = K_{22}^- \sin^2\theta, \tag{37}$$

$$K_{00}^+ = \left[\left(\frac{d^2v}{d\tau^2}\right)\left(\frac{dv}{d\tau}\right)^{-1} - \left(\frac{dv}{d\tau}\right)\left(\frac{M}{\rho^2} - \frac{Q^2}{\rho^3}\right)\right]_\Sigma, \tag{38}$$

$$K_{22}^+ = \left[\left(\frac{d^2v}{d\tau^2}\right)\left(\frac{dv}{d\tau}\right)^{-1} - \left(\frac{dv}{d\tau}\right)\left(\frac{M}{\rho^2} - \frac{Q^2}{\rho^3}\right)\right]_\Sigma, \tag{39}$$

$$K_{33}^+ = K_{22}^+ \sin^2\theta. \tag{40}$$

Making use of Eqs. (28), (33) and (34), we get

$$M \overset{\Sigma}{=} m(t, r) \iff q(r) \overset{\Sigma}{=} Q \tag{41}$$

and

$$2\left(\frac{\dot{R}'}{R} - \frac{\dot{R}A'}{RA} - \frac{\dot{B}R'}{BR}\right) \overset{\Sigma}{=} -\frac{B}{A}\left[\frac{2\ddot{R}}{R} - \left(\frac{2\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right]$$
$$+ \frac{A}{B}\left[\left(\frac{2A'}{A} + \frac{R'}{R}\right)\frac{R'}{R} - \left(\frac{B}{R}\right)^2\right], \tag{42}$$

where $q(r) = Q$ has been used. Comparing Eq. (42) with Eqs. (15) and (16), we obtain

$$p_r \overset{\Sigma}{=} 0. \tag{43}$$

The expansionfree models require the existence of internal vacuum cavity within the fluid distribution. The matching of Minkowski spacetime within cavity to the fluid distribution on $\Sigma^{(i)}$ (boundary surface between cavity and fluid) gives

$$m(t, r) \overset{\Sigma^{(i)}}{=} 0, \quad p_r \overset{\Sigma^{(i)}}{=} 0. \tag{44}$$

4 The perturbation scheme

This section is devoted to perturb the field equations, Bianchi identities and all the material quantities by using the perturbation scheme [19,20] upto first order. Initially, all the quantities have only radial dependence, i.e., fluid is in static equilibrium. After that, all the quantities and the metric functions have time dependence as well in their perturbation. These are given by

$$A(t, r) = A_0(r) + \lambda T(t)a(r), \tag{45}$$
$$B(t, r) = B_0(r) + \lambda T(t)b(r), \tag{46}$$
$$R(t, r) = R_0(r) + \lambda T(t)c(r), \tag{47}$$
$$E(t, r) = E_0(r) + \lambda T(t)e(r), \tag{48}$$
$$\mu(t, r) = \mu_0(r) + \lambda \bar{\mu}(t, r), \tag{49}$$
$$p_r(t, r) = p_{r0}(r) + \lambda \bar{p}_r(t, r), \tag{50}$$
$$p_\perp(t, r) = p_{\perp 0}(r) + \lambda \bar{p}_\perp(t, r), \tag{51}$$
$$m(t, r) = m_0(r) + \lambda \bar{m}(t, r), \tag{52}$$
$$\Theta(t, r) = \lambda \bar{\Theta}(t, r), \tag{53}$$

where $0 < \lambda \ll 1$. By the freedom allowed in radial coordinates, we choose $R_0(r) = r$. The static configuration (unperturbed) of Eqs. (14)–(17) is obtained by using Eqs. (45)–(51) as follows

$$8\pi \left(\mu_0 + 2\pi E_0^2 \right) = \frac{1}{(B_0 r)^2} \left(2r \frac{B_0'}{B_0} + B_0^2 - 1 \right), \tag{54}$$

$$8\pi \left(p_{r0} - 2\pi E_0^2 \right) = \frac{1}{(B_0 r)^2} \left(2r \frac{A_0'}{A_0} - B_0^2 + 1 \right), \tag{55}$$

$$8\pi \left(p_{\perp 0} + 2\pi E_0^2 \right) = \frac{1}{B_0^2} \left[\frac{A_0''}{A_0} - \frac{A_0'}{A_0} \frac{B_0'}{B_0} + \frac{1}{r} \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} \right) \right]. \tag{56}$$

