

# Application of time transfer function to McVittie spacetime: gravitational time delay and secular increase in astronomical unit

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**Abstract** We attempt to calculate the gravitational time delay in a time-dependent gravitational field, especially in McVittie spacetime, which can be considered as the spacetime around a gravitating body such as the Sun, embedded in the FLRW (Friedmann–Lemaître–Robertson–Walker) cosmological background metric. To this end, we adopt the time transfer function method proposed by Le Poncin-Lafitte et al. (Class Quantum Gravity 21:4463, 2004) and Teyssandier and Le Poncin-Lafitte (Class Quantum Gravity 25:145020, 2008), which is originally related to Synge’s world function  $\Omega(x_A, x_B)$  and enables to circumvent the integration of the null geodesic equation. We re-examine the global cosmological effect on light propagation in the solar system. The round-trip time of a light ray/signal is given by the functions of not only the spacial coordinates but also the emission time or reception time of light ray/signal, which characterize the time-dependency of solutions. We also apply the obtained results to the secular increase in the astronomical unit, reported by Krasinsky and Brumberg (Celest Mech Dyn Astron 90:267, 2004), and we show that the leading order terms of the time-dependent component due to cosmological expansion is 9 orders of magnitude smaller than the observed value of  $dAU/dt$ , i.e.,  $15 \pm 4$  (m/century). Therefore, it is not possible to explain the secular increase in the astronomical unit in terms of cosmological expansion.

**Keywords** Gravitation · Relativity · Ephemerides · Astronomical unit · Light propagation

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## 1 Introduction

Technological progress has significantly enhanced the accuracy of solar system observations. Presently, to archive an accuracy of 9 to 11 digits or higher, the 1st order  $\mathcal{O}(c^{-2})$  or  $\mathcal{O}(G)$  ( $c$  is the speed of light in vacuum and  $G$  is the gravitational constant) post-Newtonian/post-Minkowskian effects of general relativity must be incorporated in a number of observational models. Moreover, several space missions such as GAIA [1], Laser Astrometric Test Of Relativity (LATOR) [2], and Astrodynamical Space Test of Relativity using Optical Devices (ASTROD) [3] are planned and especially LATOR and ASTROD can potentially detect post-linear or 2nd order  $\mathcal{O}(c^{-4})$  or  $\mathcal{O}(G^2)$  post-Newtonian/post-Minkowskian effects.

Highly accurate and rigorous light propagation models are required to analyze high precision data such as the radar time delay, deflection of light rays, and frequency shift of radio signals. Although, these effects have been extensively investigated (see i.e. [4–9] and references therein, most of the previous studies were based on integrating the null geodesic equation of light rays. This approach is effective in the case of linear or 1st order approximation. However, it is generally difficult to compute the contribution of 2nd and higher order effects, even in the case of static gravitational fields; the situation becomes further complicated in the case of time-dependent spacetime.

A novel method known as the time transfer function method has been recently developed [10, 11] to overcome such circumstances. This approach is originally based on Synge's world function  $\Omega(x_A, x_B)$  [12],

$$\Omega(x_A, x_B) \equiv \frac{1}{2}(\lambda_B - \lambda_A) \int_{\lambda_B}^{\lambda_A} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda, \quad (1)$$

where  $g_{\mu\nu}$  is a metric tensor of spacetime,  $x_A = (x_A^0 = ct_A, x_A^i = \mathbf{x}_A)$  and  $x_B = (x_B^0 = ct_B, x_B^i = \mathbf{x}_B)$  are the coordinates of end points  $A$  and  $B$ , respectively, on the geodesic world-line, and  $\lambda$  is the affine parameter. Then, the world function  $\Omega(x_A, x_B)$  is defined as the half length of the world line between  $A$  and  $B$ . On the basis of (1), the time transfer functions introduced in [10, 11] define the travel time of a light rays/signal between two points, i.e.,  $t_B - t_A$ . [10] and [11] succeeded in constructing the general  $N$ -th order post-Minkowskian approximation scheme with respect to the gravitational constant  $G$  via an iterative procedure, and they showed that this method can reduce the number of calculations by taking the static Schwarzschild spacetime for instance.

