

$f(R)$ black holes

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Abstract We study the $f(R)$ -Maxwell black hole imposed by constant curvature and its all thermodynamic quantities, which may lead to the Reissner-Nordström-AdS black hole by redefining Newtonian constant and charge. Further, we obtain the $f(R)$ -Yang-Mills black hole imposed by constant curvature, which is related to the Einstein-Yang-Mills black hole in AdS space. Since there is no analytic black hole solution in the presence of Yang-Mills field, we obtain asymptotic solutions. Then, we confirm the presence of these solutions in a numerical way.

Keywords Yang-Mills black holes · $f(R)$ gravity

1 Introduction

$f(R)$ gravities as modified gravity theories [1–4] have much attentions as one of promising candidates for explaining the current and future accelerating phases in the evolution of universe [5, 6]. It is known that $f(R)$ gravities can be considered as general relativity (GR) with an additional scalar field. Explicitly, it was shown that the metric- $f(R)$ gravity is equivalent to the $\omega_{\text{BD}} = 0$ Brans-Dicke theory with the potential, while the Palatini- $f(R)$ gravity is equivalent to the $\omega_{\text{BD}} = -3/2$ Brans-Dicke theory

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with the potential [7]. Although the equivalence principle test (EPT) in the solar system imposes a strong constraint on $f(R)$ gravities, they may not be automatically ruled out if the Chameleon mechanism is employed to work. It is shown that the EPT allows $f(R)$ gravity models that are indistinguishable from the Λ CDM model (GR with positive cosmological constant) in the evolution of the universe [8]. However, this does not imply that there is no difference in the dynamics of perturbations [9].

On the other hand, the Schwarzschild-de Sitter black hole was obtained for a positively constant curvature scalar in [9] and other black hole solution was recently found for a non-constant curvature scalar [10]. A black hole solution was obtained from $f(R)$ gravities by requiring the negative constant curvature scalar $R = R_0$ [11]. If $1 + f'(R_0) > 0$, this black hole is similar to the Schwarzschild-AdS (SAdS) black hole. Also, it seems that there is no sizable difference in thermodynamic quantities between $f(R)$ and SAdS black holes when using the Euclidean action approach and replacing the Newtonian constant G by $G_{\text{eff}} = G/(1 + f'(R_0))$.

In order to obtain the constant curvature black hole solution from “ $f(R)$ gravity coupled to the matter”, the trace of its stress-energy tensor $T_{\mu\nu}$ should be zero. Hence, two candidates for the matter field are the Maxwell and Yang-Mills fields. Concerning the $f(R)$ -Maxwell black hole, the authors [11] have made a mistake to show the correct solution [12].

In this work, we study the $f(R)$ -Maxwell black hole and its all thermodynamic quantities, which are similar to the Reissner-Nordström-AdS (RNAdS) black hole when making appropriate replacements. We obtain the topological $f(R)$ -Maxwell black holes. Importantly, we obtain the topological $f(R)$ -Yang-Mills black holes, which are similar to the topological Einstein-Yang-Mills (dyonic) black holes in AdS space. Since there is no analytic black hole solution in the presence of Yang-Mills field, we obtain asymptotic solutions. Then, we confirm the presence of these solutions in a numerical way.

2 $f(R)$ -Maxwell black holes

Let us first consider the action for $f(R)$ gravity with Maxwell term in four dimensions

$$S_{fM} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [R + f(R) - F_{\mu\nu} F^{\mu\nu}]. \quad (2.1)$$

From the variation of the above action (2.1), the Einstein equation of motion for the metric can be written by

$$R_{\mu\nu}(1 + f'(R)) - \frac{1}{2}(R + f(R))g_{\mu\nu} + (g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)f'(R) = 2T_{\mu\nu} \quad (2.2)$$

with the stress-energy tensor

$$T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{g^{\mu\nu}}{4}F_{\rho\sigma}F^{\rho\sigma} \quad \text{with} \quad T^\mu{}_\mu = 0. \quad (2.3)$$

On the other hand, the Maxwell equation takes the form

$$\nabla_\mu F^{\mu\nu} = 0. \tag{2.4}$$

Considering the constant curvature scalar $R = R_0$, the trace of (2.2) leads to

$$R_0(1 + f'(R_0)) - 2(R_0 + f(R_0)) = 0 \tag{2.5}$$

which determines the negative constant curvature scalar as

$$R_0 = \frac{2f(R_0)}{f'(R_0) - 1} \equiv 4\Lambda_f < 0. \tag{2.6}$$

Substituting this expression into (2.2) leads to the Ricci tensor

$$R_{\mu\nu} = \Lambda_f g_{\mu\nu} + \frac{2}{1 + f'(R_0)} T_{\mu\nu}, \tag{2.7}$$

which implies that $R_{\mu\nu} \neq \Lambda_f g_{\mu\nu}$ (pure AdS₄ space) unless $T_{\mu\nu} = 0$.

We introduce a static spherically symmetric metric ansatz,

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2 d\Omega_2^2 \tag{2.8}$$

and a gauge field as a solution to (2.4)

$$A_t(r) = \frac{Q}{r_+} - \frac{Q}{r} \tag{2.9}$$

which provides an electrically charged black hole with $A_t(r_+) = 0$. Solving the Einstein equation (2.2) together with the condition of constant curvature scalar, we obtain the solution for a metric function

$$N(r) = 1 - \frac{2GM}{r} + \frac{Q^2}{(1 + f'(R_0))r^2} - \frac{R_0}{12}r^2. \tag{2.10}$$

We note that the topological *f*(*R*)-Maxwell black hole solution is also found to be

$$N_k(r) = k - \frac{2GM}{r} + \frac{Q^2}{(1 + f'(R_0))r^2} - \frac{R_0}{12}r^2 \tag{2.11}$$

when considering the metric ansatz [13, 14]

$$ds_k^2 = N_k(r)dt^2 + N_k^{-1}(r)dr^2 + r^2 d\Sigma_k^2, \tag{2.12}$$

with $d\Sigma_k^2 = d\theta^2 + \sigma_k^2(\theta)d\varphi^2$. Here $\sigma_k(\theta)$ denotes $\sin \theta$, θ and $\sinh \theta$ for $k = 1$ (spherical horizon), 0 (flat horizon), and $k = -1$ (hyperbolic horizon), respectively.

We could derive all thermodynamic quantities since the analytic solution was known as (2.10). First of all, the Hawking temperature is calculated to be

$$T_H(r_+, Q) = \frac{N'}{4\pi} \Big|_{r \rightarrow r_+} = \frac{1}{4\pi} \left[\frac{1}{r_+} - \frac{Q^2}{(1 + f'(R_0))r_+^3} - \frac{R_0 r_+}{4} \right]. \tag{2.13}$$

In order to compute other thermodynamic quantities, it would be better to use the Euclidean action approach [15] because we are working with $f(R)$ gravities. To make the action Euclidean, the time coordinate should be made imaginary by substituting $t = i\tau$. In this case, to eliminate the conical singularity at the horizon $r = r_+$, the coordinate τ should be periodic with the period $\beta = 1/T_H$. For this purpose, we have to calculate the Euclidean action [16, 17]

$$\Delta S_l^E = S_{fM}^E + S_{GH} + S_{ct} + S_{cF}, \tag{2.14}$$

where the Euclidean bulk action is

$$S_{fM}^E = -\frac{1}{16\pi G} \int d^4x \sqrt{g_E} [R + f(R) - F_{\mu\nu} F^{\mu\nu}]. \tag{2.15}$$

