

Formulations of the 3+1 evolution equations in curvilinear coordinates

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Abstract Following Brown (Phys Rev D79:104029, 2009), in this paper we give an overview of how to modify standard hyperbolic formulations of the 3+1 evolution equations of General Relativity in such a way that all auxiliary quantities are true tensors, thus allowing for these formulations to be used with curvilinear sets of coordinates such as spherical or cylindrical coordinates. After considering the general case for both the Nagy–Ortiz–Reula and the Baumgarte–Shapiro–Shibata–Nakamura (BSSN) formulations, we specialize to the case of spherical symmetry and also discuss the issue of regularity at the origin. Finally, we show some numerical examples of the modified BSSN formulation at work in spherical symmetry.

Keywords Numerical relativity · 3+1 Evolution equations · Curvilinear coordinates

1 Introduction

In 3+1 formalism of General Relativity one splits spacetime into a foliation of 3-dimensional (3D) spacelike hypersurfaces (assuming that the spacetime is globally hyperbolic), and projects the Einstein field equations in the normal and tangential direction to those. In this way, the 10 independent field equations are naturally separated into 4 constraint equations and 6 evolution equations for the geometric degrees of freedom. The evolution equations that are obtained directly from this projection are known as the Arnowitt–Deser–Misner (ADM) equations [1–3]. As was already

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realized in the late 80's and early 90's, these ADM evolution equations, though physically correct, have nevertheless one serious drawback: they turn out to be only weakly hyperbolic and as such are not mathematically well-posed (see e.g. [3]). By this one means that the solutions do not depend continuously on the initial data and can be unstable in the presence of constraint violations, which in practice implies that one will encounter serious stability problems in numerical evolutions based on these equations.

It turns out, however, that one can construct alternative formulations of the evolution equations by adding to them multiples of the constraints in a variety of different ways. These new systems of evolution equations will have the same physical (constraint satisfying) solutions, but will typically differ significantly in their mathematical structure. Over the last two decades, a number of different well-posed strongly hyperbolic formulations of the 3+1 evolution equations have been proposed, and several of them have been tested in numerical evolution codes.¹ In particular, the formulation proposed by Shibata and Nakamura, and Baumgarte and Shapiro, known as the BSSN formulation [4,5], has turned out to be very stable and robust in practice, and has become the standard formulation used by most 3+1 evolution codes today. This formulation has finally allowed the accurate simulation of binary black-hole systems with different masses and spins, starting from wide separations through the merger and ring-down of the final black hole [6–8] (one should mention, however, that the first successful simulation of multiple orbits of binary black holes was in fact carried out by F. Pretorius using a very different approach based on the so-called generalized harmonic formulation, an approach that is still being used today by a number of different groups [9]).

The BSSN formulation, though very successful in practice, has the drawback of involving dynamical quantities that are not true tensors, such as tensor densities and contracted Christoffel symbols. This represents no problem in most 3D simulations where one typically uses Cartesian coordinates, but becomes an important issue when one considers curvilinear coordinate systems, such as spherical or cylindrical coordinates.

Recently, Brown introduced a more general version of the BSSN system where all dynamical quantities are true tensors [10]. This “generalized BSSN” formulation is thus ideally suited for the use of curvilinear systems of coordinates, which in particular allows one to construct a BSSN version of the evolution equations for the case of spherical or cylindrical symmetry.

In this paper we give an overview of the main ideas behind Brown's approach, and apply them to both the Nagy–Ortiz–Reula (NOR) [11] and BSSN formulations. The paper is organized as follows. In Sect. 2 we give a brief review of the 3+1 formalism. Later, in Sect. 3 we discuss some important results related to the fully covariant expressions of the Riemann and Ricci curvature tensors in terms of a background metric. Section 4 then considers the case of the NOR formulation and its generalization to curvilinear coordinates. In Sect. 5 we repeat the same analysis for the BSSN formulation, and also include a brief discussion of the Gamma driver shift condition. In Sect. 6 we consider the particular case of BSSN in spherical symmetry, and discuss

¹ We will not attempt to give an exhaustive list here of such hyperbolic formulations since there are simply too many of them. But see e.g. [3] for an extended list of references.

the basic equations and the important issue of the regularization at the origin. Finally, in Sect. 7 we present some numerical examples. We conclude in Sect. 8.

Throughout the paper we will use geometric units such that $G = c = 1$. Also, Greek indices will represent all spacetime dimensions and will run from 0 to 3, while Latin indices will represent only spatial dimensions and will run from 1 to 3.

2 Basic 3+1 equations

Before considering the NOR and BSSN formulations it is convenient to first review the basic concepts and equations of the 3+1 formalism of general relativity (for a more detailed introduction see e.g. [3]).

In the 3+1 formulation spacetime is foliated into spatial hypersurfaces parametrized by a *time function* t . The basic dynamical quantities of the 3+1 formulation are then the metric of the spatial hypersurfaces γ_{ij} and the extrinsic curvature tensor of those hypersurfaces K_{ij} which is defined as

$$K_{\mu\nu} := -P_{\mu}^{\alpha} \nabla_{\alpha} n_{\nu} \quad (1)$$

where n^{μ} is the time-like normal vector to the spatial hypersurfaces, and $P_{\beta}^{\alpha} := \delta_{\beta}^{\alpha} + n^{\alpha} n_{\beta}$ the projection operator onto the hypersurfaces.

Furthermore, one also introduces the lapse function α that measures the proper time elapsed between adjacent hypersurfaces along the normal direction, and the shift vector β^i that controls how the spatial coordinates propagate from one hypersurface to the next. In more detail, an observer moving along the normal direction to the hypersurfaces (also known as an *Eulerian* observer) will have a coordinate speed given by $-\beta^i$, and will measure a proper time $d\tau = \alpha dt$. In terms of these coordinates, the unit normal vector becomes $n^{\mu} = (1/\alpha, -\beta^i/\alpha)$, and the extrinsic curvature tensor takes the form

$$K_{ij} = -\frac{1}{2\alpha} (\partial_t \gamma_{ij} - \mathcal{L}_{\beta} \gamma_{ij}), \quad (2)$$

with \mathcal{L}_{β} the Lie derivative with respect to the shift vector, and where we have only considered spatial components using the fact that the extrinsic curvature is by definition normal to the hypersurfaces.

Given the spacetime foliation just described, the Einstein field equations separate naturally into two distinct groups. The first group corresponds to those equations that have no time derivatives and results in the so-called Hamiltonian and momentum constraints

$$H := \frac{1}{2} (R + K^2 - K_{ij} K^{ij}) - 8\pi\rho = 0, \quad (3)$$

$$M^i := \nabla_j (K^{ij} - \gamma^{ij} K) - 8\pi j^i = 0. \quad (4)$$

In the above equations $R := \gamma^{ij} R_{ij}$ is the trace of the spatial Ricci tensor R_{ij} , $K := \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature, and ∇_i is the covariant derivative

associated with the spatial metric γ_{ij} , while ρ and j^i are the energy and momentum densities measured by the Eulerian observers and are given by

$$\rho := n^\mu n^\nu T_{\mu\nu} \tag{5}$$

$$j^i := -P^{i\mu} n^\nu T_{\mu\nu}, \tag{6}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the matter.

The second group of field equations corresponds to the true evolution equations of the system. In terms of the quantities introduced above these evolution equations take the form

$$\partial_t \gamma_{ij} - \mathfrak{L}_\beta \gamma_{ij} = -2\alpha K_{ij}, \tag{7}$$

$$\begin{aligned} \partial_t K_{ij} - \mathfrak{L}_\beta K_{ij} = & -\nabla_i \nabla_j \alpha + \alpha \left[R_{ij} + K K_{ij} - 2K_{ik} K_j^k \right] \\ & + 4\pi \alpha \left[\gamma_{ij} (S - \rho) - 2S_{ij} \right], \end{aligned} \tag{8}$$

where R_{ij} is the 3-dimensional Ricci tensor associated with the spatial metric γ_{ij} :

$$\begin{aligned} R_{ij} = & -\frac{1}{2} \gamma^{mn} \partial_m \partial_n \gamma_{ij} + \gamma_m \binom{\partial}{i} \binom{\partial}{j} \Gamma^m + \Gamma^m \Gamma_{(ij)m} \\ & + 2\Gamma^{mn} \binom{\Gamma}{i} \binom{\Gamma}{j} mn + \Gamma_{mni} \Gamma^{mn} j, \end{aligned} \tag{9}$$

with $\Gamma^i := \gamma^{mn} \Gamma_{mn}^i$, and where S_{ij} is the stress tensor measured by the Eulerian observers defined as

$$S_{ij} := P_i^\alpha P_j^\beta T_{\alpha\beta}, \tag{10}$$

with $S := \gamma^{ij} S_{ij}$. The evolution equations above are known in the numerical relativity community as the Arnowitt–Deser–Misner (ADM) equations [1,2].

3 Curvature tensor in terms of a background metric

It is convenient at this point to review some well-known fully covariant expressions for the Riemann and Ricci curvature tensors in terms of a background metric.

Let us assume that we have a manifold with some coordinate system and two different metric tensors defined on it: the “physical” metric γ_{ij} , and some “background” metric $\mathring{\gamma}_{ij}$ that is not necessarily flat (though in the following sections we will assume that the background metric is indeed flat). We now want to express the curvature tensor associated with the physical metric γ_{ij} in terms of the curvature associated to the background metric $\mathring{\gamma}_{ij}$, together with covariant derivatives of γ_{ij} with respect to this background. In order to do this, we start by defining the quantity:

$$\Delta^a{}_{bc} := \Gamma^a{}_{bc} - \mathring{\Gamma}^a{}_{bc}, \tag{11}$$

with Γ^a_{bc} and $\overset{\circ}{\Gamma}^a_{bc}$ the Christoffel symbols associated with γ_{ij} and $\overset{\circ}{\gamma}_{ij}$ respectively. Notice that even though neither Γ^a_{bc} nor $\overset{\circ}{\Gamma}^a_{bc}$ are components of tensors, their difference Δ^a_{bc} is in fact a proper tensor.

Having defined Δ^a_{bc} let us now calculate the covariant derivative of the physical metric γ_{ij} in the background geometry, that is $\overset{\circ}{\nabla}_a \gamma_{bc}$. Notice first that, in general

$$\nabla_a \gamma_{bc} = \overset{\circ}{\nabla}_a \overset{\circ}{\gamma}_{bc} = 0, \tag{12}$$

$$\overset{\circ}{\nabla}_a \gamma_{bc} \neq 0. \tag{13}$$

If we now take the convention that indices of Δ^a_{bc} are raised and lowered with the physical metric γ_{ij} , then we can use (12) to show that:

$$\overset{\circ}{\nabla}_a \gamma_{bc} = 2\Delta_{(bc)a}, \tag{14}$$

and equivalently

$$\overset{\circ}{\nabla}_a \gamma^{bc} = -2\Delta^{(bc)}_a. \tag{15}$$

One can now solve for Δ^a_{bc} from the above expressions to find

$$\Delta^a_{bc} = \frac{1}{2} \gamma^{am} \left(\overset{\circ}{\nabla}_b \gamma_{cm} + \overset{\circ}{\nabla}_c \gamma_{bm} - \overset{\circ}{\nabla}_m \gamma_{bc} \right). \tag{16}$$

Notice that this expression for Δ^a_{bc} is in fact identical to that for the Christoffel symbols Γ^a_{bc} , but with the partial derivatives replaced with covariant derivatives on the background. In particular, if the background is flat and we use Cartesian coordinates we will have $\Delta^a_{bc} = \Gamma^a_{bc}$ and $\overset{\circ}{\nabla}_a = \partial_a$, so that the last expression reduces to the standard definition of the Christoffel symbols.

We can now use (16) to show that the physical Riemann curvature tensor can be written in terms of the Δ^a_{bc} as:

$$R^a_{bcd} = \overset{\circ}{R}^a_{bcd} + 2\overset{\circ}{\nabla}_{[c} \Delta^a_{d]b} + 2\Delta^a_{m[c} \Delta^m_{d]b}, \tag{17}$$

where $\overset{\circ}{R}^a_{bcd}$ is the curvature tensor of the background. Again, the second term has the same structure as the standard expression for the Riemann tensor, but with the Γ^a_{bc} replaced with Δ^a_{bc} , and the partial derivatives replaced with covariant derivatives on the background. The expression again reduces to the usual one for a flat background in Cartesian coordinates.

Next, let us lower the first index in the Riemann tensor. This is not as trivial as it might seem since now γ_{ij} can not be brought inside the operator $\overset{\circ}{\nabla}_a$ directly. After a somewhat lengthy algebra, where one needs to use the expression for the commutator of the covariant derivatives of a rank 2 tensor in terms of the Riemann, one finally

finds that:

$$R_{abcd} = \frac{1}{2} \left[\gamma_{am} \mathring{R}^m{}_{bcd} - \gamma_{bm} \mathring{R}^m{}_{acd} + \mathring{\nabla}_c \mathring{\nabla}_b \gamma_{ad} - \mathring{\nabla}_d \mathring{\nabla}_b \gamma_{ac} + \mathring{\nabla}_d \mathring{\nabla}_a \gamma_{bc} - \mathring{\nabla}_c \mathring{\nabla}_a \gamma_{bd} \right] + \Delta_{mad} \Delta^m{}_{bc} - \Delta_{mac} \Delta^m{}_{bd}. \tag{18}$$

Notice that in the last expression we *do not* lower the first index of $\mathring{R}^a{}_{bcd}$ with γ_{ij} , since by convention it should be lowered with $\mathring{\gamma}_{ij}$.