On the other hand, in the field of experimental relativity, astrometric observations in the solar system have played a crucial role in the verification of gravity. It has been shown that the main PPN (parametrized post-Newtonian) parameters  $\beta$  and  $\gamma$  are equal to unity for general relativity, ( $\beta = \gamma = 1$ ); as increasing accuracy,  $\beta$  and  $\gamma$  are strictly constrained to unity (see Fig. 1 of [13]). The enhanced accuracy of observations accounts for the existence of unexplained phenomena by means of the framework of general relativity; such phenomena include the anomalous acceleration of the Pioneer spacecraft [14], Earth flyby anomaly [15], the anomalous perihelion precession of Saturn [16], and the secular increase in the astronomical unit [17]. Presently, the

origins of these anomalies are far from clear; however, they may be attributable to those fundamental properties of gravitation (see [18, 19] and the references therein).

The secular increase in the astronomical unit is particularly interesting. Here, the astronomical unit implies that the conversion constant between two length units and the current best-fit value is [20]

$$\frac{1 [\text{AU}]}{1 (\text{m})} \equiv \text{AU} = 1.495978706960 \pm 0.1. \quad (2)$$

The determination of astronomical unit is directly related to the light propagation formula, i.e., the equation of light time. Without loss of generality, the secular increase in the astronomical unit was found by the relation

$$t_{\text{theo}} = \frac{d_{\text{theo}}}{c} \left[ \text{AU} + \frac{d\text{AU}}{dt}(t - t_0) \right] (\text{s}), \quad (3)$$

where  $t_{\text{theo}}$  is the computed round-trip time (theoretical value),  $d_{\text{theo}}$  is the interplanetary distance obtained from planetary ephemerides (note that the length unit of  $d_{\text{theo}}$  is [AU]), AU and  $d\text{AU}/dt$  are the astronomical unit and its time variation, respectively, and  $t_0$  is the initial epoch.  $t_{\text{theo}}$  is compared with the observed round-trip time  $t_{\text{obs}}$ . Krasinsky and Brumberg estimated  $d\text{AU}/dt$  as  $15 \pm 4 (\text{m}/\text{cy})$ .<sup>1</sup>

The influence of cosmological expansion has been investigated as a potential cause of secular variation in AU [17, 21, 22]. In particular, [17] and [22] examined its contribution to light propagation. The approach of [17] is somewhat qualitative, and the discussion of [22] is essentially reduced to a static model that is equivalent to Schwarzschild-de Sitter spacetime. These circumstances are due to the fact that the time-dependent null geodesic equation must be solved in order to study the cosmological effect on light propagation; however it is generally difficult to solve this equation. Nonetheless, the time transfer function method proposed in [10, 11] is applicable not only to the static case but also to the time-dependent case. Therefore, it is useful to re-examine the effect of cosmological expansion on the basis of the time transfer function and to compare the theoretically obtained results with the observed  $d\text{AU}/dt$ .

In this paper, according to the time transfer function method, we consider light propagation in a time-dependent gravitational field, and we focus our discussion on the gravitational time delay. We adopt the McVittie model [23, 24] as the time-dependent spacetime; it can be considered as the spacetime around a gravitating body such as the Sun, embedded in the FLRW (Friedmann–Lemaître–Robertson–Walker) cosmological background metric. The time-dependency in the solution is characterized by either the emission time or the reception time of a light ray/signal. Then, the obtained results is applied to the secular (time) variation in the astronomical unit, reported in [17].

The remainder of this paper is organized as follows. In Sect. 2, we briefly summarize the time transfer function, and in Sect. 3, we investigate the gravitational time delay in McVittie spacetime. In Sect. 4, we discuss the secular increase in the astronomical unit. Finally, in Sect. 5, we conclude the paper with a summary of our study.

<sup>1</sup> In this paper, cy denotes century according to Krasinsky and Brumberg (2004) [17].

## 2 Time transfer function

Before discussing the gravitational time delay in McVittie spacetime, we shall briefly summarize the time transfer function method (see [10,11] for the details).