Here S_{GH} is the Gibbons-Hawking term to make the variation at the boundary clear and S_{ct} is the counter term for asymptotic AdS₄ space. We are working with the canonical ensemble as the fixed charge ensemble. In this case, we need to introduce the charge-fixing (cF) as a boundary surface term [16]

$$S_{cF} = \frac{1}{4\pi G} \int d^3x \sqrt{h} F^{\mu\nu} n_\nu A_\nu \tag{2.16}$$

where h_{ij} is the induced metric on the boundary surface and n_μ is a radial unit vector pointing outwards. If one does not introduce S_{cF} , one is working with the grand canonical ensemble. Also, the extremal black hole whose horizon is degenerate is considered to be the ground state in the canonical ensemble [18]. The location $r_+ = r_e$ of extremal horizon is determined by the condition of $T_H(r_e, Q) = 0$. That is, in order to derive the Helmholtz free energy, we must subtract the extremal mass $M^f = M_e^f$ from (2.14). Taking into account all leads to

$$\begin{aligned} \Delta S_l^E - M_e^f &= -\frac{\beta(1 + f'(R_0))}{48G} \left[-12r_+ - R_0 r_+^3 - \frac{36Q^2}{(1 + f'(R_0))r_+} \right. \\ &\quad \left. + 24 \left(r_e + \frac{Q^2}{(1 + f'(R_0))r_e} - \frac{R_0}{12} r_e^3 \right) \right] \tag{2.17} \end{aligned}$$

$$\equiv \beta F^f = \beta E^f - S_{BH}^f. \tag{2.18}$$

Here F^f is the Helmholtz free energy, β is the inverse of the Hawking temperature, and S_{BH}^f is the Bekenstein-Hawking entropy. The energy, Bekenstein-Hawking entropy,

and heat capacity are given as

$$E^f(r_+, Q) = \frac{\partial (\Delta S_E - M_e^f)}{\partial \beta} = M^f(r_+, Q) - M_e^f$$

$$= \frac{(1 + f'(R_0))r_+}{2G} \left[1 + \frac{Q^2}{(1 + f'(R_0))r_+^2} - \frac{R_0}{12}r_+^2 \right] - M_e^f, \tag{2.19}$$

$$S_{BH}^f = \beta E^f - \beta F^f = (1 + f'(R_0))\frac{A(r_+)}{4G}, \tag{2.20}$$

$$C^f(r_+, Q) = \left(\frac{\partial E^f}{\partial T} \right)_Q$$

$$= \frac{2(1 + f'(R_0))\pi r_+^2}{G} \left[\frac{-4r_+^2 + \frac{4Q^2}{1+f'(R_0)} + R_0r_+^4}{4r_+^2 - \frac{12Q^2}{1+f'(R_0)} + R_0r_+^4} \right], \tag{2.21}$$

where

$$M_e^f = M^f(r_e, Q) = \frac{1 + f'(R_0)}{3G} \left[r_e + \frac{2Q^2}{r_e(1 + f'(R_0))} \right] \tag{2.22}$$

is the mass of the extremal black hole and $A(r_+) = 4\pi r_+^2$ is the horizon area. In this case, E measures the energy above the ground state.

Considering replacements of

$$\frac{G}{1 + f'(R_0)} \rightarrow G_{\text{eff}}, \quad R_0 = 4\Lambda_f \rightarrow -\frac{12}{\ell^2}, \quad \frac{Q^2}{1 + f'(R_0)} \rightarrow Q_f^2, \tag{2.23}$$

the $f(R)$ -Maxwell black hole becomes the RNAdS black hole exactly. In this case, the ADM mass M^f , Hawking temperature T_H , and the Bekenstein-Hawking entropy S_{BH}^f take compact forms

$$M^f(r_+, Q_f) = \frac{r_+}{2G_{\text{eff}}} \left[1 + \frac{Q_f^2}{r_+^2} + \frac{r_+^2}{\ell^2} \right], \tag{2.24}$$

$$T_H(r_+, Q_f) = \frac{1}{4\pi} \left[\frac{1}{r_+} - \frac{Q_f^2}{r_+^3} + \frac{3r_+}{\ell^2} \right], \quad S_{BH}^f = \frac{\pi r_+^2}{G_{\text{eff}}}.$$

Finally, the heat capacity C^f and Helmholtz free energy F^f are given by

$$C^f(r_+, Q_f) = \frac{2\pi r_+^2}{G_{\text{eff}}} \left[\frac{3r_+^4 + \ell^2 (r_+^2 - Q_f^2)}{3r_+^4 + \ell^2 (-r_+^2 + 3Q_f^2)} \right], \tag{2.25}$$

$$F^f(r_+, Q_f) = \frac{1}{4G_{\text{eff}}r_+} \left[r_+^2 + 3Q_f^2 - \frac{r_+^4}{\ell^2} \right] - M_e. \quad (2.26)$$

At this stage, we have to mention the other thermodynamic quantities obtained directly from the metric function $N(r)$ in (2.10). In this case, all thermodynamic quantities of M , S_{BH} , C , F are obtained from the replacements as

$$R_0 \rightarrow -\frac{12}{\ell^2}, \quad \frac{Q^2}{1+f'(R_0)} \rightarrow Q_f^2. \quad (2.27)$$

There exists a slight difference in M , S_{BH} , C , F between G in the direct method and G_{eff} in the Euclidean action approach. The first law of thermodynamics is satisfied for both cases

$$dM^f = T_H S_{BH}^f, \quad dM = T_H S_{BH}. \quad (2.28)$$

One curious quantity derived from $f(R)$ gravities is the Bekenstein-Hawking entropy

$$S_{BH}^f = [1 + f'(R_0)] \frac{\pi r_+^2}{G} \quad (2.29)$$

which was also derived from the Wald method [7]. On the other hand, the conventional Bekenstein-Hawking entropy is

$$S_{BH} = \frac{\pi r_+^2}{G}. \quad (2.30)$$

For example, if one uses S_{BH}^f to check the first law of thermodynamics, one immediately finds that it is not satisfied as follows

$$dM \neq T_H S_{BH}^f. \quad (2.31)$$

At this stage, there is no way to test which approach provides the correct thermodynamic quantities for $f(R)$ -Maxwell black holes. Anyway, the entropy issue should be resolved.

The global features of thermodynamic quantities are shown in Fig. 1 for $G_{\text{eff}} = 1 = G$. Under this setting, there is no difference between two approaches: $M^f = M$, $S_{BH}^f = S_{BH}$, $C^f = C$, $F^f = F$. From the first and second graphs, we observe the local minimum $T_H = T_0$ (C blows up) at $r_+ = r_0$, in addition to the zero temperature $T_H = 0$ ($C = 0$) at the extremal point of $r_+ = r_e$ and the maximum value $T_H = T_m$ (C blows up) at $r_+ = r_m$ known as the Davies point. We note a sequence of $r_e < r_m < r_0$. For $r_e < r_+ < r_m$, the black hole is locally stable because of $C > 0$, while for $r_m < r_+ < r_0$ it is locally unstable ($C < 0$). For $r_+ > r_0$, the black hole becomes stable because of $C > 0$. Based on the local stability, the $f(R)$ -Maxwell black holes are split into small black hole (SBH) with $C > 0$ being in the

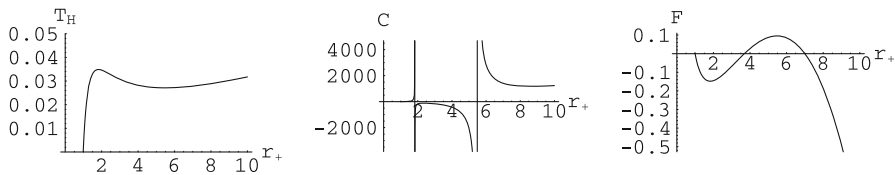


Fig. 1 Thermodynamic quantities of the $f(R)$ -Maxwell black hole as function of horizon radius r_+ with fixed $Q_f = 1$, $\ell = 10$, and $G_{\text{eff}} = 1$: temperature T_H , heat capacity C , and Helmholtz free energy F

region of $r_e < r_+ < r_m$, intermediate black hole (IBH) with $C < 0$ in the region of $r_m < r_+ < r_0$, and large black hole (LBH) with $C > 0$ in the region of $r_+ > r_0$.