Finally, let us find the expression for the Ricci tensor $R_{ab} := \gamma^{cd} R_{acbd}$. Using (18) we find, after some algebra:

$$R_{ab} = -\frac{1}{2} \gamma^{mn} \left[\mathring{\nabla}_m \mathring{\nabla}_n \gamma_{ab} + \mathring{\nabla}_a \mathring{\nabla}_b \gamma_{mn} - \mathring{\nabla}_a \mathring{\nabla}_m \gamma_{bn} - \mathring{\nabla}_b \mathring{\nabla}_m \gamma_{an} \right] + \Delta_{mna} \Delta^{mn}{}_b - \Delta_{mab} \Delta^m - \gamma^{mn} \mathring{R}^c{}_{mn(a} \gamma_{b)c}, \tag{19}$$

where we have defined

$$\Delta^m := \gamma^{ab} \Delta^m{}_{ab} = \Gamma^m - \gamma^{ab} \mathring{\Gamma}^m{}_{ab}. \tag{20}$$

We can in fact rewrite the Ricci tensor in terms of derivatives of the quantity Δ^m just defined. Using (16) one finds that (19) is entirely equivalent to

$$R_{ab} = -\frac{1}{2} \gamma^{mn} \mathring{\nabla}_m \mathring{\nabla}_n \gamma_{ab} + \gamma_m \left(\begin{smallmatrix} \mathring{\nabla} \\ a \end{smallmatrix} \begin{smallmatrix} \\ b \end{smallmatrix} \right) \Delta^m + \Delta^m \Delta_{(ab)m} + 2\Delta^{mn} \left(\begin{smallmatrix} \Delta \\ a \end{smallmatrix} \begin{smallmatrix} \\ b \end{smallmatrix} \right)_{mn} + \Delta^{mn}{}_a \Delta_{mnb} \gamma^{mn} \mathring{R}^c{}_{mn} \left(\begin{smallmatrix} \gamma \\ a \end{smallmatrix} \begin{smallmatrix} \\ b \end{smallmatrix} \right)_c, \tag{21}$$

For a flat background in Cartesian coordinates, this last expression clearly reduces to the standard expression given in (9).

4 The NOR formulation

4.1 Standard formulation

The Nagy–Ortiz–Reula (NOR) formulation [11] is in essence a generalization of the Bona–Masso formulation (BM) of the early 1990s [12–16]. This formulation is based on first writing the three-dimensional Ricci tensor that appears in the ADM evolution equations as:

$$R_{ij} = -\frac{1}{2} \gamma^{mn} \partial_m \partial_n \gamma_{ij} + \gamma_k \left(\begin{smallmatrix} \partial \\ i \end{smallmatrix} \begin{smallmatrix} \\ j \end{smallmatrix} \right) \Gamma^k + \Gamma^k \Gamma_{(ij)k} + 2\Gamma^{mn} \left(\begin{smallmatrix} \Gamma \\ i \end{smallmatrix} \begin{smallmatrix} \\ j \end{smallmatrix} \right)_{mn} + \Gamma^{mn}{}_i \Gamma_{mnj}, \tag{22}$$

where $\Gamma^i := \gamma^{mn} \Gamma^i{}_{mn}$.

The crucial difference with the ADM formulation is the fact that the quantities Γ^i that appear in the Ricci tensor above are now promoted to independent quantities and evolved separately. To find the evolution equations for the Γ^i we first note that from their definition we have

$$\Gamma^i = -\partial_m \gamma^{im} - \frac{1}{2} \gamma^{im} \partial_m \ln \gamma, \tag{23}$$

with γ the determinant of γ_{ij} . From this one can easily show, after some algebra, that:

$$\begin{aligned} \partial_t \Gamma^i - \mathfrak{L}_\beta \Gamma^i &= \gamma^{lm} \partial_l \partial_m \beta^i - \frac{2}{\gamma^{1/2}} \partial_m \left(\alpha K^{im} \gamma^{1/2} \right) \\ &\quad + \gamma^{im} \partial_m (\alpha K), \end{aligned} \tag{24}$$

where the Lie derivative of Γ^i that appears in the last expression should be understood as that of a vector:

$$\mathfrak{L}_\beta \Gamma^i := \beta^m \partial_m \Gamma^i - \Gamma^m \partial_m \beta^i. \tag{25}$$

In fact, one can now add a multiple of the momentum constraints (4) to this equation to obtain a final evolution equation of the form:

$$\begin{aligned} \partial_t \Gamma^i - \mathfrak{L}_\beta \Gamma^i &= \gamma^{mn} \partial_m \partial_n \beta^i - \left[2K^{im} - \gamma^{im} K \right] \nabla_m \alpha \\ &\quad - \alpha \nabla_m \left[(2 - \xi) K^{im} - (1 - \xi) \gamma^{im} K \right] \\ &\quad + 2 \alpha K^{mn} \Gamma^i_{mn} - 8 \pi \alpha \xi j^i, \end{aligned} \tag{26}$$

with ξ an arbitrary parameter. The importance of adding a multiple of the momentum constraints to the evolution equation for Γ^i comes from the fact that, if one chooses a slicing condition of the Bona–Masso family

$$\partial_t \alpha - \mathfrak{L}_\beta \alpha = -\alpha^2 f(\alpha) K, \tag{27}$$

with $f(\alpha)$ an arbitrary function of α , then the NOR formulation can be shown to be strongly hyperbolic (and thus well-posed) if one takes $\xi = 2$ and $f > 0$, or more generally if one takes $\xi > 0$, $f > 0$ and $f \neq 1$ (see e.g. reference [3]).

Instead of using the Bona–Masso slicing condition, one can assume that the *densitized lapse* defined as $\tilde{\alpha} := \alpha \gamma^{-f/2}$, with f a constant, is an a priori known function of spacetime, $\tilde{\alpha} = F(t, x^i)$. The same results about hyperbolicity then follow.

The standard NOR formulation in fact also adds an arbitrary multiple of the Hamiltonian constraint of the form $\alpha \eta \gamma_{ij} H$ to the evolution equation of K_{ij} , with η another free parameter (for $\eta \neq 0$ one finds a new region of parameter space where the system is also strongly hyperbolic). However, this point is of no consequence for the discussion that follows, so we will ignore it from now on.

The evolution equation for Γ^i given above is quite general, but it has the serious disadvantage that it involves quantities that are not tensors, such as Γ^i_{mn} and Γ^i itself. In the next section we will address this issue.

4.2 Curvilinear coordinates

The NOR formulation just described can in principle be used with any type of coordinates. However, when dealing with curvilinear coordinates, that is coordinates that are non-trivial even in flat space, one can easily find that the conformal connection functions Γ^i are singular at some points and generally do not behave as a vector would do. For example, in spherical coordinates the quantity Γ^r turns out to be singular in flat space, while Γ^θ is non-zero even if we assume spherical symmetry. This in itself is not necessarily a major problem, as the equations are quite general and are consistent in any set of coordinates. However, dealing with singular quantities numerically can be troublesome, and also dealing with non-tensor quantities makes it difficult to compare evolutions done with the same slicing conditions but different spatial coordinate systems. It would then seem like a good idea to replace the non-covariant quantities Γ^i with a true vector.

In order to do this we will start from the tensor Δ^i_{jk} defined in (11) above, and furthermore we will also assume that the background metric is the flat metric in the same curvilinear coordinates we are considering. Notice, in particular, that in Cartesian coordinates we have $\hat{\Gamma}^i_{jk} = 0$, so that in that case Δ^i_{jk} and Γ^i_{jk} are identical.

Just as we did before, we will again define the quantities Δ^i as in (20). We now want to calculate the evolution equation for Δ^i . From the definition above we immediately find

$$\partial_t \Delta^i = \partial_t \Gamma^i - \hat{\Gamma}^i_{mn} \partial_t \gamma^{mn}, \tag{28}$$

where we have used the fact that the flat background does not evolve. Using now (26) and the ADM evolution equation for γ_{ij} one can easily find that

$$\begin{aligned} \partial_t \Delta^i - \xi_\beta \Delta^i &= \gamma^{mn} \partial_m \partial_n \beta^i + \gamma^{mn} \xi_\beta \hat{\Gamma}^i_{mn} - \left[2K^{im} - \gamma^{im} K \right] \nabla_m \alpha \\ &\quad - \alpha \nabla_m \left[(2 - \xi) K^{im} - (1 - \xi) \gamma^{im} K \right] + 2\alpha K^{mn} \Delta^i_{mn}, \end{aligned} \tag{29}$$

where the term $\xi_\beta \hat{\Gamma}^i_{mn}$ must be calculated as if $\hat{\Gamma}^i_{mn}$ where a true tensor. In the previous equation Δ^i is clearly a vector, and so is $\partial_t \Delta^i$, but the right hand side is not manifestly covariant since it involves partial derivatives of the shift and terms containing $\hat{\Gamma}^i_{mn}$. However, this can be easily fixed since one can show that, quite generally,

$$\gamma^{mn} \hat{\nabla}_m \hat{\nabla}_n \beta^i = \gamma^{mn} \partial_m \partial_n \beta^i + \gamma^{mn} \xi_\beta \hat{\Gamma}^i_{mn} + \beta^l \gamma^{mn} \hat{R}^i_{mnl}, \tag{30}$$

with \hat{R}^i_{mnl} the curvature tensor of the background. Since in our case the background is flat by construction, we can use the last result to rewrite the evolution equation for

Δ^i in the following way

$$\begin{aligned} \partial_t \Delta^i - \mathfrak{L}_\beta \Delta^i &= \gamma^{mn} \overset{\circ}{\nabla}_m \overset{\circ}{\nabla}_n \beta^i - \left[2K^{im} - \gamma^{im} K \right] \nabla_m \alpha \\ &\quad - \alpha \nabla_m \left[(2 - \xi) K^{im} - (1 - \xi) \gamma^{im} K \right] + 2\alpha K^{mn} \Delta_{mn}^i. \end{aligned} \tag{31}$$

The last equation is now manifestly covariant.

In summary, in order to use the NOR formulation in curvilinear coordinates we need to express the 3-dimensional Ricci tensor that appears in the evolution equations for the extrinsic curvature as (confront this with Eq. (22)):

$$\begin{aligned} R_{ab} &= -\frac{1}{2} \gamma^{mn} \overset{\circ}{\nabla}_m \overset{\circ}{\nabla}_n \gamma_{ab} + \gamma_m \left(\overset{\circ}{\nabla} \begin{matrix} a \\ b \end{matrix} \right) \Delta^m + \Delta^m \Delta_{(ab)m} \\ &\quad + 2\Delta^{mn} \left(\overset{\circ}{\nabla} \begin{matrix} a \\ b \end{matrix} \right)_{mn} + \Delta^{mn} \overset{\circ}{\nabla}_a \Delta_{mnb}, \end{aligned} \tag{32}$$

with Δ^a_{bc} and Δ^a defined in (11) and (20), promote the Δ^a to independent quantities, and evolve them through (31).

5 The BSSN formulation

5.1 Standard formulation

The BSSN formulation is a reformulation of the ADM evolution equations, based on the work of Shibata and Nakamura [4] and Baumgarte and Shapiro [5], that has proven to be particularly robust in the numerical evolution of a large variety of spacetimes. This formulation is based on a conformal decomposition of the metric of the form

$$\tilde{\gamma}_{ij} = e^{-4\phi} \gamma_{ij}, \tag{33}$$

where the conformal factor ϕ is chosen in such a way that the determinant of the conformal metric is unity $\tilde{\gamma} = 1$, which implies:

$$\phi = \frac{1}{12} \ln \gamma. \tag{34}$$

From the definition above and the ADM evolution equation for the spatial metric (7), one can easily find the following evolution equation for ϕ :

$$\partial_t \phi = -\frac{1}{6} (\alpha K - \partial_m \beta^m) + \beta^m \partial_m \phi. \tag{35}$$

The last equation can in fact be rewritten as

$$\partial_t \phi - \mathfrak{L}_\beta \phi = -\frac{1}{6} \alpha K, \tag{36}$$

where the Lie derivative of ϕ is given by

$$\mathfrak{L}_\beta\phi = \beta^m \partial_m \phi + \frac{1}{6} \partial_m \beta^m. \tag{37}$$

Notice that, strictly speaking, ϕ is not a true scalar density since its definition involves a logarithm, but $\psi := e^\phi = \gamma^{1/12}$ is a well defined scalar density of weight 1/6, so that the Lie derivative of ϕ is just $\mathfrak{L}_\beta\phi = \mathfrak{L}_\beta\psi/\psi$, which reduces to the expression above.

The BSSN formulation also separates the extrinsic curvature into its trace K and its trace-free part

$$A_{ij} = K_{ij} - \frac{1}{3} \gamma_{ij} K. \tag{38}$$

We further make a conformal rescaling of the traceless extrinsic curvature of the form

$$\tilde{A}_{ij} = e^{-4\phi} A_{ij} = e^{-4\phi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right). \tag{39}$$

Just as we did in the case of the NOR formulation, the BSSN formulation also introduces three auxiliary variables known as the *conformal connection functions* and defined as

$$\tilde{\Gamma}^i := \tilde{\gamma}^{jk} \tilde{\Gamma}_{jk}^i = -\partial_j \tilde{\gamma}^{ij}, \tag{40}$$

where $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of the conformal metric, and where the second equality comes from the fact that the determinant $\tilde{\gamma}$ is equal to 1.