It is generally difficult to acquire the world function  $\Omega(x_A, x_B)$  concretely; nevertheless, in the case of Minkowskian flat spacetime, the world function is easily obtained by using the parameter equation  $x(\lambda) = (x_B - x_A)\lambda + x_A$  and by setting  $\lambda_A = 0$  and  $\lambda_B = 1$  [10,12]:

$$\Omega^{(0)}(x_A, x_B) = \frac{1}{2}\eta_{\mu\nu} (x_B^\mu - x_A^\mu) (x_B^\nu - x_A^\nu), \tag{4}$$

where the  $x^\mu$  ( $\mu = 0, 1, 2, 3$ ) are Minkowskian coordinates with respect to the Minkowski metric  $\eta_{\mu\nu} = \text{diag}(-1; 1; 1; 1)$ . For the null geodesic, the world function  $\Omega(x_A, x_B)$  satisfies the condition

$$\Omega(x_A, x_B) = 0 \tag{5}$$

because  $ds^2 = 0$ . Hence, from (4) and (5), the travel time between  $A$  and  $B$ , i.e.,  $t_B - t_A$ , in Minkowski spacetime becomes

$$c^2(t_B - t_A)^2 = \delta_{ij} (x_B^i - x_A^i) (x_B^j - x_A^j) = R_{AB}^2, \tag{6}$$

where  $\delta_{ij}$  is Kronecker’s delta. The time transfer function approach starts from (6), and the weak-field approximation is recursively developed with respect to the gravitational constant  $G$ .

If the metric has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1, \tag{7}$$

where  $h_{\mu\nu}$  is a perturbation to  $\eta_{\mu\nu}$ , the time transfer functions that give the travel time of the light ray/signal are formally expressed as follows:

$$t_B - t_A = \mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B) = \frac{1}{c} [R_{AB} + \Delta_e(t_A, \mathbf{x}_A, \mathbf{x}_B)] \tag{8}$$

$$= \mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B) = \frac{1}{c} [R_{AB} + \Delta_r(\mathbf{x}_A, t_B, \mathbf{x}_B)], \tag{9}$$

where  $\mathcal{T}_e(t_A, \mathbf{x}_A, \mathbf{x}_B)$  is the emission time transfer function in spacial coordinates  $\mathbf{x}_A, \mathbf{x}_B$  and  $t_A$  is the emission time of the signal,  $\mathcal{T}_r(\mathbf{x}_A, t_B, \mathbf{x}_B)$  is the reception time transfer function in spacial coordinates  $\mathbf{x}_A, \mathbf{x}_B$  and  $t_B$  is the reception time of the signal,  $R_{AB} = |\mathbf{x}_B - \mathbf{x}_A|$ , and  $\Delta_e$  and  $\Delta_r$  are the emission time delay function and reception time delay function, respectively.  $\Delta_e$  and  $\Delta_r$  characterize the gravitational time delay. In (8) and (9),  $R_{AB}$  comes from (6). Henceforth,  $A$  denotes the emission and  $B$  denotes the reception.

$\Delta_e$  and  $\Delta_r$  can be iteratively calculated up to any order of approximation [10, 11]; nevertheless, in this paper, we only need the 1st order formulae, i.e.,

$$\Delta_e^{(1)} = \frac{R_{AB}}{2} \int_1^0 \left[ g_{(1)}^{00} - 2N_{AB}^i g_{(1)}^{0i} + N_{AB}^i N_{AB}^j g_{(1)}^{ij} \right] d\mu \tag{10}$$

$$\Delta_r^{(1)} = \frac{R_{AB}}{2} \int_1^0 \left[ g_{(1)}^{00} - 2N_{AB}^i g_{(1)}^{0i} + N_{AB}^i N_{AB}^j g_{(1)}^{ij} \right] d\lambda, \tag{11}$$

where  $g_{(1)}^{\mu\nu}$  indicates the 1st order perturbation with respect to  $\eta^{\mu\nu}$  and  $N_{AB}^i = (x_B^i - x_A^i) / R_{AB}$ . In (10) and (11), integration is carried out along the straight line

$$t(\mu) = t_A + \mu T_{AB}, \quad x(\mu) = x_A + \mu(x_B - x_A) \quad \text{for } \Delta_e^{(1)}, \tag{12}$$

$$t(\lambda) = t_B - \lambda T_{AB}, \quad x(\lambda) = x_B - \lambda(x_B - x_A) \quad \text{for } \Delta_r^{(1)}, \tag{13}$$

where  $T_{AB}$  is the time lapse between  $A$  and  $B$  along the straight line. Then, we can put  $T_{AB} = R_{AB}/c$ .