Importantly, the free energy from the last graph in Fig. 1 plays a crucial role to test the phase transition. A black hole is globally stable when $C > 0$ and $F < 0$. We note that $F = 0$ at $r_+ = r_e$, because of $F = M - M_e - T_H S_{BH}$ with $T_H(r_e, Q_f) = 0$. We observe two extremal points for free energy: the local minimum $F = F_{\text{min}}$ at $r_+ = r_m$ and the maximum value $F = F_{\text{max}}$ at $r_+ = r_0$. The free energy is negative for $r_e < r_+ < r_m$ and it increases in the region of $r_m < r_+ < r_0$. For a point of $r_+ = r_1 > r_0$, it is zero and remains negative for $r_+ > r_1$. The temperature of $T = T_1$ (determined from the condition of $F = 0$) at $r_+ = r_1$ may play a role of the critical temperature in Hawking-Page phase transitions. The related phase transition was discussed in Ref. [19]. It can be shown that the Hawking-Page phase transition II between SBH and LBH unlikely occurs in the $f(R)$ -Maxwell black holes.

3 $f(R)$ -Yang-Mills black holes

We consider the action of $f(R)$ gravity coupled to $SU(2)$ Yang-Mills field in four dimensions

$$S_{fYM} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \{R + f(R) - F_{\mu\nu}^a F^{\mu\nu a}\}, \tag{3.1}$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c$. From the action (3.1), the Einstein equation of motion can be written by

$$R_{\mu\nu}(1 + f'(R)) - \frac{1}{2}(R + f(R))g_{\mu\nu} + (g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu)f'(R) = 2T_{\mu\nu}^{\text{YM}} \tag{3.2}$$

with $T_{\mu\nu}^{\text{YM}}$ the stress-energy tensor for the Yang-Mills field. For the constant curvature scalar $R = R_0$, taking the trace of (3.2) leads to

$$R_0(1 + f'(R_0)) - 2(R_0 + f(R_0)) = 0 \tag{3.3}$$

which determines the constant curvature scalar as

$$R_0 = \frac{2f(R_0)}{f'(R_0) - 1} \equiv 4\Lambda_f < 0. \tag{3.4}$$

Now we consider the topological metric ansatz

$$ds_k^2 = -e^{2\phi(r)}N(r)dt^2 + N^{-1}(r)dr^2 + r^2d\Sigma_k^2, \tag{3.5}$$

with $d\Sigma_k^2 = d\theta^2 + \sigma_k^2(\theta)d\varphi^2$. A dyonic solution ansatz for Yang-Mills gauge field is given by

$$A = \{u(r)\tau_3dt + \omega(r)\tau_1d\theta + [\partial_\theta\sigma(\theta)\tau_3 + \sigma(\theta)\omega(r)\tau_2]d\varphi\}, \tag{3.6}$$

where $u(r)$ [$\omega(r)$] describe the electric [magnetic] charged configurations and τ_i is the Pauli spin matrices for $SU(2)$.

Substituting (3.5) and (3.6) into the action (3.1), and after variations with respect to N , ϕ , ω , u , one finds their equations of motion:

$$\begin{aligned} \delta_N S; \quad & 2r(1 + f'(R))\phi' - r^2 \left\{ f'''(R)(R'(r))^2 + f''(R)(-\phi'R'(r) + R''(r)) \right\} \\ & - 4(\omega')^2 - \frac{4e^{-2\phi}u^2\omega^2}{N^2} = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \delta_\phi S; \quad & -2k + 2rN' + 2N - r^2f(R) + 4N(\omega')^2 + \frac{2(\omega^2 - k)^2}{r^2} \\ & + 2e^{-2\phi} \left(r^2(u')^2 + \frac{2u^2\omega^2}{N} \right) + 2r^2Nf'''(R)(R'(r))^2 \\ & + rf''(R) \{ 4NR'(r) + 2rNR''(r) + rN'R'(r) \} \\ & + rf'(R) \left\{ -2rN(\phi')^2 - 4N\phi' - rN'' - 2rN\phi'' - 2N' \right. \\ & \quad \left. - 3rN'\phi' \right\} = 0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \delta_\omega S; \quad & r^2N\omega'' + r^2(N' + \phi'N)\omega' - \omega(\omega^2 - k) + \frac{e^{-2\phi}r^2u^2\omega}{N} \\ & = 0, \end{aligned} \tag{3.9}$$

$$\delta_u S; \quad r^2u'' + (-r^2\phi' + 2r)u' - \frac{2\omega^2u}{N} = 0, \tag{3.10}$$

where the curvature scalar $R(r)$ is given by

$$R(r) = -\frac{1}{r^2} \left[-2 + rN'(4 + 3r\phi') + r^2N'' + N(2 + 2r^2(\phi')^2 + 4r\phi' + 2r^2\phi'') \right]. \tag{3.11}$$

Note that the prime ($'$) in $f(R)$ and N , ω , ϕ denotes the differentiation with respect to R and r , respectively. It is a formidable task to solve the above four equations directly. Therefore, we consider the constant curvature scalar which implies that

$$R(r) = R_0, \quad R'(r) = R''(r) = 0. \tag{3.12}$$

Actually, we have used the condition (3.12) to derive the $f(R)$ -Maxwell black holes in the previous section. Plugging (3.12) into the four equations leads to simplified

equations

$$\delta_N S; \quad r(1 + f'(R_0))\phi' - 2(\omega')^2 - \frac{2e^{-2\phi}u^2\omega^2}{N^2} = 0, \tag{3.13}$$

$$\begin{aligned} \delta_\phi S; & -2k + 2rN' + 2N - r^2f(R_0) + 4N(\omega')^2 + \frac{2(\omega^2 - k)^2}{r^2} \\ & + 2e^{-2\phi} \left(r^2(u')^2 + \frac{2u^2\omega^2}{N} \right) \\ & + rf'(R_0)(-2rN(\phi')^2 - 4N\phi' - rN'' - 2rN\phi'' - 2N' - 3rN'\phi') = 0, \end{aligned} \tag{3.14}$$

$$\delta_\omega S; \quad r^2N\omega'' + r^2(N' + \phi'N)\omega' - \omega(\omega^2 - k) + \frac{e^{-2\phi}r^2u^2\omega}{N} = 0, \tag{3.15}$$

$$\delta_u S; \quad r^2u'' + (-r^2\phi' + 2r)u' - 2\frac{\omega^2u}{N} = 0. \tag{3.16}$$

Note that for $f(R_0) = f'(R_0) = 0$, they reduce to those derived from the topological Einstein-Yang-Mills theory [22]. It is well known that there is no analytic black hole solution to the Einstein-Yang-Mills theory. Hence, we can find either the asymptotic solution with finite terms or the numerical solution.