The corresponding perturbed field equations become

$$8\pi \bar{\mu} + 32\pi^2 E_0 T e = -\frac{2T}{B_0^2} \left[\left(\frac{c}{r} \right)'' - \frac{1}{r} \left(\frac{b}{B_0} \right)' - \left(\frac{B_0'}{B_0} - \frac{3}{r} \right) \left(\frac{c}{r} \right)' \right.$$
$$\left. - \left(\frac{b}{B_0} - \frac{c}{r} \right) \left(\frac{B_0}{r} \right)^2 \right] - 16\pi \frac{Tb}{B_0} \left(\mu_0 + 2\pi E_0^2 \right), \tag{57}$$

$$0 = 2\frac{\dot{T}}{A_0 B_0}\left[\left(\frac{c}{r}\right)' - \frac{b}{r B_0} - \left(\frac{A_0'}{A_0} - \frac{1}{r}\right)\frac{c}{r}\right], \tag{58}$$

$$8\pi \bar{p}_r - 32\pi^2 E_0 T e = -\frac{2\ddot{T}}{A_0^2}\frac{c}{r} + \frac{2T}{r B_0^2}\left[\left(\frac{a}{A_0}\right)' + \left(r\frac{A_0'}{A_0} + 1\right)\left(\frac{c}{r}\right)'\right.$$
$$\left. - \frac{B_0^2}{r}\left(\frac{b}{B_0} - \frac{c}{r}\right)\right] - 16\pi\frac{Tb}{B_0}\left(p_{r0} - 2\pi E_0^2\right), \tag{59}$$

$$8\pi \bar{p}_\perp + 32\pi^2 E_0 T e = -\frac{\ddot{T}}{A_0^2}\left[\frac{b}{B_0} + \frac{c}{r}\right] + \frac{T}{B_0^2}\left[\left(\frac{a}{A_0}\right)'' + \left(\frac{c}{r}\right)''\right.$$
$$+ \left(\frac{2A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)' - \left(\frac{A_0'}{A_0} + \frac{1}{r}\right)$$
$$\times \left(\frac{b}{B_0}\right)' + \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{2}{r}\right)\left(\frac{c}{r}\right)'\right]$$
$$- 16\pi\frac{Tb}{B_0}\left(p_{\perp 0} + 2\pi E_0^2\right). \tag{60}$$

The Bianchi identities (23) and (25) for the static configuration yields

$$\frac{1}{B_0}\left[p_{r0}' + (\mu_0 + p_{r0})\frac{A_0'}{A_0} + \frac{2}{r}(p_{r0} - p_{\perp 0})\right] - \frac{4\pi E_0}{B_0 r}\left[2E_0 + r E_0'\right] = 0, \tag{61}$$

which can be rewritten as

$$\frac{A_0'}{A_0} = -\frac{1}{\mu_0 + p_{r0}}\left[p_{r0}' + \frac{2}{r}(p_{r0} - p_{\perp 0}) - \frac{4\pi E_0}{r}(2E_0 + r E_0')\right]. \tag{62}$$

The perturbed configurations imply

$$\frac{1}{A_0}\left[\dot{\bar{\mu}} + (\mu_0 + p_{r0})\dot{T}\frac{b}{B_0} + 2(\mu_0 + p_{\perp 0})\dot{T}\frac{c}{r}\right] = 0, \tag{63}$$

$$\frac{1}{B_0}\left[\bar{p}_r' + (\mu_0 + p_{r0})T\left(\frac{a}{A_0}\right)' + (\bar{\mu} + \bar{p}_r)\frac{A_0'}{A_0}\right.$$
$$\left. + 2(p_{r0} - p_{\perp 0})T\left(\frac{c}{r}\right)' + 2(\bar{p}_r - \bar{p}_\perp)\frac{1}{r}\right]$$
$$- \frac{4\pi E_0 T}{B_0 r}\left(4e + 2r E_0\left(\frac{c}{r}\right)' + re' + re\frac{E_0'}{E_0}\right) = 0. \tag{64}$$