The evolution equation for ϕ was already found above, while those for $\tilde{\gamma}_{ij}$, K and \tilde{A}_{ij} can be obtained directly from the standard ADM equations. The system of evolution equations then takes the form

$$\partial_t \tilde{\gamma}_{ij} - \mathfrak{L}_\beta \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij}, \tag{41}$$

$$\partial_t \phi - \mathfrak{L}_\beta \phi = -\frac{1}{6} \alpha K, \tag{42}$$

$$\begin{aligned} \partial_t \tilde{A}_{ij} - \mathfrak{L}_\beta \tilde{A}_{ij} = e^{-4\phi} \{ & -\nabla_i \nabla_j \alpha + \alpha R_{ij} + 4\pi\alpha [\gamma_{ij} (S - \rho) - 2S_{ij}] \}^{\text{TF}} \\ & + \alpha (K \tilde{A}_{ij} - 2\tilde{A}_{ik} \tilde{A}^k_j), \end{aligned} \tag{43}$$

$$\partial_t K - \mathfrak{L}_\beta K = -\nabla^2 \alpha + \alpha \left(\tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right) + 4\pi\alpha (\rho + S), \tag{44}$$

with $\nabla^2 := \nabla^m \nabla_m$ the spatial Laplacian operator associated with the full physical metric, and where TF denotes the trace-free part of the expression inside the brackets. Notice also that indices of conformal quantities are assumed to be raised and lowered

with the conformal metric. Here it is important to mention that the Hamiltonian constraint has already been used in the evolution equation for K in order to eliminate the Ricci scalar.

In the evolution equation for \tilde{A}_{ij} above one needs to calculate the Ricci tensor associated with the physical metric, which can be separated into two contributions in the following way:

$$R_{ij} = \tilde{R}_{ij} + R_{ij}^\phi, \tag{45}$$

where \tilde{R}_{ij} is the Ricci tensor associated with the conformal metric $\tilde{\gamma}_{ij}$, which we write in terms of the $\tilde{\Gamma}^i$ as

$$\begin{aligned} \tilde{R}_{ij} = & -\frac{1}{2}\tilde{\gamma}^{mn}\partial_m\partial_n\tilde{\gamma}_{ij} + \tilde{\gamma}_k\left(\begin{matrix} \partial \\ i \end{matrix} \begin{matrix} j \\ \end{matrix}\right)\tilde{\Gamma}^k + \tilde{\Gamma}^k\tilde{\Gamma}_{(ij)k} \\ & + 2\tilde{\Gamma}^{mn}\left(\begin{matrix} \tilde{\Gamma} \\ i \end{matrix} \begin{matrix} j \\ \end{matrix}\right)_{mn} + \tilde{\Gamma}^{mn}{}_i\tilde{\Gamma}_{mj}, \end{aligned} \tag{46}$$

(this is just the standard expression (9) for the conformal metric), and where R_{ij}^ϕ denotes additional terms that depend on derivatives of ϕ :

$$R_{ij}^\phi = -2\tilde{\nabla}_i\tilde{\nabla}_j\phi - 2\tilde{\gamma}_{ij}\tilde{\nabla}^k\tilde{\nabla}_k\phi + 4\tilde{\nabla}_i\phi\tilde{\nabla}_j\phi - 4\tilde{\gamma}_{ij}\tilde{\nabla}^k\phi\tilde{\nabla}_k\phi, \tag{47}$$

with $\tilde{\nabla}_i$ the covariant derivative associated with the conformal metric.

Notice also that the evolution equations for \tilde{A}_{ij} and K involve covariant derivatives of the lapse function with respect to the physical metric γ_{ij} (i.e. covariant derivatives with no tilde). One must also be careful with the fact that in the evolution equations above we need to calculate Lie derivatives with respect to the shift vector β^i of tensor densities. In particular, $\tilde{\gamma}_{ij}$ and \tilde{A}_{ij} are tensor densities of weight $-2/3$.

We are still missing an evolution equation for the $\tilde{\Gamma}^i$. This equation can be obtained directly from the definition, Eq. (40). One finds:

$$\partial_t\tilde{\Gamma}^i - \mathfrak{L}_\beta\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{1}{3}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k - 2\left(\alpha\partial_j\tilde{A}^{ij} + \tilde{A}^{ij}\partial_j\alpha\right). \tag{48}$$

In the above equation the Lie derivative of $\tilde{\Gamma}^i$ should be calculated as if $\tilde{\Gamma}^i$ where a vector density of weight $2/3$:

$$\mathfrak{L}_\beta\tilde{\Gamma}^i = \beta^j\partial_j\tilde{\Gamma}^i - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{3}\tilde{\Gamma}^i\partial_j\beta^j. \tag{49}$$

Again, just as we did in the case of NOR, we will modify the evolution equation for $\tilde{\Gamma}^i$ given above by adding to it a multiple of the momentum constraints. In order to do this, let us first rewrite the Hamiltonian and momentum constraints in terms of

the conformally rescaled quantities. One finds:

$$H := \frac{1}{2} \left(R + \frac{2}{3} K^2 - \tilde{A}_{ij} \tilde{A}^{ij} \right) - 8\pi\rho = 0, \tag{50}$$

$$\begin{aligned} M^i &:= \nabla_j A^{ij} - \frac{2}{3} \gamma^{ij} \partial_j K - 8\pi j^i \\ &= e^{-4\phi} \left(\tilde{\nabla}_j \tilde{A}^{ij} - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K + 6\tilde{A}^i \partial_j \phi \right) - 8\pi j^i = 0. \end{aligned} \tag{51}$$

where $\tilde{\nabla}_i$ now denotes covariant derivative with respect to the conformal metric. Notice also that the fact that the covariant metric has unit determinant implies that the term $\tilde{\nabla}_j \tilde{A}^{ij}$ can be written as:

$$\tilde{\nabla}_j \tilde{A}^{ij} = \partial_j \tilde{A}^{ij} + \tilde{\Gamma}^i_{jk} \tilde{A}^{jk}. \tag{52}$$

The Hamiltonian constraint above was in fact already used in order to eliminate the Ricci scalar from the evolution equation for the trace of the extrinsic curvature K above (Eq. (44)).

Adding now a multiple of the momentum constraint to the evolution equation for $\tilde{\Gamma}^i$, Eq. (48) above, we find:

$$\begin{aligned} \partial_t \tilde{\Gamma}^i - \xi_\beta \tilde{\Gamma}^i &= \tilde{\gamma}^{jk} \partial_j \partial_k \beta^i + \frac{1}{3} \tilde{\gamma}^{ij} \partial_j \partial_k \beta^k - 2\tilde{A}^{ij} \partial_j \alpha \\ &\quad - \alpha (2 - \xi) \partial_j \tilde{A}^{ij} + \alpha \xi \left(\tilde{\Gamma}^i_{jk} \tilde{A}^{jk} \right. \\ &\quad \left. + 6\tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K - 8\pi \tilde{j}^i \right). \end{aligned} \tag{53}$$

with ξ an arbitrary parameter, and where $\tilde{j}^i := e^{4\phi} j^i$. The standard BSSN formulation usually takes $\xi = 2$, which seems to be an optimal choice (the reason for this is that in that case the constraint violating modes can be shown to propagate at the speed of light).

Just as in the case of the NOR formulation, The BSSN formulation just described can be shown to be strongly hyperbolic for $\xi > 1/2$ [3]. Standard BSSN with $\xi = 2$ has turned out to be particularly robust in practice, and leads to stable and well behaved numerical simulations. In conjunction with the Bona-Masso slicing condition (27), and the so-called ‘‘Gamma driver’’ shift condition [17] (see Sect. 5.3 below), it has allowed for the accurate simulation of the inspiral collision of black holes with different masses and spins [6–8, 18]. Today, most 3-dimensional production numerical relativity codes use the BSSN formulation in one way or another, the notable exception being codes that use the ‘‘generalized harmonic formulation’’ (see e.g. [19]).

5.2 Curvilinear coordinates

When adapting the standard BSSN formulation to curvilinear coordinates we are faced with two problems. The first one is essentially the same problem that we had with the NOR formulation, namely that the quantities $\tilde{\Gamma}^i$ are not vectors (or more specifically vector densities, but we will come back to that point below). The second problem is the fact that in curvilinear coordinates the determinant of the flat metric is generally different from unity, so that asking for $\tilde{\gamma} = 1$ is not a good idea. Consider, for example, flat space in spherical coordinates (r, θ, φ) for which the spatial metric is:

$$ds^2 = dr^2 + r^2 d\Omega^2, \tag{54}$$

with $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$ the standard solid angle element. We then find that $\gamma = r^4 \sin^2 \theta$.

In curvilinear coordinates it is in fact much better to ask for the determinant of the conformal metric to reduce to its value in flat space (see e.g. [10]). In order to avoid confusion between the conformal metric of the standard BSSN formulation and the one we will use here, from now on we will denote conformal quantities with a hat instead of a tilde. Also, for the conformal factor we will use χ instead of ϕ , so that we will in fact have

$$\hat{\gamma}_{ij} = e^{-4\chi} \gamma_{ij}, \tag{55}$$

and we will ask for $\hat{\gamma}(t = 0) = \hat{\gamma}$, with $\hat{\gamma}$ the determinant of the flat metric background in the same curvilinear coordinates. This change introduces two new features into the BSSN formulation. In the first place, in general we will find that $\hat{\gamma}$ will not be constant in space so that we can no longer ignore its spatial derivatives. But more importantly, it is now not immediately clear how $\hat{\gamma}$ should evolve in time.

Following Brown [10], one can suggest at least two “natural” choices for the evolution of $\hat{\gamma}$:

1. $\partial_t \hat{\gamma} = 0$. This is called a “Lagrangian” condition since the determinant of the conformal metric is constant along time lines.
2. $\partial_t \hat{\gamma} - \mathfrak{L}_\beta \hat{\gamma} = 0$. This is instead an “Eulerian” condition, since the determinant of the conformal metric is now constant along the normal lines (i.e. it remains constant in time as seen by the Eulerian observers), so that it can in fact evolve along the time lines.

Standard BSSN then corresponds to the Lagrangian case in Cartesian coordinates. Using now the fact that:

$$\mathfrak{L}_\beta \hat{\gamma} = \beta^m \partial_m \hat{\gamma} + 2\hat{\gamma} \partial_m \beta^m = 2\hat{\gamma} \hat{\nabla}_m \beta^m, \tag{56}$$

we can write in general for the evolution of $\hat{\gamma}$:

$$\partial_t \hat{\gamma} = s \left(2\hat{\gamma} \hat{\nabla}_m \beta^m \right), \tag{57}$$

with:

$$s = \begin{cases} 0 & \text{Lagrangian,} \\ 1 & \text{Eulerian.} \end{cases} \quad (58)$$

On the other hand, since now $\hat{\gamma} \neq 1$, we find for the conformal factor χ :

$$\chi = \frac{1}{12} \ln(\gamma/\hat{\gamma}). \quad (59)$$

We can now use this to find the evolution equation for χ :

$$\begin{aligned} \partial_t \chi &= \frac{1}{12} \left(\frac{\partial_t \gamma}{\gamma} - \frac{\partial_t \hat{\gamma}}{\hat{\gamma}} \right) \\ &= \frac{1}{12} \left(-2\alpha K + \frac{\mathfrak{L}_\beta \gamma}{\gamma} - s \frac{\mathfrak{L}_\beta \hat{\gamma}}{\hat{\gamma}} \right), \end{aligned} \quad (60)$$

which implies:

$$\begin{aligned} \partial_t \chi - \mathfrak{L}_\beta \chi &= -\frac{1}{6} \alpha K + \frac{1}{6} (1-s) \hat{\nabla}_m \beta^m \\ &= -\frac{1}{6} \alpha K + \frac{1}{6} \sigma \hat{\nabla}_m \beta^m, \end{aligned} \quad (61)$$

where we have used Eq. (56) above, and where $\mathfrak{L}_\beta \chi := \mathfrak{L}_\beta \gamma / \gamma - \mathfrak{L}_\beta \hat{\gamma} / \hat{\gamma}$. In the above equation we have also introduced the shorthand $\sigma = (1-s)$, so that $\sigma = 1$ now corresponds to a Lagrangian evolution and $\sigma = 0$ to an Eulerian evolution.

There is an important point regarding the tensorial character of χ that should be mentioned here. Notice that because of the new definition of χ , Eq. (59), we now have:

$$\begin{aligned} \mathfrak{L}_\beta \chi &= \frac{1}{12} \left(\frac{\mathfrak{L}_\beta \gamma}{\gamma} - \frac{\mathfrak{L}_\beta \hat{\gamma}}{\hat{\gamma}} \right) \\ &= \frac{1}{6} \left(\nabla_m \beta^m - \hat{\nabla}_m \beta^m \right) \\ &= \frac{1}{12} \beta^m \partial_m \ln(\gamma/\hat{\gamma}), \end{aligned} \quad (62)$$

so that finally

$$\mathfrak{L}_\beta \chi = \beta^m \partial_m \chi. \quad (63)$$

In other words, the Lie derivative of χ is now that of a scalar function with *no density weight*. We will see below that this will be the case for all dynamical quantities. This is another important difference between the standard BSSN formulation and the generalization we are introducing here, and it can be traced back to the fact that the definition (59) of the conformal factor χ now involves the ratio of two volume elements, so that χ is a true scalar.