First, we wish to derive the asymptotic solution at infinity of $r \rightarrow \infty$. Equations (3.13)–(3.16) can be solved by considering asymptotic forms for metric and gauge field functions up to $\frac{1}{r^5}$ -order

$$N(r) = k - \frac{2m(r)}{r} - \frac{R_0}{12}r^2, \tag{3.17}$$

$$m(r) = M + \frac{M_1}{r} + \frac{M_2}{r^2} + \frac{M_3}{r^3} + \frac{M_4}{r^4} + \frac{M_5}{r^5} + O\left(\frac{1}{r^6}\right), \tag{3.18}$$

$$\omega(r) = \omega_\infty + \frac{\omega_1}{r} + \frac{\omega_2}{r^2} + \frac{\omega_3}{r^3} + \frac{\omega_4}{r^4} + \frac{\omega_5}{r^5} + O\left(\frac{1}{r^6}\right), \tag{3.19}$$

$$u(r) = u_\infty + \frac{u_1}{r} + \frac{u_2}{r^2} + \frac{u_3}{r^3} + \frac{u_4}{r^4} + \frac{u_5}{r^5} + O\left(\frac{1}{r^6}\right), \tag{3.20}$$

where $M, \omega_\infty, \omega_1, u_\infty$, and u_1 are five constants evaluated at infinity and other M_i, ω_i , and u_i are expressed in terms of these constants and $1 + f'(R_0)$ appeared in Appendix A. Let us compare $f(R)$ -Yang-Mills (fYM) black holes with Einstein-Yang-Mills (EYM) black holes. We observe the relations of coefficients between two black holes

$$M_i^{\text{fYM}} = \frac{M_i^{\text{EYM}}}{1 + f'(R_0)}, \quad \text{for } i = 1, 2, 3, 4 \tag{3.21}$$

$$\omega_i^{\text{fYM}} = \omega_i^{\text{EYM}}, \quad u_i^{\text{fYM}} = u_i^{\text{EYM}}, \quad \text{for } i = 2, 3, 4. \tag{3.22}$$

It seems that there is no longer simple relations between two black holes for $i \geq 5$. Hence, it is not easy to derive any concrete form for thermodynamic quantities of $f(R)$ -Yang-Mills black holes. Exceptionally, the form of Hawking temperature can be derived to be

$$T_H^{\text{fYM}} = \frac{1}{4\pi r_+} \left[k - \frac{R_0}{4} r_+^2 - 2m'(r_+) \right] \tag{3.23}$$

because it will be determined by the variables defined at horizon. Using (4.11), it takes the form

$$T_H^{\text{fYM}} = \frac{1}{4\pi r_+} \left[k - \frac{R_0}{4} r_+^2 - \frac{(k - \omega_+^2)^2 + r_+^4 u_0^2 e^{-2\phi_+}}{r_+^2 (1 + f'(R_0))} \right]. \tag{3.24}$$

In the case of purely magnetic charged black hole with $u_0 = 0$ and $f'(R_0) = 0$, it reduces to Eq. (28) in Ref. [22]. Here $u_0 = u'(r_+)$ may be considered as a counterpart of $A'_i(r_+) = Q/r_+^2$ in the $f(R)$ -Maxwell black holes. Furthermore, one finds the metric function $N(m(r)) \simeq M + M_1/r$ up to $\frac{1}{r^2}$ -order and gauge field functions ω and u up to $\frac{1}{r}$ -order

$$N(r) \simeq k - \frac{2M}{r} + \left\{ \frac{Q_M^2 + Q^2 - \frac{R_0 J^2}{6} - 24 \frac{u_\infty^2 \omega_\infty^2}{R_0}}{(1 + f'(R_0))} \right\} \frac{1}{r^2} - \frac{R_0}{12} r^2, \tag{3.25}$$

$$\omega \simeq \omega_\infty + \frac{J}{r}, \quad u(r) \simeq u_\infty - \frac{Q}{r} \tag{3.26}$$

with the Yang-Mills magnetic charge $Q_M = k - \omega_\infty^2$ [22] and the Yang-Mills electric charge Q . Here we reset $\omega_1 = J$ and $u_1 = -Q$ to make a connection to holographic super-conducting models using the AdS/CFT correspondence [23] and [24] for higher dimensional cases. Using the holographic interpretation with $\omega_\infty = 0$, u_∞ is the chemical potential, Q is the electric charge, and J is the component of the current J_i on the boundary at infinity which is connected with the spontaneously broken part of the bulk gauge symmetry.

We note that (3.25) and (3.26) with $Q_M^2 = 0$ and $u_\infty = Q/r_+$ [imposed by $u(r_+) = 0$] reduce to those of the topological $f(R)$ -Maxwell black holes when turning off the magnetic charge gauge potential and setting $u_i = 0 (i \geq 2)$. Especially, the constant $\omega_1 = J$ corresponds to an order parameter describing the deviation from the Abelian solution of $f(R)$ -Maxwell black holes.

Finally, it is also interesting to explore the other case of purely magnetic charged black holes obtained by choosing $k = 1$, $u(r) = 0$. Its asymptotic solution appeared in Appendix B. In the case of Einstein-Yang-Mills black holes, these black holes are stable against gravitational and sphaleronic perturbations for $\omega_+ > 1/\sqrt{3} = 0.577$ for large $|\Lambda|$ [20]. Actually, the stability condition corresponds to that a gauge field $\omega(r)$ has no zero. Hence, we conjecture that purely magnetic charged black holes in the $f(R)$ -Yang-Mills theory has a similar property because for constant curvature scalar

and $1 + f'(R_0) > 0$ (no ghost condition), the $f(R)$ -modification to the Einstein-Yang-Mills black hole will be minimized. We will check in the next section that the zero of $\omega(r)$ appears only for $\omega_+ < 1/\sqrt{3} = 0.577$. Furthermore, there exist nodeless solutions for $k = 0, 1$ Einstein-Yang-Mills black holes [22], which means that these black holes are stable.

In the next section, we will find numerical solutions for $k = 1, 0$ dyonic black holes and $k = 1$ purely magnetic charged black holes.

4 Numerical results

We numerically solve (3.13)–(3.16) with boundary conditions of (3.25) and (3.26) using a standard shooting method in *Mathematica*[®]7. The $k = 1$ Einstein-Yang-Mills black holes was discussed in Ref. [20,21], while $k = 0, -1$ Einstein-Yang-Mills black holes was found numerically in Ref. [22].