Integration of Eq. (63) yields

$$\bar{\mu} = -\left[(\mu_0 + p_{r0})\frac{b}{B_0} + 2(\mu_0 + p_{\perp 0})\frac{c}{r}\right]T. \tag{65}$$

The expansion scalar turns out to be

$$\bar{\Theta} = \frac{\dot{T}}{A_0} \left(\frac{b}{B_0} + \frac{2c}{r} \right). \tag{66}$$

Using expansionfree condition, it follows

$$\frac{b}{B_0} = -2\frac{c}{r}. \tag{67}$$

Inserting this value in Eq. (58), we obtain

$$c = k\frac{A_0}{r^2}, \tag{68}$$

where k is an integration constant. Using Eq. (67) in (65), we get

$$\bar{\mu} = 2(p_{r0} - p_{\perp 0})T\frac{c}{r}. \tag{69}$$

This shows that perturbed energy density comes from the static configuration of pressure anisotropy. Similarly, the unperturbed and perturbed configuration for Eq. (18) lead to

$$m_0 = \frac{r}{2} \left(1 - \frac{1}{B_0^2} \right) + 8\pi^2 E_0^2 r^3, \tag{70}$$

$$\bar{m} = -\frac{T}{B_0^2} \left[r \left(c' - \frac{b}{B_0} \right) + (1 - B_0^2)\frac{c}{2} \right] + 8\pi^2 E_0 T \left(2r^3 + 3r^2 c E_0 \right). \tag{71}$$

Using the matching condition (43), Eq. (50) implies

$$p_{r0} \overset{\Sigma}{=} 0, \quad \bar{p}_r \overset{\Sigma}{=} 0. \tag{72}$$

Inserting these values in Eq. (59), we obtain

$$\ddot{T}(t) - \alpha(r)T(t) \overset{\Sigma}{=} 0, \tag{73}$$

where

$$\alpha(r) \overset{\Sigma}{=} \left(\frac{A_0}{B_0} \right)^2 \left[\left(\frac{a}{A_0} \right)' + \left(r\frac{A_0'}{A_0} + 1 \right) \left(\frac{c}{r} \right)' \right.$$
$$\left. - \frac{B_0^2}{r} \left(\frac{b}{B_0} - \frac{c}{r} \right) + 16\pi^2 r E_0 B_0 \left(e B_0 + b E_0 \right) \right] \frac{1}{c}. \tag{74}$$

In order to explore instability region, all the functions involved in the above equation are taken such that α_Σ is positive. The corresponding solution of Eq. (73) is given by

$$T(t) = -\exp(\sqrt{\alpha_\Sigma}t). \tag{75}$$

This shows that the system starts collapsing at $t = -\infty$ with $T(-\infty) = 0$ keeping it in static position. It goes on collapsing with the increase of t.

5 Newtonian and post Newtonian terms and dynamical instability

This section investigates the terms corresponding to Newtonian (N), post Newtonian (pN) and post post Newtonian (ppN) regimes. This is done by converting relativistic units into c.g.s. units and expanding upto order c^{-4} in the dynamical equation. For the N approximation, it is assumed that

$$\mu_0 \gg p_{r0}, \quad \mu_0 \gg p_{\perp 0}.$$

For the metric coefficients expanded upto pN approximation, we take

$$A_0 = 1 - \frac{Gm_0}{c^2}, \quad B_0 = 1 + \frac{Gm_0}{c^2}, \tag{76}$$

where G is the gravitational constant and c is the speed of light. Using Eqs. (55) and (70), it follows that

$$\frac{A_0'}{A_0} = \frac{8\pi p_{r0}r^3 + 2m_0 - 32\pi^2 E_0^2 r^3}{2r(r - 2m_0 + 16\pi^2 E_0^2 r^3)}, \tag{77}$$

which together with Eq. (61) leads to

$$p_{r0}' = -\left[\frac{8\pi p_{r0}r^3 + 2m_0 - 32\pi^2 E_0^2 r^3}{2r(r - 2m_0 + 16\pi^2 E_0^2 r^3)}\right](\mu_0 + p_{r0})$$
$$+ \frac{2}{r}(p_{\perp 0} - p_{r0}) + \frac{4\pi E_0}{r}(2E_0 + rE_0'). \tag{78}$$

