Gen Relativ Gravit (2011) 43:1037–1045
DOI 10.1007/s10714-010-0961-0

RESEARCH ARTICLE

Energy of the Kerr–Newman-AdS black hole by using approximate Lie symmetries

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Received: 23 December 2009 / Accepted: 3 March 2010 / Published online: 28 March 2010

Abstract Using Lie approximate symmetry methods for differential equations second-order approximate symmetries of the geodesic equations for the Kerr–Newman-AdS (KN-AdS) spacetime are investigated. For this purpose the KN-AdS metric is considered as a second perturbation of the AdS metric. A rescaling of the arc length parameter for consistency of the trivial second-order approximate symmetries of the geodesic equations indicates that the energy in the KN-AdS spacetime has to be rescaled. There is an extra contribution to the energy of the KN-AdS spacetime due to the cosmological constant. This energy expression is compared with that for the Kerr–Newman (KN) spacetime.

Keywords KN-AdS black hole · Second-order approximate symmetries · Energy

1 Introduction

If the spacetime is static there is a timelike *isometry* or *Killing Vector* (KV). Further, energy conservation in a spacetime is guaranteed in the frame using a timelike KV to define the time direction. However, energy is not guaranteed in a more general spacetime [1]. Thus there is no good definition for energy in General Relativity (GR) [2]. This leads to problems with the definition of mass [2].

To resolve this problem different scientists including Einstein [3], Landau and Lifshitz [2], Papapetrou [4] and Weinberg [5] introduced different pseudo-tensors

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(a brief review of this is given in [6]). These pseudo-tensor have been criticized because all of these are coordinate dependent and hence *non-tensorial* or *gauge dependent*. This violates the basic principles of GR. Because of the coordinate dependence, many others, including Möller [7,8], Komar [9], Ashtekar and Hansen [10], Penrose [11], Chirstodoulou [12] and Isaacson [13] have proposed coordinate independent definitions. However, each of these has its own drawbacks [14–16]. For different definitions of gravitational energy see [16] and [17].

Another approach tried different concepts of *approximate symmetry* to define energy in GR. In this regard an attempt was to assume that conservation of energy holds *asymptotically* and examine whether it would work for gravitational waves [18,19]. An altogether different approach was adopted to provide a measure of the extent of break-down of symmetry by the integral of the square of the symmetrized derivative of a vector field divided by its mean square norm [13,20]. This led to an *almost symmetric space* and the corresponding vector field was called an *almost KV* [21]. This measure of "non-symmetry" was applied to the Taub cosmological solution [22] and to study gravitational radiation. It provides a choice of gauge that makes calculations simpler and was used for this purpose [23]. Effectively based on the almost symmetry, the concept of an "approximate symmetry group" was presented [24,25]. A method for finding approximate KVs on closed 2-surfaces was established and used to study the distortion of the horizon geometry of black holes [26]. Using an eigenvalue approach, this latter idea was related to the earlier work of Matzner [20] to calculate the approximate Killing fields , so as to define a meaningful spin for non-symmetric black holes in GR [27]. However none of the above discussed attempts provide a clear answer to the problem.

The notion of an "approximate or slightly broken symmetry" seems promising but merely providing simplicity of calculations is not physically persuasive. Other ideas need to be tried to find one that is drastically better than the others. We use approximate symmetry methods for differential equations (DEs) [28] to look at the solution of the problem. It is evident that we need to learn how to physically interpret the results that will emerge from the approximate symmetry calculations. Since there is a connection between energy and geodesic equations i.e. the metric tensor represents the gravitational field and the equations for the free fall are the geodesic equations which give a constant of motion that corresponds to the energy [29]. Therefore, using the approximate symmetry methods for DEs first and second order approximate symmetries of the perturbed geodesic equation have been investigated in some spacetimes to look at their energy content [30–33]. In this regard first the first-order approximate symmetries of the perturbed geodesic equations for the Schwarzschild spacetime (taken as a first perturbation of the Minkowski spacetime) have been considered [30]. Then second-order approximate symmetries of the perturbed geodesic equations for the Reissner-Nordström (RN) spacetime [32] and KN spacetime [33] (taken as a second perturbation of the Minkowski spacetime), and some of the gravitational wave spacetimes (taken as second perturbation of some static spacetimes) [31] were studied. A rescaling of the arc length parameter for consistency of the trivial second-order approximate symmetries of the perturbed geodesic equations for these spacetimes indicates that the energy in these spacetimes has to be rescaled. The approximation entail a small parameter whose powers, higher than some chosen value, is neglected. All these scaling factors (and the one obtained here for the KN-AdS black hole, discussed later), do not depend on the strength of the perturbation parameter, which means that we can take the limit as it approaches to zero. This is reminiscent of the d'Alembert principle of virtual work to obtain results for statics from dynamical considerations [34]. Thus using this approximation methods i.e. approximate symmetry methods for DEs, we get an exact expression of energy.