### 3 Gravitational time delay in McVittie spacetime

#### 3.1 McVittie spacetime

The McVittie metric is expressed in standard comoving form [23, 24] as

$$ds^2 = - \left[ \frac{1 - \frac{GM}{2c^2 ra(t)}}{1 + \frac{GM}{2c^2 ra(t)}} \right]^2 c^2 dt^2 + \left[ 1 + \frac{GM}{2c^2 ra(t)} \right]^4 a^2(t) (dr^2 + r^2 d\Omega^2), \tag{14}$$

where  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ ,  $M$  is the mass of the central gravitating body, and  $a(t)$  is a scale factor. (14) reduces to the Schwarzschild solution when  $a(t) = \text{constant}$ , and it reduces to the FLRW cosmological model for the curvature parameter  $k = 0$  when  $M = 0$ .

The various observational models of the solar system are currently formulated in some kind of proper coordinate system such as the barycentric celestial reference system (BCRS) based on the post-Newtonian framework [25], instead of the cosmological comoving frame. Hence, to compare the effects formulated using the proper coordinates with the cosmological ones within the same framework, we adopt the radial transformation [23, 26–30]

$$R = a(t)r \left[ 1 + \frac{GM}{2c^2 ra(t)} \right]^2, \tag{15}$$

then (14) is rewritten as

$$\begin{aligned}
 ds &= -\left(1 - \frac{2GM}{c^2 R}\right) c^2 dt^2 + \left(\frac{dR}{\sqrt{1 - \frac{2GM}{c^2 R}}} - \frac{HR}{c} cdt\right)^2 + R^2 d\Omega^2 \\
 &= -\left(1 - \frac{2GM}{c^2 R} - \frac{H^2 R^2}{c^2}\right) c^2 dt^2 - \frac{2HR}{c} \left(1 + \frac{GM}{c^2 R} + \mathcal{O}(M^2)\right) cdt dR \\
 &\quad + \left(1 + \frac{2GM}{c^2 R} + \mathcal{O}(M^2)\right) dR^2 + R^2 d\Omega^2,
 \end{aligned} \tag{16}$$

where  $H = H(t) = \dot{a}(t)/a(t)$  is the Hubble parameter.

As mentioned in previous section, the light path used in the computation is rectilinear so that the rectangular coordinate system can be used instead of the spherical coordinate system. By coordinate transformation,

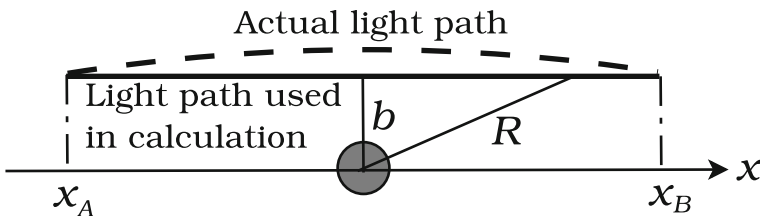
$$x = R \sin \theta \cos \phi, \quad y = R \sin \theta \sin \phi, \quad z = R \cos \theta, \tag{17}$$

and (16) becomes (see [31])

$$\begin{aligned}
 ds^2 &= -\left(1 - \frac{2GM}{c^2 R} - \frac{H^2 R^2}{c^2}\right) c^2 dt^2 - \frac{2Hx^i}{c} \left(1 + \frac{2GM}{c^2 R} + \mathcal{O}(M^2)\right) cdt dx^i \\
 &\quad + \left(\delta_{ij} + \frac{2GM}{c^2 R^3} x^i x^j + \mathcal{O}(M^2)\right) dx^i dx^j,
 \end{aligned} \tag{18}$$

where  $R = \sqrt{x^2 + y^2 + z^2}$ . To simplify the computation, the straight line used in integration is parallel to the  $x$ -axis (see Fig. 1). Hence, (18) reduces to

$$\begin{aligned}
 ds^2 &= -\left(1 - \frac{2GM}{c^2 R} - \frac{H^2 R^2}{c^2}\right) c^2 dt^2 - \frac{2Hx}{c} \left(1 + \frac{2GM}{c^2 R} + \mathcal{O}(M^2)\right) cdt dx \\
 &\quad + \left(1 + \frac{2GM}{c^2 R^3} x^2 + \mathcal{O}(M^2)\right) dx^2,
 \end{aligned} \tag{19}$$