In view of dimensional analysis, this equation can be written in c.g.s. units as follows

$$p_{r0}' = -G\left[\frac{c^{-2}8\pi p_{r0}r^3 + 2m_0 - 32c^{-2}\pi^2 E_0^2 r^3}{2r(r - 2Gc^{-2}m_0 + 16Gc^{-4}\pi^2 E_0^2 r^3)}\right](\mu_0 + c^{-2}p_{r0})$$
$$+ \frac{2}{r}(p_{\perp 0} - p_{r0}) + \frac{4\pi E_0}{r}(2E_0 + rE_0'). \tag{79}$$

When we expand this equation upto c^{-4} order and rearrange lengthy calculations, we have

$$
\begin{aligned}
p'_{r0} = & -G\frac{\mu_0 m_0}{r^2} + \frac{2}{r}(p_{\perp 0} - p_{r0}) + \frac{4\pi}{r}\left(2E_0^2 + rE_0E'_0\right) \\
& - \frac{G}{c^2 r^3}\left(2G\mu_0 m_0^2 + p_{r0}m_0 r + 4\pi\mu_0 p_{r0}r^4 - 16\pi^2 E_0^2\mu_0 r^4\right) \\
& - \frac{G}{c^4 r^4}\left(4G^2\mu_0 m_0^2 + 2Gp_{r0}m_0^2 r + 4\pi\mu_0 p_{r0}r^4\right. \\
& \left. - 32\pi^2 GE_0^2 m_0\mu_0 r^4 - 16\pi^2 E_0^2 p_{r0}r^5\right).
\end{aligned}
\tag{80}
$$

Here the terms with coefficient c^0 are called N order terms, coefficient with c^{-2} of pN order and with c^{-4} are of ppN order terms. The relationship between $\bar{\mu}$ and $\bar{p}_r$ is given by [19,20]

$$
\bar{p}_r = \Gamma\frac{p_{r0}}{\mu_0 + p_{r0}}\bar{\mu}.
\tag{81}
$$

It is noted that the fluid under the expansionfree condition evolves without being compressed [29]. Thus the adiabatic index Γ (which measures the variation of pressure for a given variation of density) is irrelevant here for the case of expansionfree evolution as the perturbed energy density depends on the static configuration. Using Eq. (69) in the above equation, it follows that

$$
\bar{p}_r = 2\Gamma\frac{p_{r0}}{\mu_0 + p_{r0}}(p_{r0} - p_{\perp 0})T\frac{c}{r}.
\tag{82}
$$

From Eqs. (54) and (70), we get

$$
\frac{B'_0}{B_0} = \frac{8\pi\mu_0 r^3 - 2m_0 - 32\pi^2 E_0^2 r^3}{2r(r - 2m_0 + 16\pi^2 E_0^2 r^3)}.
\tag{83}
$$

Next, we develop dynamical equation by substituting Eq. (60) along with Eqs. (67) and (75) in Eq. (64) and neglecting the ppN order terms $\bar{p}_r$, $\bar{\mu}\frac{A'_0}{A_0}$, it follows that

$$
\begin{aligned}
& 8\pi(\mu_0 + p_{r0})r\left(\frac{a}{A_0}\right)' + 16\pi(p_{r0} - p_{\perp 0})r\left(\frac{c}{r}\right)' \\
& - 64\pi(p_{\perp 0} + 2\pi E_0^2)\frac{c}{r} - 32\pi^2 E_0\left(2e + 2rE_0\left(\frac{c}{r}\right)' + re' + re\frac{E'_0}{E_0}\right) \\
& - \frac{2}{B_0^2}\left[\left(\frac{a}{A_0}\right)'' + \left(\frac{c}{r}\right)'' + \left(2\frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right. \\
& \left. + \left(3\frac{A'_0}{A_0} - \frac{B'_0}{B_0} + \frac{4}{r}\right)\left(\frac{c}{r}\right)'\right] - 2\frac{\alpha_\Sigma}{A_0^2}\frac{c}{r} = 0.
\end{aligned}
\tag{84}
$$