The next step is to find the evolution equation for $\hat{\gamma}_{ij}$. Starting from the definition (55) above, and using the evolution equation for χ (61), together with the ADM evolution equation for γ_{ij} given by (7), we now find

$$\partial_t \hat{\gamma}_{ij} - \mathcal{L}_\beta \hat{\gamma}_{ij} = -2\alpha \hat{A}_{ij} - \frac{2}{3} \sigma \hat{\gamma}_{ij} \hat{\nabla}_m \beta^m, \tag{64}$$

where \hat{A}_{ij} is now defined as:

$$\hat{A}_{ij} := e^{-4\chi} \left(K_{ij} - \frac{1}{3} \gamma_{ij} K \right). \tag{65}$$

Again, with the definition of χ above, both $\hat{\gamma}_{ij}$ and \hat{A}_{ij} are true tensors and not tensor densities, and their Lie derivatives should be calculated accordingly.

Similarly, the evolution equations for \hat{A}_{ij} and K become:

$$\begin{aligned} \partial_t \hat{A}_{ij} - \mathcal{L}_\beta \hat{A}_{ij} = e^{-4\chi} \{ & -\nabla_i \nabla_j \alpha + \alpha R_{ij} + 4\pi\alpha [\gamma_{ij} (S - \rho) - 2S_{ij}] \}^{\text{TF}} \\ & + \alpha \left(K \hat{A}_{ij} - 2\hat{A}_{ik} \hat{A}^k_j \right) - \frac{2}{3} \sigma \hat{A}_{ij} \hat{\nabla}_m \beta^m, \end{aligned} \tag{66}$$

$$\partial_t K - \mathcal{L}_\beta K = -\nabla^2 \alpha + \alpha \left(\hat{A}_{ij} \hat{A}^{ij} + \frac{1}{3} K^2 \right) + 4\pi\alpha (\rho + S), \tag{67}$$

Again, the Lie derivatives on the left-hand side are now those of proper tensors with no density weight. Notice that there is no term with $\sigma \hat{\nabla}_m \beta^m$ in the evolution equation for K since it is a scalar.

Just as before, the Ricci tensor that appears in the evolution equation for \hat{A}_{ij} is now separated into two contributions in the following way

$$R_{ij} = \hat{R}_{ij} + R_{ij}^\chi, \tag{68}$$

where \hat{R}_{ij} is the Ricci tensor associated with the conformal metric $\hat{\gamma}_{ij}$, and where R_{ij}^χ denotes the terms that depend on derivatives of χ :

$$R_{ij}^\chi = -2\hat{\nabla}_i \hat{\nabla}_j \chi - 2\hat{\gamma}_{ij} \hat{\nabla}^k \hat{\nabla}_k \chi + 4\hat{\nabla}_i \chi \hat{\nabla}_j \chi - 4\hat{\gamma}_{ij} \hat{\nabla}^k \chi \hat{\nabla}_k \chi, \tag{69}$$

with $\hat{\nabla}_i$ the covariant derivative associated with the conformal metric $\hat{\gamma}_{ij}$.

Following what we did in the case of the NOR formulation, we now want to write a fully covariant expression for the conformal Ricci tensor \hat{R}_{ij} . In order to do so we will again introduce the quantities

$$\hat{\Delta}^a_{bc} := \hat{\Gamma}^a_{bc} - \hat{\Gamma}^a_{bc}, \tag{70}$$

$$\hat{\Delta}^i := \hat{\gamma}^{mn} \hat{\Delta}^i_{mn} = \hat{\Gamma}^i - \hat{\gamma}^{mn} \hat{\Gamma}^i_{mn}. \tag{71}$$

With these definitions the conformal Ricci tensor can be written as

$$\begin{aligned} \hat{R}_{ab} = & -\frac{1}{2} \hat{\gamma}^{mn} \hat{\nabla}_m \hat{\nabla}_n \hat{\gamma}_{ab} + \hat{\gamma}_m \left(\hat{\nabla}_a \hat{\nabla}_b \right) \hat{\Delta}^m + \hat{\Delta}^m \hat{\Delta}_{(ab)m} \\ & + 2 \hat{\Delta}^{mn} \left(\hat{\Delta}_a \hat{\Delta}_b \right)_{mn} + \hat{\Delta}_a^{mn} \hat{\Delta}_{mnb}, \end{aligned} \tag{72}$$

The next step is to promote the $\hat{\Delta}^i$ to independent variables and find an evolution equation for them. In order to do this we must first find the evolution equation for the $\hat{\Gamma}^i$. Notice that, since now we have $\hat{\gamma} \neq 1$, the $\hat{\Gamma}^i$ now take the form

$$\hat{\Gamma}^i = \hat{\gamma}^{mn} \hat{\Gamma}^i_{mn} = -\partial_m \hat{\gamma}^{im} - \frac{1}{2} \hat{\gamma}^{im} \partial_m \ln \hat{\gamma}. \tag{73}$$

Using Eqs. (57) and (64) we now find, after some algebra:

$$\begin{aligned} \partial_t \hat{\Gamma}^i = & \mathfrak{L}_\beta \hat{\Gamma}^i + \hat{\gamma}^{mn} \partial_m \partial_n \beta^i - \frac{2}{\hat{\gamma}^{1/2}} \partial_m \left(\alpha \hat{A}^{im} \hat{\gamma}^{1/2} \right) \\ & + \frac{\sigma}{3} \left[\hat{\gamma}^{im} \partial_m \left(\hat{\nabla}_n \beta^n \right) + 2 \hat{\Gamma}^i \hat{\nabla}_n \beta^n \right], \end{aligned} \tag{74}$$

where $\mathfrak{L}_\beta \hat{\Gamma}^i$ is calculated as the Lie derivative of a vector:

$$\mathfrak{L}_\beta \hat{\Gamma}^i = \beta^m \partial_m \hat{\Gamma}^i - \hat{\Gamma}^m \partial_m \beta^i. \tag{75}$$

The evolution equation for $\hat{\Delta}^i$ can now be obtained from the last equation using the fact that

$$\partial_t \hat{\Delta}^i = \partial_t \hat{\Gamma}^i - \hat{\Gamma}^i_{mn} \partial_t \hat{\gamma}^{mn}, \tag{76}$$

where, just as we did in the case of the NOR formulation, we have assumed that the flat background does not evolve. One finds

$$\begin{aligned} \partial_t \hat{\Delta}^i = & \mathfrak{L}_\beta \hat{\Delta}^i + \hat{\gamma}^{mn} \partial_m \partial_n \beta^i + \hat{\gamma}^{mn} \mathfrak{L}_\beta \hat{\Gamma}^i_{mn} \\ & - 2 \hat{\nabla}_m \left(\alpha \hat{A}^{im} \right) + 2 \alpha \hat{A}^{mn} \hat{\Delta}^i_{mn} \\ & + \frac{\sigma}{3} \left[\hat{\nabla}^i \left(\hat{\nabla}_n \beta^n \right) + 2 \hat{\Delta}^i \hat{\nabla}_n \beta^n \right], \end{aligned} \tag{77}$$

where again the Lie derivative of $\hat{\Delta}^i$ is that of a true vector with no density weight, and the term $\mathfrak{L}_\beta \hat{\Gamma}^i_{mn}$ should be calculated as the Lie derivative of a tensor.

Just as it happened in the case of the NOR formulation, the right hand side of the last equations contains terms that involve partial derivatives of the shift, and also terms containing $\hat{\Gamma}^i_{mn}$, so the expression is not explicitly covariant. We can again fix this

by using the fact that on a flat background the following relation holds:

$$\hat{\gamma}^{mn} \hat{\nabla}_m \hat{\nabla}_n \beta^i = \hat{\gamma}^{mn} \partial_m \partial_n \beta^i + \hat{\gamma}^{mn} \mathfrak{L}_\beta \hat{\Gamma}^i_{mn}, \tag{78}$$

so that the evolution equation for $\hat{\Delta}^i$ takes the final form

$$\begin{aligned} \partial_t \hat{\Delta}^i - \mathfrak{L}_\beta \hat{\Delta}^i &= \hat{\gamma}^{mn} \hat{\nabla}_m \hat{\nabla}_n \beta^i - 2 \hat{\nabla}_m (\alpha \hat{A}^{im}) + 2\alpha \hat{A}^{mn} \hat{\Delta}^i_{mn} \\ &+ \frac{\sigma}{3} \left[\hat{\nabla}^i (\hat{\nabla}_n \beta^n) + 2 \hat{\Delta}^i \hat{\nabla}_n \beta^n \right], \end{aligned} \tag{79}$$

which is now manifestly covariant.

The last step is to add a multiple of the momentum constraints to the evolution equation for $\hat{\Delta}^i$ above. Doing that we finally find:

$$\begin{aligned} \partial_t \hat{\Delta}^i - \mathfrak{L}_\beta \hat{\Delta}^i &= \hat{\gamma}^{mn} \hat{\nabla}_m \hat{\nabla}_n \beta^i - 2 \hat{A}^{im} \partial_m \alpha - \alpha (2 - \xi) \hat{\nabla}_m \tilde{A}^{im} + 2\alpha \hat{A}^{mn} \hat{\Delta}^i_{mn} \\ &+ \alpha \xi \left(6 \tilde{A}^{ij} \partial_j \phi - \frac{2}{3} \tilde{\gamma}^{ij} \partial_j K - 8\pi \tilde{j}^i \right) + \frac{\sigma}{3} \left[\hat{\nabla}^i (\hat{\nabla}_n \beta^n) + 2 \hat{\Delta}^i \hat{\nabla}_n \beta^n \right], \end{aligned} \tag{80}$$

with ξ is an arbitrary constant such that $\xi > 1/2$ for the final system to be strongly hyperbolic.

We can now ask which would be the preferred choice for σ in the above evolution equations, that is, should we take a Lagrangian or an Eulerian approach? Looking at the evolution equation for χ , Eq. (61), one might first think that the simplest choice would be to take $\sigma = 0$ ($s = 1$), that is an Eulerian approach, since in that case the evolution equation simplifies. However, from the discussion above about the scalar character of χ we see that if we choose $\sigma = 0$, the evolution equation for χ *does not reduce* to the standard BSSN evolution equation for ϕ given by Eq. (36) in the case of Cartesian coordinates (for which $\hat{\gamma} = 1$). This statement might seem somewhat puzzling since for $\sigma = 0$ Eqs. (36) and (61) look identical. However, one must remember that χ is a true scalar, while ϕ is a scalar density, so that their Lie derivatives are different. The same is true for the evolution equations of $\hat{\gamma}_{ij}$, \hat{A}_{ij} and $\hat{\Delta}^i$.

It is in fact not difficult to convince oneself that if we want to recover the standard BSSN evolution equations in the case of Cartesian coordinates we must choose $\sigma = 1$, i.e. the Lagrangian approach, and simply remember that all dynamical quantities are now true tensors with *no* density weight. The terms corresponding to the Lie derivatives of tensor densities in standard BSSN now appear explicitly on the right-hand side of the evolution equations through the terms proportional to $\hat{\nabla}_m \beta^m$.

5.3 The Gamma driver shift condition

The equations presented in the previous section are completely general, and can be used with any gauge condition, both for the lapse and the shift. When we mentioned the issue of hyperbolicity we specified a particular slicing condition (the Bona–Masso

condition), and the use of a shift known a priori, but this was just in order to make the discussion concrete.

One particularly important shift condition that has turned out to be extremely robust in practice, and in the last few years has allowed for the stable and accurate simulation of inspiraling black holes is the so-called “Gamma driver” shift condition [17]. This shift condition is particularly well-adapted to the BSSN formulation, and comes in two versions (with several variations). The first possibility is the “parabolic” Gamma driver which takes the form:

$$\partial_t \beta^i = c_1 \partial_t \hat{\Gamma}^i, \quad (81)$$

with c_1 some positive constant. The reason for the name “parabolic” is that, since the time derivative of $\hat{\Gamma}^i$ that appears in the right-hand side involves second derivatives of the shift, the above equation results in a generalized heat-like equation for the shift components. In numerical simulations, the above condition has the same problem as any other parabolic equation, namely that for numerical stability when using an explicit scheme the time step must be proportional to the square of the spatial grid spacing, resulting in a prohibitive use of computational resources in the case of high resolution simulations (for implicit schemes this restriction can be lifted, but such schemes are more complex to code, particularly in more than one spatial dimension).

The second version of the Gamma driver is the so-called “hyperbolic” Gamma driver which can be written in two alternative forms. The first form is simply

$$\partial_t \beta^i = c_2 \hat{\Gamma}^i, \quad (82)$$

while the second is

$$\partial_t^2 \beta^i = c_2 \partial_t \hat{\Gamma}^i. \quad (83)$$

In both cases we end up with a generalized wave equation for the shift, which justifies the name “hyperbolic”. The second version is usually preferred since it allows one to add a damping term that has been found to be very important in numerical simulations:

$$\partial_t^2 \beta^i = c_2 \partial_t \hat{\Gamma}^i - \eta \partial_t \beta^i. \quad (84)$$

For reasons that we will not go into here, typical values of the parameters are $c_2 = 3/4$ and $\eta = 2/M_{\text{ADM}}$ (notice that η has dimensions of inverse distance, so it is usually scaled with the total ADM mass of the spacetime).