The cosmological constant has been a subject of active research since the early stages of the development of GR. The interest in the cosmological constant has been revived by connecting it with the vacuum energy of a quantum field [35]. It has also been recently studied by the researchers in the context of dark energy for which the cosmological constant is the simplest candidate (see, e.g. [36]). The energy of the KN spacetime was discussed in [33]. Here we want to explore the effect of the cosmological constant on that energy expression. Consequently we investigate the second-order approximate symmetries of the geodesic equations for KN-AdS spacetime to look at its energy expression. We then compare the energy expressions in both the spacetimes. For the approximate symmetries of KN-AdS spacetime we introduce the mass, spin and charge of the gravitating source as a small parameter, ϵ. We retain terms of order ϵ^2 in the second-order perturb geodesic equations and neglect its higher powers.

The plan of the paper is as follows. In the next section mathematical formalism to be used is give. In Sect. 3, second-order approximate symmetries of the geodesic equations for the KN-AdS spacetime are considered. Finally a summary and discussion are given in Sect. 4.

2 Mathematical preliminaries

First we give the definition of the second-order approximate symmetries of a system of ODEs under point symmetry transformations defined on a real parameter fibre bundle over the manifold [28]

$$\mathbf{X} = \xi(s, x^\mu)\frac{\partial}{\partial s} + \eta^\nu(s, x^\mu)\frac{\partial}{\partial x^\nu}, \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$. A vector field

$$\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2 + O(\epsilon^3), \tag{2}$$

is called a second-order approximate symmetry of the system of perturbed ODEs

$$\mathbf{E} = \mathbf{E}_0 + \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2 + O(\epsilon^3), \tag{3}$$

if the following condition ([32] and references given therein) holds

$$\left(\mathbf{X}_0 + \epsilon\mathbf{X}_1 + \epsilon^2\mathbf{X}_2\right)\left(\mathbf{E}_0 + \epsilon\mathbf{E}_1 + \epsilon^2\mathbf{E}_2\right)\Big|_{\mathbf{E}_0+\epsilon\mathbf{E}_1+\epsilon^2\mathbf{E}_2=O(\epsilon^3)} = O(\epsilon^3), \tag{4}$$

where $\mathbf{X}_0$ is the exact symmetry generator of the system of ODEs $\mathbf{E_0}$, i.e.

$$(\mathbf{X}_0)\,(\mathbf{E}_0)|_{\mathbf{E}_0=0} = 0. \tag{5}$$

Here $\mathbf{X}_1$, $\mathbf{X}_2$ are called the first-order and second-order approximate parts of the approximate symmetry generator respectively, $\mathbf{E}_1$ is the first-order perturbed part and $\mathbf{E}_2$ the second order perturbed part of the system of ODEs, respectively. The second-order approximate symmetry is called non-trivial if at least one of the lower order symmetries are non-zero for it, that is if (for second-order at least) any one of $\mathbf{X}_0$, $\mathbf{X}_1$, is non-zero and $\mathbf{X}_2$ is not proportional to it. In the case of trivial symmetries it is also possible that lower order symmetries cancel out in the set of determining equations. It should be noted that the scaling factor comes from the use of the perturbed system of ODEs in the subscript of (4), as required.