**Fig. 1** Light ray/signal path. We assume that the path used in calculation is rectilinear and parallel to the  $x$ -axis (*bold solid line*);  $b$  is the impact parameter and  $R = \sqrt{x^2 + b^2}$ . The actual path is denoted by the *bold dashed line*

where  $y = b = \text{constant}$  ( $b$  is the impact factor),  $z = 0$ ,  $R = \sqrt{x^2 + b^2}$ , and in this case, we may put  $N_{AB}^x = (x_B - x_A)/R_{AB}$ ,  $N_{AB}^y = N_{AB}^z = 0$ . This approach is similar to the one described in [32], (see Sect. 40.4 and Fig. 40.3 of [32]). In (19), we see the Hubble parameter  $H(t)$ , which generally is an arbitrary function of time  $t$ . Hence, we suppose that  $H(t)$  changes adiabatically because the time lapse of a light ray/signal observed in the solar system is much shorter than the age of the universe,  $T_U \approx 10^{10}$  (year):

$$\begin{aligned}
 H(t) &= H_0 + \left. \frac{dH}{dt} \right|_0 t + \mathcal{O}\left(\left. \frac{d^2H}{dt^2} \right|_0\right), \\
 \left| \left. \frac{dH}{dt} \right|_0 \right| &\approx \frac{H_0}{T_U} \approx 10^{-24} [1/(\text{s year})],
 \end{aligned}
 \tag{20}$$

where  $H_0 \approx 10^{-17}$  (1/s).

### 3.2 Gravitational time delay

The time delay functions  $\Delta_e$  and  $\Delta_r$  can be resolved into the following components:

$$\begin{aligned}
 \Delta_e(x_A, t_A, x_B) &= \hat{\Delta}(x_A, x_B) + \bar{\Delta}(x_A, x_B) \\
 &\quad + \bar{\Delta}_e(x_A, t_A, x_B) + \mathcal{O}(M^2, (\dot{H}|_0)^2, \ddot{H}|_0),
 \end{aligned}
 \tag{21}$$

$$\begin{aligned}
 \Delta_r(x_A, x_B, t_B) &= \hat{\Delta}(x_A, x_B) + \bar{\Delta}(x_A, x_B) \\
 &\quad + \bar{\Delta}_r(x_A, x_B, t_B) + \mathcal{O}(M^2, (\dot{H}|_0)^2, \ddot{H}|_0),
 \end{aligned}
 \tag{22}$$

where  $\hat{\Delta}(x_A, x_B)$  corresponds to the Shapiro time delay due to central mass  $M$ ,  $\bar{\Delta}(x_A, x_B)$  is due to the static component of cosmological expansion ( $H_0$  in (20)), and  $\bar{\Delta}_e(x_A, t_A, x_B)$ ,  $\bar{\Delta}_r(x_A, x_B, t_B)$  are due to the time-dependent (secular) component of cosmological expansion ( $dH/dt|_0$  in (20)). By the straightforward calculations from (10), (11), (19), and (20), we obtain following results:

$$\begin{aligned}
 \hat{\Delta}(x_A, x_B) &= \frac{GM}{c^2} \left[ 2 \ln \left| \frac{x_B + \sqrt{x_B^2 + b^2}}{x_A + \sqrt{x_A^2 + b^2}} \right| \right. \\
 &\quad \left. - \left( \frac{x_B}{\sqrt{x_B^2 + b^2}} - \frac{x_A}{\sqrt{x_A^2 + b^2}} \right) \right] + \mathcal{O}(M^2),
 \end{aligned}
 \tag{23}$$

$$\begin{aligned}
 \bar{\Delta}(x_A, x_B) &= -\varepsilon \frac{H_0}{c} (x_B^2 - x_A^2) + \frac{H_0^2}{6c^2} [x_B^3 - x_A^3 + 3b^2(x_B - x_A)] \\
 &\quad - \varepsilon \frac{4GMH_0}{c^3} \left( \sqrt{x_B^2 + b^2} - \sqrt{x_A^2 + b^2} \right) + \mathcal{O}(M^2, (\dot{H}|_0)^2, \ddot{H}|_0)
 \end{aligned}
 \tag{24}$$