The main problem with the above shift conditions from the point of view of our present discussion is that, while they work well in Cartesian coordinates, they are seriously flawed in curvilinear coordinates since on the left-hand side we have a proper vector, while on the right-hand side we have contracted Christoffel symbols. But we see that this problem can now be easily solved by choosing conditions of the form

$$\partial_t \beta^i = c_1 \partial_t \hat{\Delta}^i \quad \text{parabolic}, \quad (85)$$

$$\partial_t^2 \beta^i = c_2 \partial_t \hat{\Delta}^i - \eta \partial_t \beta^i \quad \text{hyperbolic}. \quad (86)$$

When working in curvilinear coordinates, one must then use these modified Gamma driver conditions (or rather “Delta driver” conditions) in order to keep everything consistent.

6 The case of spherical symmetry

Having found the general form of the NOR and BSSN equations in curvilinear coordinates, we will now consider the special case of spherical symmetry. Here we will concentrate on the case of the BSSN formulation, for the NOR formulation the analysis is entirely analogous (and in fact somewhat simpler).

6.1 BSSN in spherical symmetry

6.1.1 Main equations

We start by writing the general form of the spatial metric in spherical symmetry as

$$dl^2 = e^{4\chi} \left(a(r, t) dr^2 + r^2 b(r, t) d\Omega^2 \right), \quad (87)$$

with $a(r, t)$ and $b(r, t)$ positive metric functions, $d\Omega^2$ the solid angle element $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$, and where χ is the BSSN conformal factor introduced in Sect. 5.2 above. Notice that with this notation the components of the conformal metric are $\hat{\gamma}_{rr} = a$, $\hat{\gamma}_{\theta\theta} = r^2 b$, and $\hat{\gamma}_{\varphi\varphi} = (r \sin \theta)^2 b$.

The determinants of the physical and conformal metric take the form:

$$\gamma = ab^2 \left(r^4 e^{12\chi} \sin^2 \theta \right), \quad (88)$$

$$\hat{\gamma} = ab^2 \left(r^4 \sin^2 \theta \right). \quad (89)$$

The determinant of the flat metric in spherical coordinates can be easily found by setting $a = b = 1$ in the expression for $\hat{\gamma}$ above:

$$\hat{\gamma} = r^4 \sin^2 \theta. \quad (90)$$

The condition that $\hat{\gamma}(t = 0) = \hat{\gamma}$ now implies that initially we must ask for $ab^2 = 1$.

Notice in particular that for a Lagrangian evolution ($\sigma = 1$) the metric components a and b are in fact not independent of each other. This is because in that case the determinant of the conformal metric $\hat{\gamma}$ remains constant in time, so that the relation $ab^2 = 1$ will always hold. In the Eulerian case ($\sigma = 0$) the quantity ab^2 does evolve, but its evolution is entirely controlled by the shift, and is independent of both the lapse and the extrinsic curvature.

Let us now consider the shift vector. Since we are in spherical symmetry, the shift (as well as any other vector) will only have a radial component: $\beta^i = (\beta^r, 0, 0)$. Since

the different evolution equations in BSSN involve the conformal divergence of the shift it is convenient at this point to calculate it. One finds, after some algebra:

$$\begin{aligned} \hat{\nabla}_m \beta^m &= \partial_r \beta^r + \beta^r \left(\frac{\partial_r a}{2a} + \frac{\partial_r b}{b} + \frac{2}{r} \right) \\ &= \partial_r \beta^r + \beta^r \left(\frac{\partial_r (ab^2)}{2ab^2} + \frac{2}{r} \right). \end{aligned} \tag{91}$$

Consider next the auxiliary vector $\hat{\Delta}^i$. Since this is a true vector (this is the whole idea), it will again only have a radial component: $\hat{\Delta}^i = (\hat{\Delta}^r, 0, 0)$. One can show that this is indeed the case from the definition (71). For the radial component we find

$$\hat{\Delta}^r = \frac{1}{a} \left[\frac{\partial_r a}{2a} - \frac{\partial_r b}{b} - \frac{2}{r} \left(1 - \frac{a}{b} \right) \right]. \tag{92}$$

One must remember, however, that in what follows $\hat{\Delta}^r$ will be promoted to an independent variable and the equation above will be considered a constraint.

Let us now find the specific form of the evolution equation for the conformal factor χ and the components of the conformal metric a and b . From Eq. (61) we find

$$\partial_t \chi = \beta^r \partial_r \chi + \frac{\sigma}{6} \hat{\nabla}_m \beta^m - \frac{1}{6} \alpha K, \tag{93}$$

where the divergence of the shift is given by (91) above. For the conformal metric components we find

$$\partial_t a = \beta^r \partial_r a + 2a \partial_r \beta^r - \frac{2}{3} \sigma a \hat{\nabla}_m \beta^m - 2\alpha a A_a, \tag{94}$$

$$\partial_t b = \beta^r \partial_r b + 2b \frac{\beta^r}{r} - \frac{2}{3} \sigma b \hat{\nabla}_m \beta^m - 2\alpha b A_b. \tag{95}$$

where we have introduced the quantities

$$A_a := \hat{A}_r^r, \quad A_b := \hat{A}_\theta^\theta. \tag{96}$$

The reason for using the mixed components of the traceless extrinsic curvature instead of the fully covariant ones is that in spherical symmetry such a choice simplifies considerably the evolution equations. In particular, the fact that the tensor \hat{A}_{ij} must be traceless implies that

$$A_a + 2A_b = 0. \tag{97}$$

Notice also that in fact one has

$$\hat{A}_r^r = \hat{\gamma}^{rr} \hat{A}_{rr} = \gamma^{rr} A_{rr} = A_r^r, \tag{98}$$

and similarly for the angular component, so when we use mixed components of second-rank tensors the “conformal” and “physical” versions are identical.

Consider next the evolution equation for the trace of the extrinsic curvature K . We find:

$$\begin{aligned} \partial_t K &= \beta^r \partial_r K - \nabla^2 \alpha + \alpha \left(A_a^2 + 2A_b^2 + \frac{1}{3} K^2 \right) \\ &\quad + 4\pi \alpha (\rho + S_a + 2S_b), \end{aligned} \tag{99}$$

with S_a and S_b the mixed components of the stress tensor

$$S_a := S_r^r, \quad S_b := S_\theta^\theta, \tag{100}$$

and where the physical Laplacian of the lapse is given by

$$\begin{aligned} \nabla^2 \alpha &= \frac{1}{ae^{4\chi}} \left[\partial_r^2 \alpha \right. \\ &\quad \left. - \partial_r \alpha \left(\frac{\partial_r a}{2a} - \frac{\partial_r b}{b} - 2\partial_r \chi - \frac{2}{r} \right) \right]. \end{aligned} \tag{101}$$

The evolution equation for the traceless part of the extrinsic curvature is somewhat more complicated. Remember first that we only need an evolution equation for A_a , since the traceless condition implies $A_b = -A_a/2$. Rewriting Eq. (66) for the case of spherical symmetry we find

$$\begin{aligned} \partial_t A_a &= \beta^r \partial_r A_a - \left(\nabla^r \nabla_r \alpha - \frac{1}{3} \nabla^2 \alpha \right) + \alpha \left(R_r^r - \frac{1}{3} R \right) \\ &\quad + \alpha K A_a - 16\pi \alpha (S_a - S_b), \end{aligned} \tag{102}$$

where $\nabla^2 \alpha$ was given above and

$$\nabla^r \nabla_r \alpha = \frac{1}{ae^{4\chi}} \left[\partial_r^2 \alpha - \partial_r \alpha \left(\frac{\partial_r a}{2a} + 2\partial_r \chi \right) \right], \tag{103}$$

and where the mixed radial component of the Ricci tensor R_r^r and its trace R are given by

$$\begin{aligned} R_r^r &= -\frac{1}{ae^{4\chi}} \left[\frac{\partial_r^2 a}{2a} - a \partial_r \hat{\Delta}^r - \frac{3}{4} \left(\frac{\partial_r a}{a} \right)^2 \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\partial_r b}{b} \right)^2 - \frac{1}{2} \hat{\Delta}^r \partial_r a + \frac{\partial_r a}{rb} + \frac{2}{r^2} \left(1 - \frac{a}{b} \right) \left(1 + \frac{r \partial_r b}{b} \right) \right. \\ &\quad \left. + 4 \partial_r^2 \chi - 2 \partial_r \chi \left(\frac{\partial_r a}{a} - \frac{\partial_r b}{b} - \frac{2}{r} \right) \right], \end{aligned} \tag{104}$$

$$R = -\frac{1}{ae^{4\chi}} \left[\frac{\partial_r^2 a}{2a} + \frac{\partial_r^2 b}{b} - a\partial_r \hat{\Delta}^r - \left(\frac{\partial_r a}{a}\right)^2 + \frac{1}{2} \left(\frac{\partial_r b}{b}\right)^2 + \frac{2}{rb} \left(3 - \frac{a}{b}\right) \partial_r b + \frac{4}{r^2} \left(1 - \frac{a}{b}\right) + 8 \left(\partial_r^2 \chi + (\partial_r \chi)^2\right) - 8\partial_r \chi \left(\frac{\partial_r a}{2a} - \frac{\partial_r b}{b} - \frac{2}{r}\right) \right]. \tag{105}$$

It is interesting to notice that in Eq. (102) there is no contribution from the divergence of the shift (and hence σ plays no role). The reason for this is that even though such terms do appear in the evolution equation for A_{rr} , once we raise the index to find the evolution equation for $A_a = A^r_r$ they cancel out. In fact, the shift contribution reduces to a pure advection term $\beta^r \partial_r A_a$. This is one of the reasons why working with the mixed components is useful.

Finally, we need an evolution equation for $\hat{\Delta}^r$. Writing (80) for the case of spherical symmetry we find

$$\begin{aligned} \partial_t \hat{\Delta}^r &= \beta^r \partial_r \hat{\Delta}^r - \hat{\Delta}^r \partial_r \beta^r + \frac{1}{a} \partial_r^2 \beta^r + \frac{2}{b} \partial_r \left(\frac{\beta^r}{r}\right) \\ &+ \frac{\sigma}{3} \left(\frac{1}{a} \partial_r (\hat{\nabla}_m \beta^m) + 2\hat{\Delta}^r \hat{\nabla}_m \beta^m\right) - \frac{2}{a} (A_a \partial_r \alpha + \alpha \partial_r A_a) \\ &+ 2\alpha \left(A_a \hat{\Delta}^r - \frac{2}{rb} (A_a - A_b)\right) + \frac{\alpha \xi}{a} \left[\partial_r A_a - \frac{2}{3} \partial_r K + 6A_a \partial_r \chi \right. \\ &\left. + (A_a - A_b) \left(\frac{2}{r} + \frac{\partial_r b}{b}\right) - 8\pi j_r \right], \end{aligned} \tag{106}$$

with j_r the (physical) covariant component of the momentum density, and where as before ξ is an arbitrary parameter such that $\xi > 1/2$, with preferred value $\xi = 2$.

Finally, it is also convenient to write the specific form of the Hamiltonian and momentum constraints. One finds

$$H = R - \left(A_a^2 + 2A_b^2\right) + \frac{2}{3} K - 16\pi \rho = 0, \tag{107}$$

$$\begin{aligned} M^r &= \partial_r A_a - \frac{2}{3} \partial_r K + 6A_a \partial_r \chi \\ &+ (A_a - A_b) \left(\frac{2}{r} + \frac{\partial_r b}{b}\right) - 8\pi j_r = 0. \end{aligned} \tag{108}$$

6.1.2 Regularization

As has already been discussed in [20,21], unless special care is taken, in spherical symmetry the coordinate singularity at the origin can be a source of serious problems caused by the lack of regularity of the geometric variables there. The problem arises because of the presence of terms in the evolution equations that go as $1/r$ near the origin. At the analytic level, for a regular spacetime one can show that such terms cancel exactly at the origin, thus ensuring well-behaved solutions. However, this exact cancellation usually fails to hold for numerical solutions. One then finds that the numerical solution becomes ill-behaved near $r = 0$.

There are in fact two different types of regularity conditions that the geometric variables must satisfy at $r = 0$. The first type of conditions are simply those imposed by the requirement that the different variables should have a well defined parity at the origin, and imply the following behavior for small r :

$$\alpha \sim \alpha^0 + \mathcal{O}(r^2), \quad (109)$$

$$\beta^r \sim \mathcal{O}(r), \quad (110)$$

$$a \sim a^0 + \mathcal{O}(r^2), \quad (111)$$

$$b \sim b^0 + \mathcal{O}(r^2), \quad (112)$$

$$A_a \sim A_a^0 + \mathcal{O}(r^2), \quad (113)$$

$$A_b \sim A_b^0 + \mathcal{O}(r^2), \quad (114)$$

$$\hat{\Delta}^r \sim \mathcal{O}(r), \quad (115)$$

with $\{\alpha^0, a^0, b^0, A_a^0, A_b^0\}$ perhaps functions of time, but not of r .

Notice, however, that the above parity conditions are not enough to guarantee regularity of the system of equations described in the previous section. Although most terms involving divisions by powers of r are indeed manifestly regular given the different parity conditions (because they involve various derivatives of the geometric quantities), there are in fact two types of terms that would seem to remain ill-behaved at the origin. In particular, the expression for $\hat{\Delta}^r$ (Eq. (92)) involves the term $(1 - a/b)/r$, while the expressions for the radial component of the Ricci tensor R_r^r and its trace R (Eqs. (104) and (105)) have terms of the form $(1 - a/b)/r^2$, which apparently blow up at the origin. Similarly, in the momentum constraint (108) we have a term of the type $(A_a - A_b)/r$, which again would seem to be ill-behaved at the origin.