In order to obtain numerical solutions, we transform Eqs. (3.13)–(3.16) into

$$\phi'(r) = \frac{2(\omega^2 u^2 e^{-2\phi} + (\omega')^2 N^2)}{r(1 + f'(R_0))N^2}, \tag{4.1}$$

$$m'(r) = \frac{2r^2(\omega^2 u^2 e^{-2\phi} + (\omega')^2 N^2) + N((k - \omega^2)^2 + r^4((ue^{-\phi})')^2)}{2r^2(1 + f'(R_0))N} + \frac{2ruu'(\omega^2 u^2 e^{-2\phi} + (\omega')^2 N^2)}{(1 + f'(R_0))^2 N^2} + \frac{2u^2(\omega^2 u^2 e^{-2\phi} + (\omega')^2 N^2)^2}{(1 + f'(R_0))^3 N^4}, \tag{4.2}$$

$$\omega''(r) = -\phi'\omega' - \frac{N'\omega'}{N} - \frac{\omega(k - \omega^2)}{r^2 N} - \frac{\omega u^2 e^{-2\phi}}{N^2}, \tag{4.3}$$

$$u''(r) = \phi'u' - \frac{2u'}{r} + \frac{2u\omega^2}{r^2 N}, \tag{4.4}$$

where we used the relation (3.4) to include $1 + f'(R_0)$ only as $f(R)$ -gravity effects. First, let us develop the solution forms near the non-degenerate horizon at $r = r_+$. Because of $N(r_+) = 0$, we derive a relation of $u(r_+)\omega(r_+) = 0$ from Eq. (4.4). Choosing $\omega(r_+) = 0$, it is easily shown that $\omega'(r_+) = \omega''(r_+) = 0$, which implies that $\omega(r) = 0$. This is not the case. So we choose $u(r_+) = 0$ instead, and the solution near the non-degenerate horizon can be expanded as

$$\phi(r) = \phi_+ + \phi'(r_+)(r - r_+) + O(r - r_+)^2, \tag{4.5}$$

$$m(r) = m_+ + m'(r_+)(r - r_+) + O(r - r_+)^2, \tag{4.6}$$

$$\omega(r) = \omega_+ + \omega'(r_+)(r - r_+) + O(r - r_+)^2, \tag{4.7}$$

$$u(r) = u_0(r - r_+) + \frac{u''(r_+)}{2}(r - r_+)^2 + O(r - r_+)^3, \tag{4.8}$$

where the coefficients are determined by equations

$$m_+ = m(r_+) = \frac{r_+}{2} \left(1 - \frac{R_0 r_+^2}{12} \right), \quad (4.9)$$

$$\phi'(r_+) = \frac{2\omega_+^2 \left((k - \omega_+^2)^2 + r_+^4 u_0^2 e^{-2\phi_+} \right)}{r_+ (N'(r_+))^2 (1 + f'(R_0))}, \quad (4.10)$$

$$m'(r_+) = \frac{(k - \omega_+^2)^2 + r_+^4 u_0^2 e^{-2\phi_+}}{2r_+^2 (1 + f'(R_0))}, \quad (4.11)$$

$$\omega'(r_+) = -\frac{\omega_+ (k - \omega_+^2)}{r_+^2 N'(r_+)}, \quad (4.12)$$

$$u''(r_+) = -\frac{2u_0}{r_+} \left(1 - \frac{r_+^2 \omega_+^2 (4k - R_0 r_+^2)}{4(N'(r_+))^2} \right) \quad (4.13)$$

which satisfy at the horizon $r = r_+$. Since the metric function N is zero at the horizon $r = r_+$ and it should be positive outside the horizon, we have to choose a condition of $N'(r_+) > 0$. This restricts the range of ω_+ through the inequality

$$2m'(r_+) < k - \frac{R_0 r_+^2}{4} \quad (4.14)$$

which yields a positiveness of $\omega'(r_+) > 0$ for $\omega_+ (\omega_+^2 - k) > 0$, while $m'(r_+) > 0$ for $1 + f'(R_0) > 0$. Note that $\phi'(r_+)$, $m'(r_+)$, $\omega'(r_+)$, and $u''(r_+)$ depend on r_+ , ϕ_+ , ω_+ , and u_0 , which means that they are four independent parameters describing the near horizon geometry of (4.5)–(4.8). On the other hand, there are five independent parameters of M , ω_∞ , ω_1 , u_∞ , and u_1 describing asymptotic region of (3.18)–(3.20). Remembering that (4.1)–(4.4) are two first- and two second-order differential equations, we need six initial parameters to solve the equations numerically at each boundary of horizon and asymptotic infinity. However, considering $N(r_+) = 0$, we choose $u(r_+) = 0$ and then, (4.3) leads to a first-order differential equation. This is why four independent parameters is enough to specify the near horizon geometry of $f(R)$ -Yang-Mills black hole. In addition, we have time-rescaling symmetry so that we can replace ϕ by $\phi + \phi_0$ without loss of generality. This means that either ϕ_+ or $\phi(\infty)$ can be cast to zero. Here we choose $\phi(\infty) = 0$ to achieve an asymptotic AdS_4 space which is similar to that of $f(R)$ -Maxwell black hole solution. Hence, the asymptotic solution has five parameters less than six of general analysis. In this manner, we show that ϕ_+ is not an arbitrary parameter.

We are now in a position to solve the initial value problem, for given r_+ , R_0 , ω_+ , and u_0 , by introducing a specific form of $f(R)$ gravity as

$$f(R) = -\frac{\alpha c_1 \left(\frac{R}{\alpha}\right)^n}{1 + \beta \left(\frac{R}{\alpha}\right)^n} \quad (4.15)$$

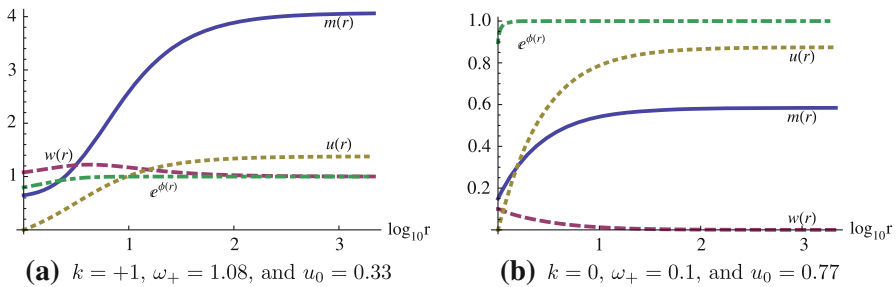


Fig. 2 The numerically solved functions m (solid lines), ω (dashed lines), u (dotted lines), and e^{ϕ} (dot-dashed lines) are depicted with respect to radius $\log_{10} r$ for $r_+ = 1$ and $R_0 = -3.6$ ($\alpha = -0.9, \beta = 1.125, c_1 = 3.025$). **a** For $k = 1$, we have $m_+ = 0.65$ and $\phi_+ = -0.23$ with $\phi(\infty) = 0$. Then, we obtain $\omega_\infty = 1.0$ so that $Q_M = (1 - \omega_\infty^2) = 0$. **b** For $k = 0$, we get $m_+ = 0.15$ and $\phi_+ = -0.11$ with $\phi(\infty) = 0$. Then, we obtain $\omega_\infty = 0$ so that $Q_M = 0$

proposed in Ref. [25] and setting $n = 1$ for simplicity in this work. Imposing the constant curvature scalar (3.4), Eq.(4.15) implies

$$c_1 = \frac{\left(1 + \frac{\beta R_0}{\alpha}\right)^2}{1 + \frac{2\beta R_0}{\alpha}}. \tag{4.16}$$

The Einstein limit exists in the constant curvature case, which shows that $c_1 \rightarrow 2\beta\Lambda_f/\alpha$ as $\alpha \rightarrow 0$. Then, $f(R)$ becomes a negative cosmological constant ($f(R) \rightarrow -2\Lambda_f$). In what follows, however, we consider only a nonvanishing α case.