In order to discuss instability conditions of this equation upto pN order, we evaluate the following terms of dynamical equation. Under expansionfree condition, Eq. (59) can be written as

$$\left(\frac{a}{A_0}\right)' = -\frac{kA_0}{r^2}\left[16\pi(p_{r0} - 2\pi E_0^2)B_0^2 - \alpha_\Sigma \left(\frac{B_0}{A_0}\right)^2 \right.$$
$$\left. + \left(\frac{A_0'}{A_0}\right)^2 - \frac{2}{r}\frac{A_0'}{A_0} + \frac{3}{r^2}(B_0^2 - 1)\right] - 16\pi^2 er B_0^2 E_0, \qquad (85)$$

where Eqs. (68) and (73) has been used. We can write two more equations by using Eqs. (85) and (68) as follows

$$\left(\frac{a}{A_0}\right)'' + \left(2\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'$$
$$= k\frac{A_0}{r^2}\left[16\pi p_{r0}B_0^2\left(\frac{1}{r} - \frac{B_0'}{B_0}\right) - 16\pi p_{r0}'B_0^2 + \frac{2}{r}\left(\frac{A_0'}{A_0}\right)'\right.$$
$$\left. - \frac{2}{r}\frac{A_0'}{A_0}\frac{B_0'}{B_0} + \frac{1}{r^2}\frac{A_0'}{A_0}(5 - 9B_0^2) - \frac{3}{r^2}\frac{B_0'}{B_0}(B_0^2 + 1) + \frac{9}{r^3}(B_0^2 - 1)\right]$$
$$+ \alpha_\Sigma k\frac{A_0}{r^2}\left(\frac{B_0}{A_0}\right)^2\left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r}\right) - 16\frac{kA_0\pi^2}{r^2}\left[12E_0^2\frac{A_0'}{A_0}\right.$$
$$\left. + 2E_0^2\left(2\frac{B_0'}{B_0} - \frac{1}{r}\right) + 8E_0E_0'\right] + 32\pi^2 r\left(3eE_0\frac{B_0'}{B_0} + eE_0' + e'E_0\right)$$
$$- 64\pi^2 er E_0\frac{A_0'}{A_0}. \qquad (86)$$

$$\left(\frac{c}{r}\right)'' + \left(3\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{4}{r}\right)\left(\frac{c}{r}\right)'$$
$$= k\frac{A_0}{r^3}\left[\left(\frac{A_0'}{A_0}\right)' - \frac{A_0'}{A_0}\frac{B_0'}{B_0} - \frac{11}{r}\frac{A_0'}{A_0} + \frac{3}{r}\frac{B_0'}{B_0}\right]. \qquad (87)$$

Combining Eqs. (86) and (87), it follows that

$$-\frac{2}{B_0^2}\left[\left(\frac{a}{A_0}\right)'' + \left(\frac{c}{r}\right)'' + \left(2\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right.$$
$$\left. - \left(3\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{4}{r}\right)\left(\frac{c}{r}\right)'\right] - 2\frac{\alpha_\Sigma}{A_0^2}\frac{c}{r}$$
$$= 32\pi k\frac{A_0}{r^2}\left[p_{r0}' + p_{r0}\left(\frac{B_0'}{B_0} - \frac{1}{r}\right)\right] - 6k\frac{A_0}{B_0^2 r^3}\left[\left(\frac{A_0'}{A_0}\right)'\right.$$
$$\left. - \frac{A_0'}{A_0}\frac{B_0'}{B_0} - (3B_0^2 + 2)\frac{1}{r}\frac{A_0'}{A_0} - \frac{1}{r}B_0B_0' + \frac{3}{r^2}(B_0^2 - 1)\right]$$