The reason why these apparently ill-behaved terms turn out to be regular after all is a consequence of a second type of regularity conditions. These new conditions come from the fact that spacetime should be locally flat at the origin, and imply that for small r we must have

$$a - b \sim \mathcal{O}(r^2), \quad A_a - A_b \sim \mathcal{O}(r^2), \quad (116)$$

so that

$$a^0 = b^0, \quad A_a^0 = A_b^0. \quad (117)$$

It turns out that it is not trivial to implement numerically both the parity regularity conditions and the local flatness regularity conditions at the same time. The reason for this is that at $r = 0$ we now have three boundary conditions for just two variables: both the derivatives of a and b must vanish, plus a and b must be equal to each other (and the same thing must happen for A_a and A_b).

The regularization issue has already been discussed in some detail in several references [20–22]. Here we will introduce a regularization procedure based on the one presented in [20], but with one important modification. We will then start by introducing an auxiliary variable defined as

$$\lambda := \frac{1}{r^2} \left(1 - \frac{a}{b}\right). \tag{118}$$

Notice first that in [20] the variable λ was in fact defined defined as $\lambda := (1 - a/b)/r$. This is just a small difference of no real consequence. Now, the local-flatness regularity conditions above imply that close to the origin we must have

$$\lambda \sim \lambda^0 + \mathcal{O}(r^2). \tag{119}$$

The main difference with the regularization procedure described in [20] is that we will now also introduce a second auxiliary variable defined as

$$A_\lambda := \frac{1}{r^2} (A_a - A_b). \tag{120}$$

Again, the local-flatness regularity conditions imply that close to the origin we will have

$$A_\lambda \sim A_\lambda^0 + \mathcal{O}(r^2). \tag{121}$$

Having introduced λ and A_λ we can rewrite all apparently ill-behaved terms in the BSSN equations in terms of these quantities so that the equations now look regular. In particular, the expression for $\hat{\Delta}^r$ becomes

$$\hat{\Delta}^r = \frac{1}{a} \left[\frac{\partial_r a}{2a} - \frac{\partial_r b}{b} - 2r\lambda \right], \tag{122}$$

while R_r^r and R take the form

$$\begin{aligned} R_r^r = & -\frac{1}{ae^{4\chi}} \left[\frac{\partial_r^2 a}{2a} - a\partial_r \hat{\Delta}^r - \frac{3}{4} \left(\frac{\partial_r a}{a}\right)^2 \right. \\ & + \frac{1}{2} \left(\frac{\partial_r b}{b}\right)^2 - \frac{1}{2} \hat{\Delta}^r \partial_r a + \frac{\partial_r a}{rb} + 2\lambda \left(1 + \frac{r\partial_r b}{b}\right) \\ & \left. + 4\partial_r^2 \chi - 2\partial_r \chi \left(\frac{\partial_r a}{a} - \frac{\partial_r b}{b} - \frac{2}{r}\right) \right], \end{aligned} \tag{123}$$

$$\begin{aligned} R = & -\frac{1}{ae^{4\chi}} \left[\frac{\partial_r^2 a}{2a} + \frac{\partial_r^2 b}{b} - a\partial_r \hat{\Delta}^r - \left(\frac{\partial_r a}{a}\right)^2 + \frac{1}{2} \left(\frac{\partial_r b}{b}\right)^2 + \frac{2}{rb} \left(3 - \frac{a}{b}\right) \partial_r b \right. \\ & \left. + 4\lambda + 8 \left(\partial_r^2 \chi + (\partial_r \chi)^2\right) - 8\partial_r \chi \left(\frac{\partial_r a}{2a} - \frac{\partial_r b}{b} - \frac{2}{r}\right) \right]. \end{aligned} \tag{124}$$

Similarly, the momentum constraint now becomes

$$M^r = \partial_r A_a - \frac{2}{3} \partial_r K + 6A_a \partial_r \chi + A_\lambda \left(2r + r^2 \frac{\partial_r b}{b}\right) - 8\pi j_r = 0. \tag{125}$$

Of course, at this point we haven't really fixed the problem, all we have actually done is to define some new variables as short-hands where we have hidden the ill-behaved terms. The way to solve the problem is to promote λ and A_λ to independent variables (with initial data given through their definitions), and find evolution equations for them that are manifestly regular.

The evolution equation for λ can be found directly from its definition and the evolution equations for a and b (Eqs. (94) and (95)), and turns out to be

$$\partial_t \lambda = \beta^r \partial_r \lambda + \frac{2}{r} \left[\beta^r \lambda - \frac{a}{b} \partial_r \left(\frac{\beta^r}{r} \right) \right] + \frac{2\alpha a}{b} A_\lambda. \tag{126}$$

Notice that this equation is manifestly regular as long as $\beta^r \sim \mathcal{O}(r)$ for small r , and A_λ itself remains regular. Notice also that the equation does not involve σ , so it has the same form for Eulerian or Lagrangian evolutions.

In order to find the evolution equation for A_λ it is important to notice first that the traceless condition $A_a + 2A_b = 0$ implies that A_λ and A_a are in fact related through

$$A_\lambda = \frac{3A_a}{2r^2}. \tag{127}$$

This actually implies that for A_λ to remain regular we must ask for $A_a \sim \mathcal{O}(r^2)$ near the origin (or in other words: $A_a^0 = A_b^0 = 0$).

We can now use the evolution equation for A_a , Eq. (102), to obtain the evolution equation for A_λ . After a long algebra, in which one needs to take care of keeping together several terms that might be ill-behaved individually (particularly terms of the form $\partial_r(F/r)$, with F some function that behaves as $F \sim \mathcal{O}(r)$ near the origin), one finds

$$\begin{aligned} \partial_t A_\lambda = & \beta^r \partial_r A_\lambda + 2A_\lambda \frac{\beta^r}{r} \\ & - \frac{1}{rae^{4\chi}} \left[\partial_r \left(\frac{\partial_r \alpha}{r} \right) - \frac{\partial_r \alpha}{2r} \left(\frac{\partial_r a}{a} + \frac{\partial_r b}{b} + 8\partial_r \chi \right) \right] \\ & - \frac{\alpha}{rae^{4\chi}} \left[2\partial_r \left(\frac{\partial_r \chi}{r} \right) - \frac{\partial_r \chi}{r} \left(\frac{\partial_r a}{a} + \frac{\partial_r b}{b} + 4\partial_r \chi \right) \right] \\ & + \frac{\alpha}{ae^{4\chi}} \left[\frac{b}{2a} \partial_r^2 \lambda + \frac{a}{r} \partial_r \left(\frac{\hat{\Delta}^r}{r} \right) + \frac{\partial_r \lambda}{r} \left(1 + \frac{2b}{a} - \frac{rb}{2} \hat{\Delta}^r \right) \right. \\ & \left. + \frac{\partial_r a}{ar^2} \left(\frac{3}{4} \frac{\partial_r a}{a} - \frac{\partial_r b}{b} \right) - \frac{\lambda}{r} \left(b \hat{\Delta}^r + 2 \frac{\partial_r b}{b} \right) + \frac{b}{a} \lambda^2 \right] + \alpha K A_\lambda - 8\pi \alpha S_\lambda, \end{aligned} \tag{128}$$

where $S_\lambda := (S_a - S_b)/r^2$. The above equation is now manifestly regular in all its terms.

Notice that at this point one can in fact just choose to ignore the evolution equation for A_a , evolve A_λ using (128), and later recover A_a though the relation $A_a = 2r^2 A_\lambda/3$.

This is in fact what we do in the numerical code used for the numerical examples of the next section, since in practice it seems to reduce considerably the size of the numerical error (though evolving both A_a and A_λ independently also results in regular and stable evolutions).

7 Some numerical examples

We have constructed a numerical code using the regularized BSSN formulation in spherical symmetry described in the last section. The code uses a method of lines algorithm, with either second order 3-step iterative Crank–Nicholson (ICN) or fourth order Runge–Kutta (RK4) as the time integrator, and second or fourth order centered differences in space. This code turns out to be very robust, stable, and well behaved at the origin.

Here we will present some examples of numerical simulations using this code that involve both pure gauge dynamics in vacuum situations, and simulations with a non-zero matter field. We will also consider simulations of a Schwarzschild black hole, which is a special case since it is in fact not regular at the origin.

7.1 Pure gauge dynamics

As a first example we will consider a pure gauge pulse propagating through the numerical spacetime. We consider initial data corresponding to Minkowski spacetime:

$$\chi = 0, \quad (129)$$

$$a = b = 1, \quad (130)$$

$$A_a = A_b = K = 0, \quad (131)$$

$$\hat{\Delta}^r = 0, \quad (132)$$

which also imply $\lambda = A_\lambda = 0$. Non-trivial gauge dynamics are obtained by choosing an initial lapse with a Gaussian profile of the form

$$\alpha = 1 + \frac{\alpha_0 r^2}{1 + r^2} \left[e^{-(r-r_0)^2/\sigma^2} + e^{-(r+r_0)^2/\sigma^2} \right], \quad (133)$$

with α_0 some initial amplitude, r_0 the center of the Gaussian and σ its width. Notice that we have multiplied the whole expression with $r^2/(1+r^2)$ and have in fact added two symmetric Gaussians (centered at $r = r_0$ and $r = -r_0$). This is done in order to make sure that the initial lapse is both an even function of r , and vanishes at $r = 0$. Having set up this initial lapse we evolve the system using harmonic slicing (with zero shift):

$$\partial_t \alpha = -\alpha^2 K. \quad (134)$$

For the simulation shown below we have chosen the initial data parameters as $\alpha_0 = 0.01$, $r_0 = 5$, $\sigma = 1$, with grid parameters given by $\Delta r = 0.1$ and $\Delta t = \Delta r/2$.

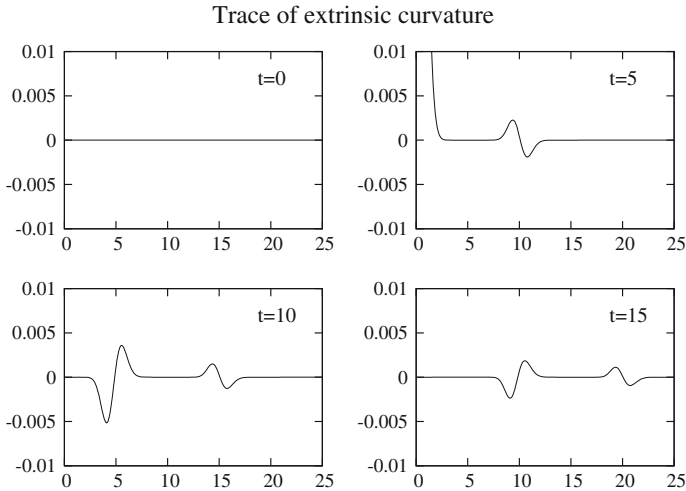


Fig. 1 Evolution of the trace of the extrinsic curvature K . The different panels correspond to times $t = 0, 5, 10, 15$

During the simulation the initial pulse first separates in two smaller pulses propagating in opposite directions (due to the time-symmetry of the initial data). The inward moving pulse later implodes through the origin at $t \sim 5$ and starts moving outward. Figure 1 shows a snapshot of the evolution of the extrinsic curvature K at times $t = 0, 5, 10, 15$. Notice that at $t = 5$ the value of K at the origin is too large for the chosen scale (it reaches a value of ~ 0.1), however the evolution always remains well-behaved and the value of K at the origin later returns to zero as the pulse moves outward.

Figure 2 shows the evolution of the Hamiltonian constraint for this simulation. Notice again that it remains well-behaved as the pulse goes through the origin. A small remnant of constraint violation that does not propagate away can also be clearly seen centered at the initial position of the pulse $r \sim 5$.

Finally, we will consider the issue of convergence of the code. Figure 3 shows the root-mean-square (RMS) norm of the Hamiltonian constraint as a function of time, for simulations at three different resolutions, $\Delta r = 0.1$, $\Delta r = 0.05$ and $\Delta r = 0.025$ (keeping always the same ratio $\Delta t/\Delta r = 0.5$). For the two highest resolutions, the norms have been rescaled by the corresponding factors expected for second order convergence, 4 and 16 respectively. The fact that all three lines coincide indicates that the code is indeed converging to second order.