$$\begin{aligned}
\bar{\Delta}_e(x_A, t_A, x_B) = & \frac{dH}{dt} \Big|_0 \left( -\frac{\varepsilon}{c} \left[ (x_B^2 - x_A^2) t_A + \frac{1}{3} (2x_B^2 - x_A x_B - x_A^2) T_{AB} \right] \right. \\
& + \frac{H_0}{3c^2} \left\{ [x_B^3 - x_A^3 + 3b^2(x_B - x_A)] t_A \right. \\
& + \frac{1}{4} [3x_B^3 - x_A x_B^2 - x_A^2 x_B - x_A^3 + 6b^2(x_B - x_A)] T_{AB} \left. \right\} \\
& - \varepsilon \frac{2GM}{c^3 R_{AB}} \left\{ [(x_B - 2x_A) T_{AB} + 2(x_B - x_A) t_A] \sqrt{x_B^2 + b^2} \right. \\
& + [x_A T_{AB} - 2(x_B - x_A) t_A] \sqrt{x_A^2 + b^2} \\
& \left. \left. - b^2 T_{AB} \ln \left| \frac{x_B + \sqrt{x_B^2 + b^2}}{x_A + \sqrt{x_A^2 + b^2}} \right| \right\} \right) + \mathcal{O}(M^2, (\dot{H}|_0)^2, \ddot{H}|_0) \quad (25)
\end{aligned}$$

$$\begin{aligned}
\bar{\Delta}_r(x_A, x_B, t_B) = & \frac{dH}{dt} \Big|_0 \left( -\frac{\varepsilon}{c} \left[ (x_B^2 - x_A^2) t_B - \frac{1}{3} (x_B^2 + x_A x_B - 2x_A^2) T_{AB} \right] \right. \\
& + \frac{H_0}{3c^2} \left\{ [x_B^3 - x_A^3 + 3b^2(x_B - x_A)] t_B \right. \\
& - \frac{1}{4} [x_B^3 + x_A x_B^2 + x_A^2 x_B - 3x_A^3 + 6b^2(x_B - x_A)] T_{AB} \left. \right\} \\
& + \varepsilon \frac{2GM}{c^3 R_{AB}} \left\{ [x_B T_{AB} - 2(x_B - x_A) t_B] \sqrt{x_B^2 + b^2} \right. \\
& - [(2x_B - x_A) T_{AB} - 2(x_B - x_A) t_B] \sqrt{x_A^2 + b^2} \\
& \left. \left. - b^2 T_{AB} \ln \left| \frac{x_B + \sqrt{x_B^2 + b^2}}{x_A + \sqrt{x_A^2 + b^2}} \right| \right\} \right) \\
& + \mathcal{O}(M^2, (\dot{H}|_0)^2, \ddot{H}|_0), \quad (26)
\end{aligned}$$

where  $\varepsilon = N_{AB}^x = 1$  for  $x_B - x_A > 0$  and  $\varepsilon = -1$  for  $x_B - x_A < 0$ .  $\varepsilon$  is derived from the term  $-2N_{AB}^i g_{(1)}^{0i}$  in (10) and (11).

### 3.3 Equation of light time

Actually,  $x_A$  and  $x_B$  are positions at  $t = t_A$  and  $t = t_B$ , respectively:

$$x_A = x_A(t_A), \quad x_B = x_B(t_B). \quad (27)$$



Then, (21) and (22) can be symbolically expressed as

$$\Delta_e(x_A(t_A), t_A, x_B(t_B)) = \hat{\Delta}(x_A(t_A), x_B(t_B)) + \bar{\Delta}(x_A(t_A), x_B(t_B)) + \bar{\Delta}_e(x_A(t_A), t_A, x_B(t_B)), \tag{28}$$

$$\Delta_r(x_A(t_A), x_B(t_B), t_B) = \hat{\Delta}(x_A(t_A), x_B(t_B)) + \bar{\Delta}(x_A(t_A), x_B(t_B)) + \bar{\Delta}_r(x_A(t_A), x_B(t_B), t_B), \tag{29}$$

Hence, the equations

$$t_B - t_A = \frac{1}{c} [R_{AB}(x_A(t_A), x_B(t_B)) + \Delta_e(x_A(t_A), t_A, x_B(t_B))], \tag{30}$$

$$t_B - t_A = \frac{1}{c} [R_{AB}(x_A(t_A), x_B(t_B)) + \Delta_r(x_A(t_A), x_B(t_B), t_B)], \tag{31}$$

can be considered as the equation of light time. Here that  $x_A$  and  $x_B$  must be known beforehand as a function of time  $t$  via the numerical integration of the equation of motion or via lunar-planetary ephemerides.