The numerical solutions to (4.1)–(4.4) are depicted in Fig. 2 graphically for $\alpha = -0.9, \beta = 1.125, c_1 = 3.025$, for which R_0 and $f'(R_0)$ are fixed to either $R_0 = -3.6$ and $f'(R_0) = -0.1$ or $R_0 = 1.8$ and $f'(R_0) = -10$. In this work, since we are interested in asymptotically AdS space with $1 + f'(R_0) > 0$, we choose the former of $R_0 = -3.6$ and $f'(R_0) = -0.1$. Choosing the horizon radius to be $r_+ = 1$, the left and right figures are distinguished by specifying remaining parameters k, ω_+ and u_0 :(a) $k = 1, \omega_+ = 1.08$ and $u_0 = 0.33$ (b) $k = 0, \omega_+ = 0.1$ and $u_0 = 0.77$. Furthermore, these numerical solutions are being used to determine the form of parameters in asymptotic AdS space. Matching its asymptotic forms (3.25) and (3.26), we find that for (a), $M \approx 4.07, \omega_\infty \approx 1.00, \omega_1 \approx 2.63, u_\infty \approx 1.38$, and $u_1 \approx -4.14$, while for (b), $M \approx 0.58, \omega_\infty \approx 0.00, \omega_1 \approx 0.13, u_\infty \approx 0.87$, and $u_1 \approx -0.87$. At this stage, we point out that that the magnetic charge $Q_M = (k - \omega_\infty^2)$ vanishes for both cases, so that their asymptotic geometry are similar to the $f(R)$ -Maxwell black holes.

It is also interesting to explore the other solution of purely magnetic charged black holes numerically by choosing $k = 1, u(r) = 0$. Considering its asymptotic solution appeared in Appendix B, we find the numerical solutions. For a given $f(R)$ -form (4.15), the numerical solutions to Eqs. (5.2)–(5.4) could be developed for the same values as in the dyonic black hole solution. Setting the horizon at $r_+ = 1$, the two graphs in Fig. 3 are distinguished by specifying a remaining parameter ω_+ : (a) $\omega_+ = 1.08$ (b) $\omega_+ = 0.1$. Furthermore, this numerical solutions are used to find the parameters in the asymptotic solutions. We find $M \approx 1.19, \omega_\infty \approx 1.17, \omega_1 \approx -2.54$ for (a), while

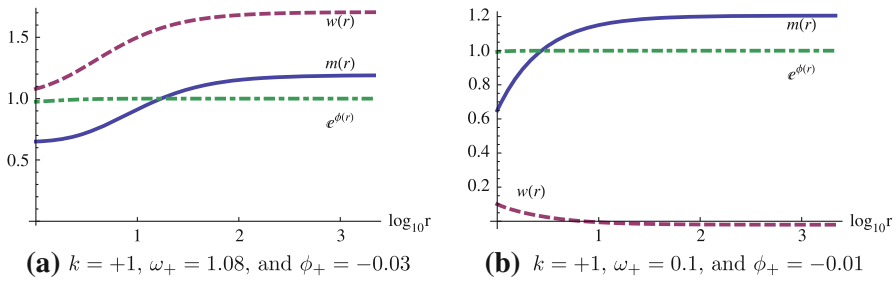


Fig. 3 The numerically solved functions m (solid lines), ω (dashed lines), and e^ϕ (dot-dashed lines) of purely magnetic case ($u(r) = 0$) are depicted with respect to radius $\log_{10} r$ for $r_+ = 1$, and $R_0 = -3.6$ ($\alpha = -0.9, \beta = 1.125, c_1 = 3.025$). For convenience, $k = 1$ case is plotted only, and we have $m_+ = 0.65$

$M \approx 1.21, \omega_\infty \approx -0.02, \omega_1 \approx 0.14$ for (b). The stability condition for this black hole corresponds to the condition that a gauge field $\omega(r)$ has no zero. As is shown in the Fig. 3, we find that the zero of $\omega(r)$ appears for $\omega_+ < 1/\sqrt{3} = 0.577$. Hence, we conjecture that the stability condition for the Einstein-Yang-Mills black hole holds for the $f(R)$ -Yang-Mills black holes with $1 + f'(R_0) > 0$.

5 Discussions

First of all, we summarize all numerical solutions to $f(R)$ -Yang-Mills black holes in Table 1. This table shows asymptotic solution forms constructed in the numerical way: two dyonic solutions for $k = 1$ and $k = 0$ black holes and two magnetically charged black holes for $k = 1$ and $\omega_+ = 1.08, 0.1$. The former was developed to compare with the topological Einstein-Maxwell black holes, while the latter was displayed to see the stability of $f(R)$ -Yang-Mills black holes. We find that for $1 + f'(R_0) > 0$, the $f(R)$ -Yang-Mills black holes are similar to Einstein-Yang-Mills black holes in AdS space. In this case, one may develop the second-order phase transition between $f(R)$ -Maxwell and $f(R)$ -Yang-Mills black holes to explain the holographic superconductor without Higgs field as in the Einstein theory [23]. The difference is that

Table 1 Asymptotic forms of numerical solution for $f(R)$ -Yang-Mills black holes

	Dyonic solution		Purely magnetic solution	
	$k = +1$ ($\omega_+ = 1.08, u_0 = 0.33$)	$k = 0$ ($\omega_+ = 0.1, u_0 = 0.77$)	$\omega_+ = 1.08$ ($k = +1$)	$\omega_+ = 0.1$ ($k = +1$)
$m(r) \approx$	$4.07 - \frac{18.87}{r}$	$0.58 - \frac{0.42}{r}$	$1.19 - \frac{4.18}{r}$	$1.21 - \frac{0.56}{r}$
$\omega(r) \approx$	$1.00 + \frac{2.63}{r}$	$0.00 + \frac{0.13}{r}$	$1.71 - \frac{2.54}{r}$	$-0.02 + \frac{0.14}{r}$
$u(r) \approx$	$1.38 - \frac{4.14}{r}$	$0.87 - \frac{0.87}{r}$	0	0
$\phi(r) \approx$	$-\frac{15.57}{r^4}$	$-\frac{0.01}{r^4}$	$-\frac{3.58}{r^4}$	$-\frac{0.01}{r^4}$

the AdS_4 space was constructed not by introducing a cosmological constant, but by choosing an appropriate $f(R)$ function in (3.4). Also, it seems that for $1 + f'(R_0) > 0$, the stability condition of magnetically charged Einstein-Yang-Mills black holes holds for magnetically charged $f(R)$ -Yang-Mills black holes.

The condition of $1 + f'(R_0) > 0$ is related to no ghost state for graviton propagations on AdS_4 space [26–29], the positiveness of effective Newton constant $G_{\text{eff}} > 0$ in cosmological implications [7, 8], and a necessary condition that $f(R)$ black hole becomes a type of Schwarzschild-AdS black hole [11]. In this work, this condition is necessary to obtain $f(R)$ -Maxwell black hole and to derive its thermodynamic quantities. Also, $f(R)$ -Yang-Mills black holes requires this condition to have asymptotic and numerical solutions. The other condition of $f''(R_0) < 0$ is not necessary to obtain the constant curvature black hole solutions. However, this condition may be needed to be free from the Dolgov-Kawasaki instability related to tachyonic mass [2, 7] in the perturbation analysis of $f(R)$ -Maxwell (Yang-Mills) black holes. We hope to make a progress on the perturbation analysis.