$$-2\frac{\alpha_\Sigma k}{A_0 r^2}\left(\frac{A_0'}{A_0}+\frac{B_0'}{B_0}\right)-16\frac{kA_0\pi^2}{r^2}\left[12E_0^2\frac{A_0'}{A_0}+2E_0^2\left(2\frac{B_0'}{B_0}-\frac{1}{r}\right)\right.$$

$$\left.+8E_0E_0'\right]+32\pi^2 r\left(3eE_0\frac{B_0'}{B_0}+eE_0'+e'E_0\right)-16\pi^2 er E_0\frac{A_0'}{A_0}. \quad (88)$$

Inserting Eqs. (85) and (88) in Eq. (84) and making use of Eqs. (62), (76) and (83), we obtain dynamical equation at pN order (with $c = G = 1$)

$$-\frac{8\pi}{r^2}\left\{\left(1-\frac{m_0}{r}\right)\left[2p_{ro}'+\frac{2}{r}(5p_{r0}-p_{\perp0})-8\pi E_0 E_0'-\frac{24\pi E_0^2}{r}\right]\right.$$

$$\left.+4\pi\mu_0 r(4p_{r0}-E_0^2)+(\mu_0+p_{r0})r\left[\frac{3}{r^2}(B_0^2-1)-\alpha_\Sigma\left(\frac{B_0}{A_0}\right)^2\right]\right\}$$

$$+(\mu_0+p_0)r\left[6\frac{m_0}{r^3}-\alpha_\Sigma\left(1+3\frac{m_0}{r}\right)\right]-\mu_0 r\left(\frac{3}{r^2}+4\alpha_\Sigma\right)\left(\frac{m_0}{r}\right)^2$$

$$+\frac{32\pi k}{r^2}\left[p_{r0}'-\frac{p_{r0}}{r}+4\pi r\mu_0 p_{r0}-\frac{m_0}{r}p_{r0}'-16\pi^2 E_0^2 r^2 p_{r0}+\frac{m_0}{r}p_{r0}\right.$$

$$\left.\times(64\pi^3 E^2 r^3\mu_0+16\pi^2 E_0^2 r^2)\right]-6\frac{k}{r^3}\left\{\frac{\mu_0'}{\mu_0^2}\left[p_{r0}'+\frac{2}{r}(p_{r0}-p_{\perp0})\right.\right.$$

$$\left.-\frac{4\pi E_0}{r}(2E_0+rE_0')\right]-\left(1-3\frac{m_0}{r}\right)\frac{1}{\mu_0}\left[p_{r0}''-\frac{2}{r^2}(p_{r0}-p_{\perp0})\right.$$

$$\left.+\frac{2}{r}(p_{r0}'-p_{\perp0}')-\frac{4\pi}{r}(3E_0 E_0'+rE_0 E_0''+E_0'^2)+\frac{4\pi E_0}{r^2}(2E_0+rE_0')\right]$$

$$+\frac{8\pi r^3\mu_0-2m_0-32\pi^2 E_0^2 r^3}{2r^2\mu_0}\left[p_{r0}'+\frac{2}{r}(p_{r0}-p_{\perp0})-\frac{4\pi E_0}{r}(2E_0\right.$$

$$\left.+rE_0')\right]+\left(5-9\frac{m_0}{r}\right)\frac{1}{r\mu_0}\left[p_{r0}'+\frac{2}{r^2}(p_{r0}-p_{\perp0})-\frac{4\pi E_0}{r}(2E_0+rE_0')\right]$$

$$-4\pi\mu_0\left[1+\frac{m_0}{r}+2\left(\frac{m_0}{r}\right)^2\right]+\frac{1}{r^2}\left[7\frac{m_0}{r}-5\left(\frac{m_0}{r}\right)^2\right]+16\pi^2 E_0^2$$