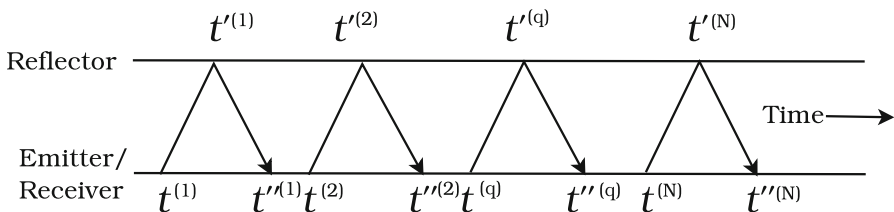
### 3.4 Time-dependency of solution

The solutions from (21) to (26) have the following form:

$$\mathcal{T}_e^{(q)}(x_A^{(q)}, t_A^{(q)}, x_B^{(q)}) = \frac{1}{c} [R_{AB}^{(q)}(x_A^{(q)}, x_B^{(q)}) + E_1^{(q)}(x_A^{(q)}, x_B^{(q)}) + E_2^{(q)}(x_A^{(q)}, x_B^{(q)})t_A^{(q)}], \tag{32}$$

$$\mathcal{T}_r^{(q)}(x_A^{(q)}, x_B^{(q)}, t_B^{(q)}) = \frac{1}{c} [R_{AB}^{(q)}(x_A^{(q)}, x_B^{(q)}) + R_1^{(q)}(x_A^{(q)}, x_B^{(q)}) + R_2^{(q)}(x_A^{(q)}, x_B^{(q)})t_B^{(q)}], \tag{33}$$

where the superscript ( $q$ ) implies the  $q$ -th observation,  $E_1^{(q)}(x_A^{(q)}, x_B^{(q)})$ ,  $R_1^{(q)}(x_A^{(q)}, x_B^{(q)})$  constitute the static component of gravitational time delay, and  $E_2^{(q)}(x_A^{(q)}, x_B^{(q)})t_A$ ,  $R_2^{(q)}(x_A^{(q)}, x_B^{(q)})t_B$  constitute the component depending on either  $t_A$  or  $t_B$ . If we carry out a series of observations,  $q = 1, 2, 3, \dots, N$  (see Fig 2),  $t_A^{(q)}$ ,  $t_B^{(q)}$  on the right-hand



**Fig. 2** Time-dependency of solutions. The emission time, reflection time, and reception time of the ( $q$ )-th observation are denoted by  $t^{(q)}$ ,  $t'^{(q)}$ , and  $t''^{(q)}$ , respectively

side of (32) and (33) can be regarded as the time-dependent (secular) component of the solution.

#### 4 Application to secular increase in astronomical unit

The Secular increase in the astronomical unit (AU), reported by Krasinsky and Brumberg (2004) [17], is an unexplained physical phenomenon observed in the solar system. This anomaly was discovered while analyzing planetary radar and spacecraft (mainly Martian landers/orbiters) ranging data and improving the various astronomical constants including AU. [17] estimated

$$\frac{dAU}{dt} = 15 \pm 4 \text{ (m/cy)}. \quad (34)$$

as the most appropriate value. Subsequently  $dAU/dt \simeq 20 \text{ (m/cy)}$  was separately evaluated by Pitjeva at the Institute of Applied Astronomy (IAA), Russia, and by Standish at the Jet Propulsion Laboratory (JPL), USA.<sup>2</sup>

Previously, some attempts have been made to explain the secular trend in terms of cosmological expansion [17, 21, 34]. In particular, [17, 22] considered its contribution to light propagation. However, as mentioned in Sect. 1, it is generally difficult to compute the time-dependent geodesic of null rays; hence, the approach of the former is somewhat qualitative, whereas that of the latter is essentially restricted to discussion in static spacetime.