At this stage, we would like to mention the close connection between $f(R)$ and Einstein black holes by rewriting the action (2.1) as

$$\tilde{S}_{fM} = \int d^4x \sqrt{-g} \left[\frac{1}{16\pi G} \{R + f(R)\} - F_{\mu\nu} F^{\mu\nu} \right]. \tag{5.1}$$

In this case, Einstein equation takes the form instead of (2.7)

$$R_{\mu\nu} = \Lambda_f g_{\mu\nu} + \frac{8\pi G}{1 + f'(R_0)} \tilde{T}_{\mu\nu} \tag{5.2}$$

with $\tilde{T}_{\mu\nu} = 4T_{\mu\nu}$. Introducing a replacement of $G_{\text{eff}} \rightarrow G/(1 + f'(R_0))$, the above equation become

$$R_{\mu\nu} = \Lambda_f g_{\mu\nu} + 8\pi G_{\text{eff}} \tilde{T}_{\mu\nu}. \tag{5.3}$$

The solution is determined by

$$\tilde{N}(r) = 1 - \frac{GM}{r} + \frac{16\pi G_{\text{eff}} Q^2}{r^2} - \frac{R_0 r^2}{12} \tag{5.4}$$

which is the same form as (2.10) in the unit $16\pi G = 1$. Also, we may make such a replacement for $f(R)$ -Yang Mills theory by rewriting (3.1) as \tilde{S}_{fYM} . In this case, the M_i , ω_i , and u_i in Appendix A including the substitution rules (3.21) and (3.22) may be conjectured by following $v = 4\pi G/e^2$ with $e^2 = 1 + f'(R_0)$ in Ref. [21], where $-F^2/4$ was used instead of $-F^2$.

Finally, we wish to comment on two points. One is to answer to the question of “is it possible to apply the reconstruction technique of Ref. [30] which was developed for pure $f(R)$ gravity to the $f(R)$ with Maxwell (Yang-Mills) field?”. The answer is “yes” because the pure metric $f(R)$ gravity is equivalent to the ω_{BD} Brans-Dicke theory with the potential term. Expressing $\phi = 1 + f'(R)$, we transform (2.1) and (3.1)

into the Brans-Dicke theory $[\phi, V(\phi) = f - Rf'(R)]$ with Maxwell (Yang-Mills) field. As far as the constant curvature scalar black hole solution is concerned, we can obtain the same black hole solution from the reconstructed Brans-Dicke theory with Maxwell (Yang-Mills) field [31,32]. The other is to answer to the question of “in the case of coupled YM- $f(R)$ theory [33],

$$S_{YMf} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ R - 4\pi G(1 + f(R))F_{\mu\nu}^a F^{\mu\nu a} \left[1 + b\tilde{g}^2 \ln \times \left[\frac{-0.5 F_{\mu\nu}^a F^{\mu\nu a}}{\mu^4} \right] \right] \right\} \tag{5.5}$$

do we expect to obtain the similar solution?”. In the case of $f(R) = 0$, the last term of the above action reduces to the effective Lagrangian of $SU(N)$ Yang-Mills theory up to one-loop order with

$$b = \frac{1}{4} \frac{1}{8\pi^2} \frac{11}{3} N. \tag{5.6}$$

Even though its equation of motions take complicated forms, we expect to have similar numerical solution found here for the constant curvature scalar black hole. This may be true because for the constant curvature scalar black hole, the replacement of $G_{eff} \rightarrow G/(1 + f'(R_0))$ is expected to make the numerical solution simple unless b log-term plays an important role.

Consequently, the $f(R)$ -Maxwell (Yang-Mills) black holes imposed by constant curvature scalar and $1 + f'(R_0) > 0$ are closely related to the Einstein-Maxwell (Yang-Mills) black holes in AdS space.

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Appendix A: Coefficients for asymptotic solution to $f(R)$ -Yang-Mills black holes

$$M_1 = -\frac{k^2 + u_1^2 - \omega_1^2 R_0/6 - 2(k + 12u_\infty^2/R_0)\omega_\infty^2 + \omega_\infty^4}{2(1 + f'(R_0))}$$

$$M_2 = -\frac{2\omega_\infty(-12u_1 u_\infty \omega_\infty/R_0 + \omega_1(-k + \omega_\infty^2))}{1 + f'(R_0)}$$

$$M_3 = \frac{16u_1 u_\infty \omega_1 \omega_\infty + R_0 \omega_1^2(k - 2\omega_\infty^2) + 8\omega_\infty^2(u_1^2 + (k - \omega_\infty^2)^2 - 6u_\infty^2(k + 2\omega_\infty^2)/R_0)}{R_0(1 + f'(R_0))}$$

$$M_4 = -\frac{1}{2(1 + f'(R_0))} \left[\left(M - \frac{8u_1 u_\infty}{R_0} \right) \omega_1^2 + \frac{48u_\infty \omega_\infty^2 \{ 3Mu_\infty - 4u_1(k - 6u_\infty^2/R_0 - 2\omega_\infty^2) \}}{R_0^2} \right. \\ \left. + 2\omega_1^3 \omega_\infty - \frac{16\omega_1 \omega_\infty(2k^2 + u_1^2 - 5k\omega_\infty^2 + 3\omega_\infty^4)}{R_0} + \frac{96\omega_1 \omega_\infty u_\infty^2(2k + \omega_\infty^2)}{R_0^2} \right]$$