$$+64\pi^3\mu_0 E_0^2 r^2+48\pi^2 E_0^2\frac{m_0}{r}\right\}-2\frac{\alpha_\Sigma k}{r^2}\left\{-\left(1+\frac{m_0}{r}\right)\frac{1}{r\mu_0}\left[p_{r0}'\right.\right.$$

$$\left.+\frac{2}{r}(p_{r0}-p_{\perp0})-\frac{4\pi E_0}{r}(2E_0+rE_0')\right]+4\pi\left[1+3\frac{m_0}{r}+\left(\frac{m_0}{r}\right)^2\right]r\mu_0$$

$$-\frac{1}{r}\left[\frac{m_0}{r}+3\left(\frac{m_0}{r}\right)^2\right]-\left(1+\frac{m_0}{r}\right)64\pi^3 E_0^2\mu_0 r^3-32\pi^2 m_0 E_0^2\right\}$$

$$-16\frac{kA_0\pi^2}{r^2}\left[16\pi\mu_0 E_0^2 r^2-\frac{2E_0^2}{r}\left(1-\frac{m_0}{r}\right)+8E_0 E_0'\right]=0. \quad (89)$$

Using the fact that $\mu_0 \gg p_{r0}$, we discard the terms belonging to pN and ppN order like $\frac{p_{r0}}{\mu_0}$ in the above equation to obtain dynamical equation at N approximation as follows

$$24\pi\mu_0 + 8\pi\mid p'_{r0}\mid r + 2\left(\alpha_\Sigma - \frac{21}{r^2}\right)\frac{m_0}{r} + \frac{32\pi^2 r^3 E_0}{k}\left(2e + re' + re\frac{E'_0}{E_0}\right)$$

$$= 32\pi(5p_{r0} - 2p_{\perp 0}) - 416\pi^2 E_0^2 - 32\pi^2 r E_0 E'_0. \tag{90}$$

Here $p'_{r0} < 0$ shows that pressure is decreasing during collapse of expansionfree fluid. Using Eq. (19) in the above equation, we get

$$\frac{4\pi}{9}\mid p'_{r0}\mid + \alpha_\Sigma \frac{m_0 r^2}{9} + \frac{16\pi^2 r^6 E_0}{9k}\left(2e + re' + re\frac{E'_0}{E_0}\right)$$

$$= \frac{16\pi}{9}(5p_{r0} - 2p_{\perp 0})r^3 + \frac{4\pi}{3}\left(7\int_{r_{\Sigma(i)}}^{r}\mu_0 r^2 dr - \mu_0 r^3\right)$$

$$+ \frac{16\pi^2}{3}\left[7\int_{r_{\Sigma(i)}}^{r}(E_0 E'_0 r^3 + 2E_0^2 r^2)dr - \frac{1}{3}(13E_0^2 r^3 - r^4 E_0 E'_0)\right]. \tag{91}$$

For the instability of expansionfree fluid, we require that each term in Eq. (91) must be positive. For this purpose, the positivity of the first term of Eq. (91) leads to $p_{r0} > (\frac{2}{5})p_{\perp 0}$ and the positivity of the last two terms is determined by considering the radial profile of the energy density and electromagnetic field in the form $\mu_0 = \gamma r^m$ and $E_0 = \delta r^n$ respectively. Here γ, δ are the positive constants and m, n are constants defined in the interval $(-\infty, \infty)$. Using these solutions, the last two terms of Eq. (91) will be positive for $m \neq -3$ and $n \neq -2, 2$ if

$$r > r_{\Sigma(i)}\left(\frac{7}{4-m}\right)^{\frac{1}{m+3}}, \tag{92}$$

and

$$r > r_{\Sigma(i)}\left(\frac{14+7n}{2-n}\right)^{\frac{1}{2n+3}}. \tag{93}$$

These two equations define the range of instability. Thus instability of the system is subject to the consistency of Eqs. (92) and (93). For $m = -3$, we obtain from Eq. (92)

$$\frac{8\pi\gamma}{6}\left[7\log\left(\frac{r}{r_{\Sigma(i)}}\right) - 1\right], \tag{94}$$

which defines the instability region for $r > r_{\Sigma(i)}1.15$. For $n = -2, 2$, the range of instability is not defined.