7.2 Scalar field

The second set of simulations are somewhat more interesting since they evolve a non-zero matter field and therefore contain true dynamics. The matter field will correspond to a real scalar field Φ with stress-energy tensor given by:

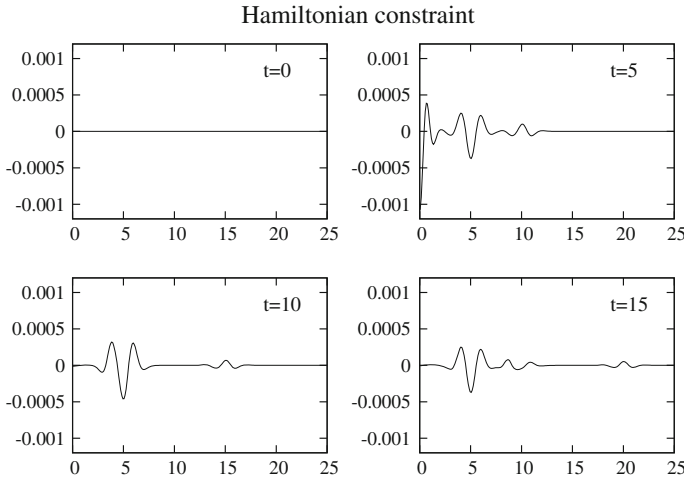


Fig. 2 Evolution of the Hamiltonian constraint. The *different panels* correspond to times $t = 0, 5, 10, 15$

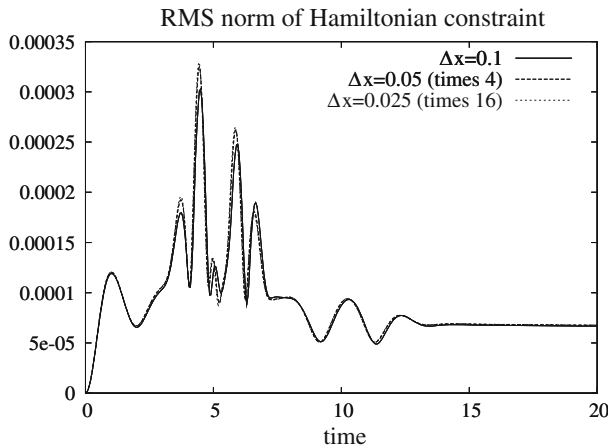


Fig. 3 RMS norm of the Hamiltonian constraint as a function of time, for simulations at three different resolutions: $\Delta r = 0.1, \Delta r = 0.05$ and $\Delta r = 0.025$. The two highest resolutions have been rescaled by factors of 4 and 16, respectively

$$T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \Phi \partial_\alpha \Phi. \tag{135}$$

Let us now consider the case of spherical symmetry, and define the auxiliary first order quantities:

$$\Pi := n^\mu \partial_\mu \Phi = \frac{1}{\alpha} (\partial_t \Phi - \beta^r \partial_r \Phi), \tag{136}$$

$$\Psi := \partial_r \Phi. \tag{137}$$

The corresponding energy density ρ , momentum density j^r , and stress tensor S_{ij} that appear in the 3+1 equations are then given by

$$\rho := n^\mu n^\nu T_{\mu\nu} = \frac{1}{2} \left(\Pi^2 + \frac{\Psi^2}{ae^{4\chi}} \right), \quad (138)$$

$$j^r := -P^{r\mu} n^\nu T_{\mu\nu} = -\Pi\Psi, \quad (139)$$

$$S_a := T_r^r = \frac{1}{2} \left(\Pi^2 + \frac{\Psi^2}{ae^{4\chi}} \right), \quad (140)$$

$$S_b := T_\theta^\theta = \frac{1}{2} \left(\Pi^2 - \frac{\Psi^2}{ae^{4\chi}} \right). \quad (141)$$

The scalar field evolves through the simple wave equation $\square\Phi = 0$, which in this case can be written as the following system of first order equations:

$$\partial_t \Phi = \beta^r \partial_r \Phi + \alpha \Pi, \quad (142)$$

$$\partial_t \Psi = \beta^r \partial_r \Psi + \Psi \partial_r \beta^r + \partial_r (\alpha \Pi), \quad (143)$$

$$\begin{aligned} \partial_t \Pi = & \beta^r \partial_r \Pi + \frac{\alpha}{ae^{4\chi}} \left[\partial_r \Psi + \Psi \left(\frac{2}{r} - \frac{\partial_r a}{2a} + \frac{\partial_r b}{b} + 2\partial_r \chi \right) \right] \\ & + \frac{\Psi}{ae^{4\chi}} \partial_r \alpha + \alpha K \Pi. \end{aligned} \quad (144)$$

For the initial data, we again choose an initial Gaussian profile for the scalar field of the form

$$\Phi = \frac{\Phi_0 r^2}{1+r^2} \left[e^{-(r-r_0)^2/\sigma^2} + e^{-(r+r_0)^2/\sigma^2} \right], \quad (145)$$

with Φ_0 the initial amplitude, r_0 the center of the Gaussian and σ its width. We also assume time symmetry, so that at $t = 0$ we have $K_{ij} = 0$ and $\Pi = 0$, which implies that the momentum constraint is identically satisfied.

Finally, we choose a conformally flat metric with $a = b = 1$, and solve for the conformal factor $\psi = e^\chi$ using the Hamiltonian constraint, which in this case takes the form

$$\partial_r^2 \psi + \frac{2}{r} \partial_r \psi + 2\pi \psi^5 \rho = 0. \quad (146)$$

Notice that when we substitute the value of ρ given above (with $\Pi = 0$), this equation reduces to

$$\partial_r^2 \psi + \frac{2}{r} \partial_r \psi + (\pi \Psi^2) \psi = 0. \quad (147)$$

This is clearly a linear equation for ψ , which simplifies considerably its numerical solution (we can simply use centered spatial differences and invert the resulting tridiagonal matrix directly).

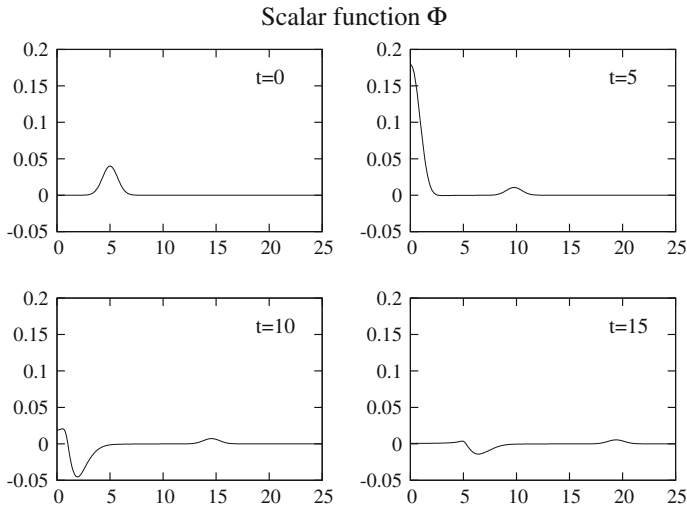


Fig. 4 Evolution of the scalar field Φ . The *different panels* correspond to times $t = 0, 5, 10, 15$

As gauge conditions we have chosen zero shift and 1+log slicing, which is given by

$$\partial_t \alpha = -2\alpha K. \quad (148)$$

For the specific simulation discussed here, we have taken as initial data parameters: $\Phi_0 = 0.04$, $r_0 = 5$, $\sigma = 1$. We have chosen this initial data to be rather strong, but not quite strong enough to collapse to a black hole. This is on purpose since we are interested in the regularity at the origin, and the collapse of the lapse associated with the formation of a black hole makes this issue less relevant as everything just freezes close to the origin. In Sect. 7.3 below we will consider the case of a single Schwarzschild black hole.

In Fig. 4 we show snapshots of the evolution of the scalar field Φ , for a numerical simulation using a grid spacing $\Delta r = 0.025$ and time step $\Delta t = \Delta r/2$. One can clearly see how the initial pulse separates into ingoing and outgoing pieces. The ingoing part then implodes through the origin and starts moving out, though significantly deformed.

Figure 5 shows the evolution of the central value of the lapse function α as a function of time (actually the value at $r = \Delta r/2$ since the origin itself is staggered). Notice how at $t \sim 7$ the central value of the lapse drops below 0.2, indicating a very strong gravitational field. However, the lapse later bounces and returns towards 1, and no black hole forms. The simulation remains well behaved throughout, and the origin remains regular.

Finally, in Fig. 6 we show again a plot of the RMS norm of the Hamiltonian constraint for three different resolutions, $\Delta r = 0.05$, $\Delta r = 0.025$ and $\Delta r = 0.0125$ (due to the large dynamical range, the plot is now logarithmic). Again, the higher resolution runs have been rescaled by factors of 4 and 16, respectively. The fact that the lines lie on top of each other shows that the code is converging to second order, as

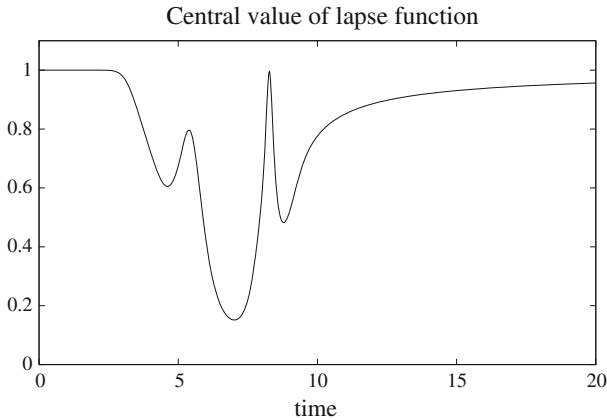


Fig. 5 Central value of the lapse as a function of time

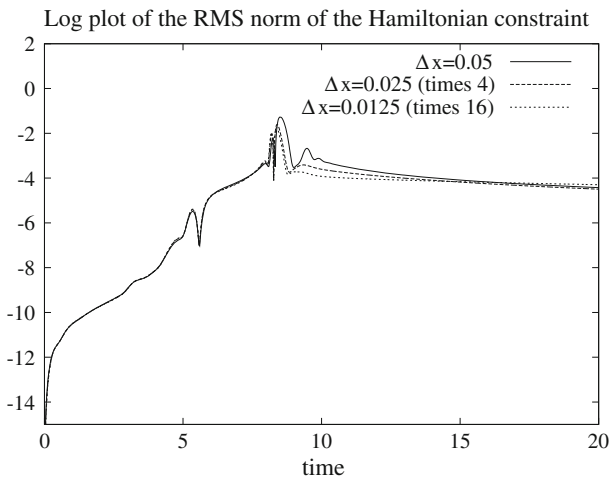


Fig. 6 Logarithm of the RMS norm of the Hamiltonian constraint as a function of time, for simulations at three different resolutions: $\Delta r = 0.05$, $\Delta r = 0.025$ and $\Delta r = 0.0125$. The two highest resolutions have been rescaled by factors of 4 and 16, respectively

expected. One can notice that from $t \sim 7$ to $t \sim 15$ the convergence is less than perfect and the lines do not align precisely. This behaviour is a reflection of the fact that the gravitational field is very strong so that very high resolution is needed to adequately capture the situation.

7.3 Schwarzschild black hole

As our final example we will choose a Schwarzschild black hole, so again we are back in vacuum. The Schwarzschild solution is static in standard coordinates, but these coordinates are ill-behaved at the horizon. We will therefore use isotropic coordinates

in which the initial spatial metric takes the form

$$dl^2 = \psi^4 \left(dr^2 + r^2 d\Omega^2 \right), \tag{149}$$

where the conformal factor ψ is given by

$$\psi = 1 + M/2r, \tag{150}$$

and with M the mass of the black hole. As is well known, the Schwarzschild solution in isotropic coordinates has the topology of a wormhole (Einstein–Rosen bridge), with the throat located at $r = M/2$ (coincident with the horizon at $t = 0$), and with a coordinate singularity at $r = 0$ which corresponds to the compactification of the asymptotic flat region on the other side of the wormhole.

The presence of the coordinate singularity at $r = 0$ implies that our regularization procedure is now wrong since it was based on the idea of space being locally flat at the origin, which is clearly not the case here. The origin is not even part of space, which is why this type of initial data is known as puncture initial data (space in fact corresponds to \mathbb{R}^3 minus the point at the origin, so it is a “punctured” \mathbb{R}^3). The main consequence of this is that using the regularization procedure now fails and the simulations quickly crash. However, we have found that if we simply turn off the regularization, and run without introducing the variables λ and A_λ , we can have very stable and accurate simulations with the exception of the first few grid points closer to the origin where the code fails to converge (but we do find convergence away from these points as the plots below show). A more careful look at the data shows that, for the simulations described below, close to the origin we have $A_a \sim r$, so that $A_\lambda \sim 1/r$ (confront Eq. (127)), which explains why the regularization procedure fails. A more detailed analysis of the behaviour at the origin for black hole simulations is clearly needed, but this is outside the scope of this paper.

For all the simulations shown here we have chosen maximal slicing. This corresponds to the condition $K = \partial_t K = 0$, which leads to the following equation for the lapse function:

$$\begin{aligned} \partial_r^2 \alpha + \left(\frac{2}{r} - \frac{\partial_r a}{2a} + \frac{\partial_r b}{b} + 2\partial_r \chi \right) \partial_r \alpha \\ - \alpha a e^{4\chi} \left[K_{ij} K^{ij} + 4\pi (\rho + S_a + 2S_b) \right] = 0, \end{aligned} \tag{151}$$

where $K_{ij} K^{ij} = A_a^2 + 2A_b^2 + K^2/3$. Notice that this is again a linear equation for α , which can be solved by direct matrix inversion. The reason for choosing maximal slicing is to make sure that the lapse collapses at $r = 0$, which would not happen with 1+log slicing. This is because the origin is in fact an infinite proper distance away, so that any slicing condition with a finite speed of propagation would never change the value of the lapse there.²

² One could still use 1+log slicing if one chooses a pre-collapsed initial lapse. This is in fact what is typically done in 3D black hole simulations.

For the shift we have chosen a Gamma driver condition of the form discussed in Sect. 5.3. In the particular case of spherical symmetry this condition reduces to:

$$\partial_t^2 \beta^r = \frac{3}{4} \partial_t \hat{\Delta}^r - \eta \partial_t \beta^i. \quad (152)$$

Here we have already chosen the coefficient of the term $\partial_t \hat{\Delta}^r$ equal to 3/4 in order to have an asymptotic gauge speed equal to 1. The condition above is solved in first order form by introducing the time derivative of the shift as an auxiliary quantity, so that we in fact solve the system:

$$\partial_t \beta^r = B^r, \quad (153)$$

$$\partial_t B^r = \frac{3}{4} \partial_t \hat{\Delta}^r - \eta B^r. \quad (154)$$

Notice that this is the same shift condition (with minor variations) that is currently being used in most 3D codes that evolve black hole spacetimes. In the simulations shown below, the damping coefficient is always taken to be $\eta = 2$.