$$\begin{aligned}
 M_5 = & \frac{1}{30(1 + f'(R_0))} \left[-6\omega_1^4 + \frac{12\omega_1^2 (21k^2 + 6u_1^2 - 99k\omega_\infty^2 + 78\omega_\infty^4)}{R_0} \right. \\
 & - \frac{576\omega_1^2 u_\infty^2 (k - 2\omega_\infty^2)}{R_0^2} + \frac{288\omega_1 \omega_\infty u_1 u_\infty (9k - 48u_\infty^2/R_0 - 28\omega_\infty^2)}{R_0^2} \\
 & - \frac{360\omega_1 \omega_\infty M (k - \omega_\infty^2)}{R_0} - \frac{4320\omega_\infty^2 M u_1 u_\infty + 144\omega_\infty^2 (13k - 12\omega_\infty^2) (k - \omega_\infty^2)^2}{R_0^2} \\
 & + \frac{144\omega_\infty^2 u_1^2 (13k - 12\omega_\infty^2)}{R_0^2} - \frac{17280\omega_\infty^2 u_1^2 u_\infty^2}{R_0^3} + \frac{41472\omega_\infty^2 u_\infty^4 (2k + 3\omega_\infty^2)}{R_0^4} \\
 & \left. - \frac{1728\omega_\infty^2 u_\infty^2 (k - \omega_\infty^2) (13k + 6\omega_\infty^2)}{R_0^3} \right] - \frac{R_0}{60(1 + f'(R_0))^2} \left[\left(\omega_1^2 + \frac{144u_\infty^2 \omega_\infty^2}{R_0^2} \right) \right. \\
 & \left. \times \left(\omega_1^2 - \frac{12(u_1^2 + k^2 - 2k\omega_\infty^2 + \omega_\infty^4)}{R_0} - \frac{144\omega_\infty^2 u_\infty^2}{R_0^2} \right) \right] \\
 \omega_2 = & \frac{6}{R_0} \omega_\infty (k - 12u_\infty^2/R_0 - \omega_\infty^2) \\
 \omega_3 = & \frac{6}{R_0} (-8u_1 u_\infty \omega_\infty / R_0 + \omega_1 (k - 4u_\infty^2/R_0 - \omega_\infty^2)) \\
 \omega_4 = & -\frac{1}{2R_0} \left[4(3M + 12u_1 u_\infty / R_0) \omega_1 - \frac{12\omega_\infty (7k^2 - 2u_1^2 + 144u_\infty^4/R_0 - 10k\omega_\infty^2 + 3\omega_\infty^4)}{R_0} \right. \\
 & \left. + 6\omega_1^2 \omega_\infty + \frac{288\omega_\infty u_\infty^2 (5k - 4\omega_\infty^2)}{R_0^2} \right] \\
 \omega_5 = & -\frac{1}{10R_0} \left[6\omega_1^3 + \frac{576\omega_\infty u_1 u_\infty (11k - 24u_\infty^2/R_0 - 7\omega_\infty^2)}{R_0^2} + \frac{144\omega_\infty M (4k - 60u_\infty^2/R_0 - 4\omega_\infty^2)}{R_0} \right. \\
 & \left. - \frac{12\omega_1 \{39k^2 - 6u_1^2 + 144u_\infty^4/R_0^2 - 66k\omega_\infty^2 + 27\omega_\infty^4 - 264u_\infty^2 (k - 2\omega_\infty^2) / R_0\}}{R_0} \right] \\
 & - \frac{\omega_1}{10(1 + f'(R_0))} \left[-\frac{24u_1^2}{R_0} + 3\omega_1^2 - \frac{24k^2 - (48k + 432u_\infty^2/R_0) \omega_\infty^2 + 24\omega_\infty^4}{R_0} \right] \\
 u_2 = & -\frac{12}{R_0} u_\infty \omega_\infty^2 \\
 u_3 = & -\frac{4}{R_0} \omega_\infty (2u_\infty \omega_1 + u_1 \omega_\infty) \\
 u_4 = & -\frac{2}{R_0} \left[2u_1 \omega_1 \omega_\infty - \frac{144u_\infty^3 \omega_\infty^2}{R_0^2} + u_\infty \omega_1^2 - \frac{24u_\infty \omega_\infty^2 (-k + \omega_\infty^2)}{R_0} \right] \\
 u_5 = & \frac{1}{5R_0} \left[48u_\infty \omega_\infty \left\{ 3M \omega_\infty + \omega_1 (-6k + 24u_\infty^2/R_0 + 7\omega_\infty^2) \right\} \right. \\
 & + \frac{u_1}{30(1 + f'(R_0))} \left[3\omega_1^2 \left\{ -1 - \frac{12(1 + f'(R_0))}{R_0} \right\} \right. \\
 & \left. \left. + \frac{144\omega_\infty^2 \{-3u_\infty^2 + (-6k + 60u_\infty^2/R_0 + 4\omega_\infty^2) (1 + f'(R_0))\}}{R_0^2} \right] \right]
 \end{aligned}$$

Appendix B: Asymptotic solution to a magnetically charged black hole for $f(R)$ -Yang-Mills theory

In this case, SU(2) Yang-Mills gauge field is given by

$$A = \{\omega(r)\tau_1 d\theta + [\partial_\theta \sigma(\theta)\tau_3 + \sigma(\theta)\omega(r)\tau_2] d\varphi\}. \quad (5.7)$$

Three equations of motion for N , ϕ and ω are given by

$$\delta_N S; \quad r(1 + f'(R_0))\phi' - 2(\omega')^2 = 0, \quad (5.8)$$

$$\begin{aligned} \delta_\phi S; \quad -2 + 2rN' + 2N - r^2 f(R_0) + 4N(\omega')^2 + \frac{2(\omega^2 - 1)^2}{r^2} \\ + rf'(R_0)(-2rN(\phi')^2 - 4N\phi' - rN'' - 2rN\phi'' - 2N' - 3rN'\phi') = 0, \end{aligned} \quad (5.9)$$

$$\delta_\omega S; \quad r^2 N\omega'' + r^2(N' + \phi'N)\omega' - \omega(\omega^2 - 1) = 0. \quad (5.10)$$

The above equations can be solved by assuming asymptotic forms up to $\frac{1}{r^5}$ -order

$$N = 1 - \frac{2m}{r} - \frac{R_0}{12}r^2, \quad (5.11)$$

$$m = M + \frac{M_1}{r} + \frac{M_2}{r^2} + \frac{M_3}{r^3} + \frac{M_4}{r^4} + \frac{M_5}{r^5} + O\left(\frac{1}{r^6}\right), \quad (5.12)$$

$$\omega = \omega_\infty + \frac{\omega_1}{r} + \frac{\omega_2}{r^2} + \frac{\omega_3}{r^3} + \frac{\omega_4}{r^4} + \frac{\omega_5}{r^5} + O\left(\frac{1}{r^6}\right), \quad (5.13)$$

where ω_∞ , ω_1 , M are three constants and the coefficients M_i and ω_i are expressed in terms of ω_∞ , ω_1 , M , and $1 + f'(R_0)$

$$M_1 = -\frac{(\omega_\infty^2 - 1)^2 - \frac{\omega_1^2}{6}R_0}{2(1 + f'(R_0))},$$

$$M_2 = \frac{2\omega_1\omega_\infty(1 - \omega_\infty^2)}{1 + f'(R_0)},$$

$$M_3 = \frac{\omega_1^2 R_0(1 - 2\omega_\infty^2) + 8\omega_\infty^2(1 - \omega_\infty^2)^2}{R_0(1 + f'(R_0))},$$

$$M_4 = \frac{-\omega_1^2(M + 2\omega_1\omega_\infty)R_0 + 16\omega_1\omega_\infty(1 - \omega_\infty^2)(2 - 3\omega_\infty^2)}{2R_0(1 + f'(R_0))},$$

$$M_5 = -\frac{1}{5R_0^2(1 + f'(R_0))} \left[R_0^2\omega_1^4 - 60MR_0\omega_1\omega_\infty(-1 + \omega_\infty^2) \right.$$

$$\left. + 24(-13 + 12\omega_\infty^2)(\omega_\infty - \omega_\infty^3)^2 \right]$$

$$\begin{aligned}
 & -6R_0\omega_1^2 \left(7 - 33\omega_\infty^2 + 26\omega_\infty^4\right) \Big] + \frac{\omega_1^2}{60(1 + f'(R_0))^2} \left\{ -R_0\omega_1^2 \right. \\
 & \left. + 12 \left(-1 + \omega_\infty^2\right)^2 \right\} \\
 \omega_2 = & \frac{6\omega_\infty \left(1 - \omega_\infty^2\right)}{R_0}, \\
 \omega_3 = & \frac{6\omega_1 \left(1 - \omega_\infty^2\right)}{R_0}, \\
 \omega_4 = & \frac{-3(2M + \omega_1\omega_\infty)\omega_1 R_0 + 6\omega_\infty \left(1 - \omega_\infty^2\right) \left(7 - 3\omega_\infty^2\right)}{R_0^2}, \\
 \omega_5 = & -\frac{3}{5R_0^2} \left[R_0\omega_1^3 - 96M\omega_\infty \left(-1 + \omega_\infty^2\right) - 6\omega_1 \left(13 - 22\omega_\infty^2 + 9\omega_\infty^4\right) \right] \\
 & - \frac{3\omega_1}{10R_0(1 + f'(R_0))} \left\{ R_0\omega_1^2 - 8 \left(-1 + \omega_\infty^2\right)^2 \right\}. \tag{5.14}
 \end{aligned}$$

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