Now we find the instability range of Eqs. (92) and (93) for the remaining values of m and n. For this purpose, we consider the following two cases.

Case (i) Here we take $m \leq 0$ and $n \leq 0$. For $m = 0$, Eq. (92) gives $r > r_{\Sigma(i)} 1.20$ which shows that the region of instability decreases from 1.20 to 1.15 as m varies from 0 to -3. When m varies from -3 to $-\infty$, the unstable region is swept out by the whole fluid, i.e., $r > r_{\Sigma(i)}$. For $n = 0$, Eq. (93) yields $r > r_{\Sigma(i)} 1.91$. This indicates that instability range varies from 1.91 to 2.33 as n varies from 0 to -1, i.e., decreases for this region and vanishes for $n \leq -2$.

Case (ii) When $m \geq 0$ and $n \geq 0$, we see from Eq. (92) that the range of instability decreases as m increases and vanishes for $m \geq 4$, while for $n \geq 0$, the range of instability in Eq. (93) varies from 1.91 to 1.83 as n varies from 0 to 1, i.e., the range of instability increases as n approaches to 1 and vanishes for $n \geq 2$. In other words, electromagnetic field reduces the instability region in the interval $(-2, 2)$.

It is mentioned here that, for the pN approximation, the physical behavior of the dynamical equation is essentially the same by considering the relativistic effects upto first order

$$24\pi\mu_0 + 8\pi |\ p'_{r0}\ |r + 2\left(\alpha_\Sigma - \frac{21}{r^2}\right)\frac{m_0}{r}$$

$$+ 16\pi |\ p'_{r0}\ |m_0 + 8\pi\alpha_\Sigma p_{r0}r^2 + 6\left(\alpha_\Sigma - \frac{5}{r^2}\right)\left(\frac{m_0}{r}\right)^2$$

$$+ \frac{32\pi^2 r^3 E_0}{k}\left(2e + re' + re\frac{E'_0}{E_0}\right) - 24\pi\mu_0\left(\frac{m_0}{r} - \alpha_\Sigma m_0^2\right)$$

$$+ 64\pi^2 E_0^2\left(1 - \frac{m_0}{r}\right) + \frac{64\pi^2 E_0^2}{r}\left(5 + \alpha_\Sigma r - 7\frac{m_0}{r}\right)$$

$$= 32\pi(5p_{r0} - 2p_{\perp0}) + 16\pi(2p_{r0} - p_{\perp0})\frac{m_0}{r} + 128\pi^3\mu_0 E_0^2 r(1 - \alpha_\Sigma r^2)$$

$$+ \frac{16\pi^2}{k}r^4 E_0\mu_0 + 32\pi^2 E_0 E'_0\left(9 - \frac{2m_0}{r}\right). \qquad (95)$$

6 Concluding remarks

This paper investigates the effects of electromagnetic field on the instability range of expansionfree fluid at Newtonian and post Newtonian regimes. In general, the instability range is defined by the adiabatic index Γ which measures the compressibility of the fluid. On the other hand, in our case, the instability range depends upon the radial profile of the energy density, electromagnetic field and the local anisotropy of pressure at N approximation, but independent of the adiabatic index Γ. This means that the stiffness of the fluid at Newtonian and post Newtonian regimes does not play any role at the instability range. It is interesting to note that independence of Γ requires the expansionfree collapse (without compression of the fluid). This shows the importance of local anisotropy, inhomogeneity energy density and electromagnetic field in the structure formation as well as evolution of self-gravitating objects.

We see from Eqs. (92) and (93) that in the absence of electromagnetic field the region of instability is taken to the whole fluid. However, with the inclusion of electromagnetic field, the region of instability decreases. Thus the system is unstable in the interval $(-2, 2)$ and stable for the remaining values of n. Also, Eqs. (92) and (93) define the instability range of the cavity associated with the expansionfree fluid. We would like to mention here that the unstable range will be customized differently for different parts of the sphere as the energy density and electromagnetic field are defined by the radial profile.

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