We still need to mention one final ingredient that goes into these simulations. Following [6, 7], we have found that the simulations are better behaved if instead of evolving the singular conformal factor χ directly, we evolve the quantity $X := e^{-2\chi}$.³

We are now ready to describe the numerical simulations. In all our simulations we have chosen the mass of the black hole to be $M = 1$. We have chosen a grid spacing of $\Delta r = 0.01$ and time step of $\Delta t = 0.005$. We have also used 10,000 grid points in order to place the boundaries sufficiently far away so as not to have large errors from the boundaries affect the evolution.⁴

Figures 7, 8, 9, 10 show snapshots of the evolution of the lapse function α , the radial component of the shift vector β^r , the radial metric component a and the conformal factor χ at times $t = 0, 5, 10, 15$. Do notice that in order to better appreciate the plots, for the conformal factor χ we have used a log plot, while for the shift we plot a smaller radial domain.

The first thing to notice from these plots is the fact that the simulation is well behaved. The lapse collapses to 0 at the origin (but with a non-zero derivative there as expected from the gauge conditions used, see e.g. [23]). The shift grows with time to counteract the slice stretching effect, but becomes almost stationary at late times. On the other hand the conformal factor, though still singular at the origin, also remains well behaved.

In order to show that the simulations remain well behaved for long times, and in fact reach an almost stationary state, in Fig. 11 below we show the evolution of the maximum value of the radial metric component a and the radial shift vector β^r up to $t = 100$. In both plots we can see that initially the maximum values grow rapidly, but

³ Notice that in [6, 7] they in fact evolve the quantity $\chi := e^{-4\phi}$. Here our notation is different, so that χ plays the role of ϕ , and X the role of χ , we also use a second power instead of fourth power since we find this to work better in our case.

⁴ The boundary conditions chosen are stable and well behaved, but we will not discuss them in any detail here. We are preparing a paper where we will concentrate on the boundary conditions.

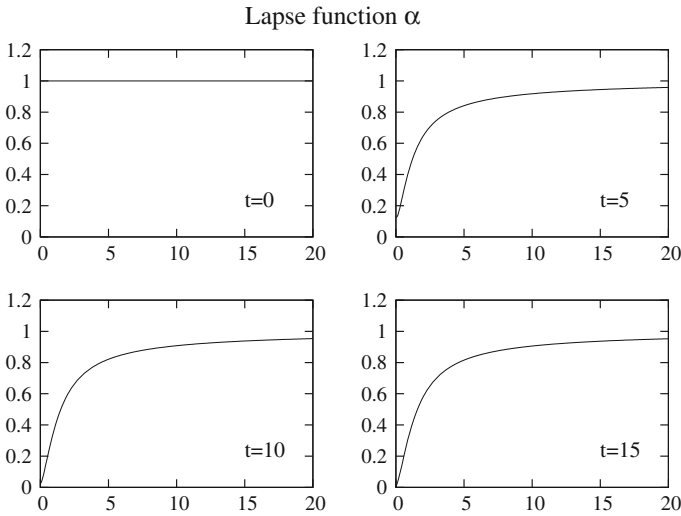


Fig. 7 Evolution of the lapse function α . The different panels correspond to times $t = 0, 5, 10, 15$

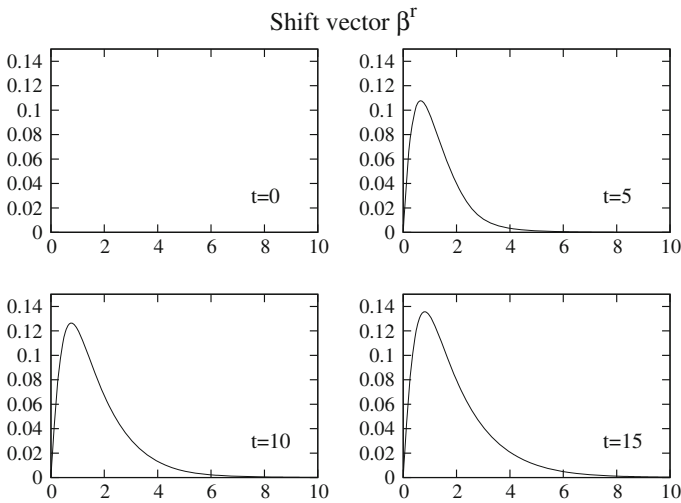


Fig. 8 Evolution of the radial component of the shift vector β^r . The different panels correspond to times $t = 0, 5, 10, 15$

this behaviour is later replaced by a very slow upward drift (this drift is well known from 3D simulations and is a consequence of the gauge conditions, particularly the damping term in the Gamma driver shift condition).

Figure 12 shows the time evolution of the coordinate position of the apparent horizon and the apparent horizon mass (defined in terms of its area A_{ah} as $M_{ah} = (A_{ah}/16\pi)^{1/2}$). One can notice how the radial position of the horizon drifts outward from $r = 0.5$ initially to $r \sim 1.1$ at the end of the simulation. On the other hand, the horizon mass remains within 0.005% of unity throughout the entire simulation.

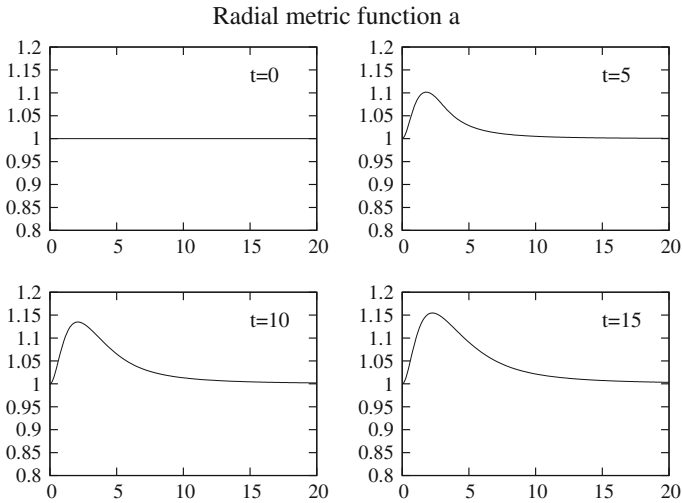


Fig. 9 Evolution of the radial metric function a . The different panels correspond to times $t = 0, 5, 10, 15$

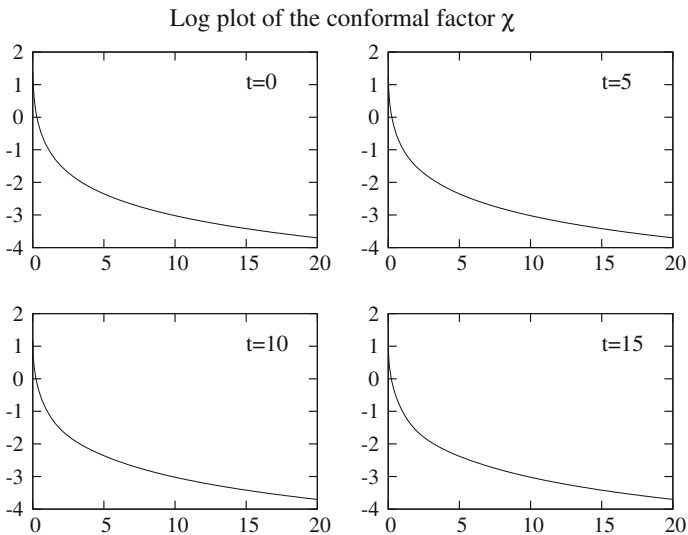


Fig. 10 Evolution of the log of the conformal factor χ . The different panels correspond to times $t = 0, 5, 10, 15$

This shows that the spherically symmetric BSSN code with maximal slicing and a Gamma driver shift condition can successfully and accurately simulate a black hole spacetime for a very long time without the need to excise the black hole interior. We have in fact done much longer evolutions up to $t \sim 10,000$, for which the system remains stable and well behaved. However, in those cases outer boundary effects become important and introduce larger errors, and for this reason we do not show them here.

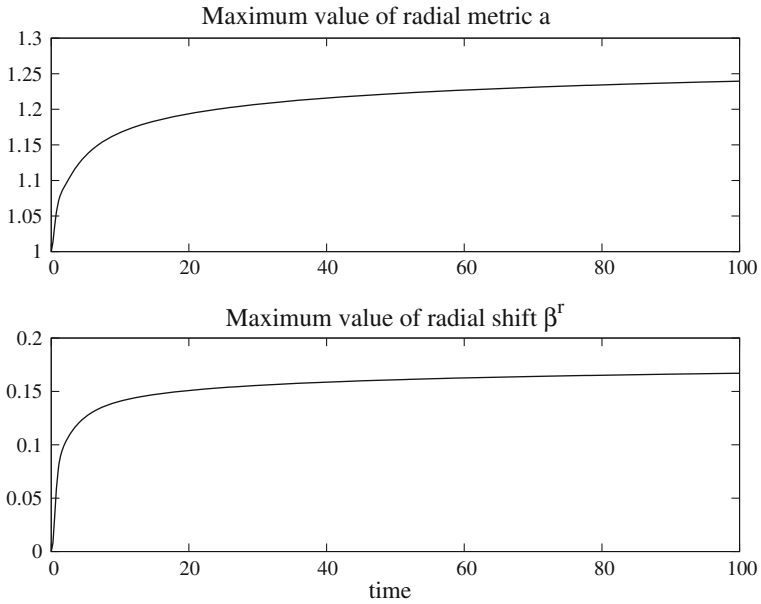


Fig. 11 Evolution of the maximum value of the radial metric component a (*upper panel*), and the radial component of the shift vector β^r (*lower panel*)

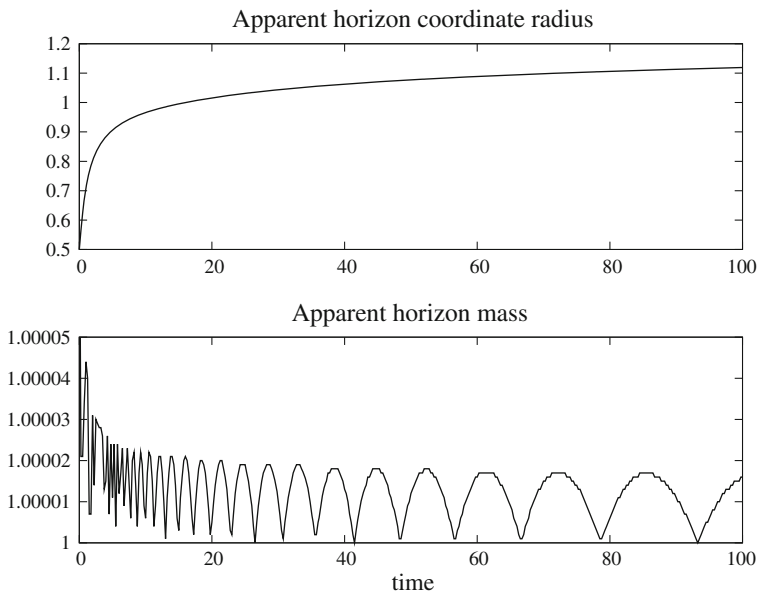


Fig. 12 Coordinate position of the apparent horizon (*upper panel*), and apparent horizon mass $M_{ah} = (A_{ah}/16\pi)^{1/2}$ (*lower panel*), as functions of time

8 Conclusions

Following Brown [10], in this paper we have described how to modify standard hyperbolic formulations of the 3+1 evolution equations of General Relativity in such a way that all auxiliary quantities are true tensors, thus allowing for these formulations to be used with curvilinear sets of coordinates. We have considered in particular both the Nagy–Ortiz–Reula (NOR) and the Baumgarte–Shapiro–Shibata–Nakamura (BSSN) formulations, but the main ideas presented here can in principle be applied in general.

The key idea has been that instead of using contracted Christoffel symbols as auxiliary variables, one should use the difference between the physical Christoffel symbols and the Christoffel symbols associated with a background flat metric in the curvilinear coordinates, since the difference of Christoffel symbols always behaves as a true tensor.

Also, in the particular case of the BSSN formulation, one should not force the conformal volume element (the metric determinant) to be equal to 1, but rather to be equal to its initial value in the curvilinear coordinates. One important consequence of this choice is that all dynamical geometric quantities now remain true tensors instead of tensor densities as in standard BSSN. One is also free to choose how this conformal volume element will evolve in time. Two “natural” choices present themselves: the Lagrangian approach where the conformal volume elements remain constant along time lines, and the Eulerian approach where they remain constant along the normal direction to the hypersurfaces. Standard BSSN can then be shown to be equivalent to the Lagrangian approach.

Having developed the general formalism, we considered as an example the particular case of BSSN in spherical symmetry, and studied in some detail the important problem of the regularity of the equations at the origin. For this we introduced extra auxiliary quantities that allowed us to impose the “local flatness” regularity condition in a consistent way. Our regularization algorithm is similar to the one presented in [20], but it has been modified in a way that makes it more general and easier to implement. It is important to mention that this regularization assumes that spacetime is regular at the origin and as such does not work for the case of black hole spacetimes where the origin is in fact a compactification of an asymptotic infinity on the other side of the Einstein–Rosen bridge.

Finally, we presented a series of numerical simulations of our BSSN code in spherical symmetry, and we showed that the code was capable of evolving both regular spacetimes (with and without matter), as well as black hole spacetimes in a stable, accurate and robust way.

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