Now, on the basis of the results obtained in previous section, let us re-examine whether the cosmological effect relates with the observed  $dAU/dt$ . It is appropriate to regard the coefficients of  $t_A$  and  $t_B$  in (32) and (33) as secular terms owing to cosmological expansion. Then, if we assume that  $dH/dt|_0 \approx 10^{-24} \text{ [1/(s year)]}$  from (20), the leading order of magnitude of coefficients is,

$$E_2(x_A, x_B) \approx R_2(x_A, x_B) \approx 10^{-10} \text{ (m/year)} = 10^{-8} \text{ (m/cy)} \quad (35)$$

because  $x_A$  and  $x_B$  are of the order of a few [AU] or  $10^{11} \text{ (m)}$  in the solar system. Unfortunately, this is approximately 9 orders of magnitude smaller than the evaluated  $dAU/dt$ , i.e.,  $15 \text{ (m/cy)}$ . Therefore, the time-dependent effect due to cosmological expansion does not induce the secular increase in AU.

#### 5 Summary

We considered light propagation in the time-dependent McVittie spacetime, which can be considered as the spacetime around a gravitating body such as the Sun, embedded in the FLRW cosmological background metric. To discuss the time-dependent null ray,

<sup>2</sup> [17] used only the radiometric observational data, whereas Pitjeva and Standish carried out analysis using various kinds of data such as optical and VLBI; each of them worked with different software. However, as it stands, their results have not been published officially because of the difficulty in interpreting the obtained value, i.e.,  $20 \text{ (m/cy)}$  [33].

we adopted the recently developed time transfer function method [10, 11] which is originally related to Synge's world function  $\Omega(x_A, x_B)$  and precludes the integration of the null geodesic equation. The first step involved the application of this method to McVittie spacetime and re-examination of the cosmological effect on the round-trip time of a light ray/signal in the solar system. The time delay functions  $\Delta_e$  and  $\Delta_r$ , which characterize the gravitational time delay, were given by the functions of not only the spacial coordinates  $x_A$  and  $x_B$  but also the emission time  $t_A$  or reception time  $t_B$  (see (21–26)); the presence of the terms  $t_A$  and  $t_B$  in  $\Delta_e$  and  $\Delta_r$  express the time-dependency of the solution.

On the basis of the obtained results, we also investigated the secular increase in the astronomical unit (AU), reported by [17]; we showed that the leading order terms of the time-dependent component due to cosmological expansion are 9 orders of magnitude smaller than the observed value of  $d\text{AU}/dt$ , i.e.,  $15 \pm 4$  (m/cy). Therefore, we explicitly asserted that it is not possible to explain the secular increase of AU in terms of cosmological expansion.

Currently, it is assumed that the most plausible cause of  $d\text{AU}/dt$  is either the lack of calibrations of internal delays in the radio signals within spacecraft or the complexity in modeling the solar corona; however, no conclusive evidence has not been reported thus far. The origin of the secular increase in AU has also been examined in terms of other physical aspects such as solar mass loss [17, 35], the time variation of the gravitational constant  $G$  [17], the influence of dark matter in the solar system [34], the multi-dimensional brane world scenario [36], the transfer of rotational angular momentum of the Sun due to solar mass loss [37], the azimuthally symmetric theory of gravitation [38], and the kinematics in Randers–Finsler geometry [39]. However, so far the cause of the secular increase in AU is not clear. Nonetheless, this phenomenon and other anomalies discovered in the solar system may be related to some fundamental properties of gravitation; therefore it is important to verify these phenomena theoretically and observationally.

On the other hand, the cosmological effect on gravitationally bound local systems has been widely investigated, see i.e. [30] and references therein. Its influence in the solar system is probably so small that one cannot detect it presently. Nevertheless, it seems that the theoretical discussion is not resolved. Therefore, from the theoretical point of view, it is interesting to construct robust theoretical model beyond the McVittie model and bring an end to the argument consistently.

Owing to the rapid enhancement of observational techniques, more accurate and rigorous formulae are required to express light propagation. [10] and [11] succeeded in developing an elegant and useful method for constructing the post-Newtonian/post-Minkowskian approximation, and they showed that Synge's world function  $\Omega(x_A, x_B)$  (which is not widely adopted or discussed) is a powerful tool for formulations in practical problems. Thus, the value of Synge's world function can be reaffirmed, and it is possible to proceed with the theoretical development of observational models based on it.

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