Noether symmetries are those infinitesimal symmetry generators that leave a Lagrangian $L(s, x^\mu, \dot{x}^\mu)$ invariant. They always form a subalgebra of the symmetries of the corresponding Euler–Lagrange (geodesic) equations [37]. It is defined as a vector field given by (1), such that

$$\mathbf{X}^{[1]}L + (D_s\xi)L = D_s A, \tag{6}$$

where $A(s, x^\mu)$, is a gauge function [28]. The total derivative operator D_s and the first prolongation (defined on the real parameter fibre bundle over the tangent bundle to the manifold) $\mathbf{X}^{[1]}$ of the vector field $\mathbf{X}$ given by (1) are

$$D_s = \frac{\partial}{\partial s} + \dot{x}^\mu \frac{\partial}{\partial x^\mu}, \tag{7}$$

and

$$\mathbf{X}^{[1]} = \mathbf{X} + \left(\eta^\nu_{,s} + \eta^\nu_{,\mu}\dot{x}^\mu - \xi_{,s}\dot{x}^\nu - \xi_{,\mu}\dot{x}^\mu\dot{x}^\nu\right)\frac{\partial}{\partial \dot{x}^\nu}. \tag{8}$$

For more general considerations see [28]. The significance of Noether symmetries is clear from the following theorem [38], proved in [39].

Theorem 1 *If* $\mathbf{X}$ *is a Noether point symmetry corresponding to a Lagrangian* $L(s, x^\mu, \dot{x}^\mu)$ *of a second-order ODE* $\ddot{x}^\mu = g(s, x, \dot{x}^\mu)$*, then*

$$I = \xi L + (\eta^\mu - \dot{x}^\mu\xi)\frac{\partial L}{\partial \dot{x}^\mu} - A, \tag{9}$$

is a first integral of the ODE associated with $\mathbf{X}$.

We briefly define geodesic equations in general. The set of these equations is given by

$$\ddot{x}^\mu + \Gamma^\mu_{\lambda\nu}\dot{x}^\lambda\dot{x}^\nu = 0, \tag{10}$$

where dot denotes derivative with respect to the geodetic parameter s and

$$\Gamma^\mu_{\lambda\nu} = \frac{1}{2}g^{\mu\sigma}(g_{\lambda\sigma,\nu} + g_{\nu\sigma,\lambda} - g_{\lambda\nu,\sigma}). \tag{11}$$

3 Approximate symmetries of the geodesic equations for the KN-AdS black hole

The line element for the KN-AdS spacetime in Boyer–Lindqust coordinates is given by ($G = c = 1$) [40]

$$ds^2 = \frac{\Delta_r}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi}d\phi\right)^2 - \frac{\rho^2}{\Delta_r}dr^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 - \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left(adt - \frac{r^2+a^2}{\Xi}d\phi\right)^2, \tag{12}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Xi = 1 - \frac{a^2}{l^2}, \quad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta,$$
$$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2mr + Q^2, \tag{13}$$

m is the mass, a is the angular momentum per unit mass, Q is the total charge of the gravitating source. The AdS radius l is related to a (negative) cosmological constant Λ by $\Lambda = -3/l^2$. This metric reduces to the KN metric when $l \to \infty$. This spacetime has two KVs $\partial/\partial t$ and $\partial/\partial\phi$, which give the energy and azimuthal angular momentum conservation. The Noether symmetry generator for this spacetime are the two KVs and the generator $\partial/\partial s$.

For the second-order approximate symmetries of the geodesic equations for the KN-AdS black hole we introduce the small parameter, ϵ as

$$\epsilon = 2m, \quad a^2 = k_1\epsilon^2, \quad Q^2 = k_2\epsilon^2, \tag{14}$$

where $0 < k_1 + k_2 \leq 1/4$. Retaining second power of ϵ and neglecting higher powers the metric coefficients in (12) become

$$\begin{aligned}
g_{00} &= 1 + \frac{r^2}{l^2} - \frac{\epsilon}{r} + \epsilon^2\left(\frac{k_1}{l^2}\sin^2\theta - \frac{k_2}{r^2}\right) + O(\epsilon^3),\\
g_{11} &= \frac{-l^2}{l^2+r^2}\left[+1 + \frac{\epsilon l^2}{r(l^2+r^2)} - \frac{\epsilon^2 l^2}{r^2}\right.\\
&\quad \left.\times\left\{\frac{k_1}{l^2}\sin^2\theta + \frac{k_2}{l^2+r^2} - \frac{l^2}{(l^2+r^2)^2}\right\}\right] + O(\epsilon^3),\\
g_{22} &= -r^2\left[1 + \epsilon^2 k_1\left(\frac{1}{r^2} + \frac{1}{l^2}\right)\cos^2\theta\right] + O(\epsilon^2),\\
g_{33} &= -r^2\sin^2\theta - k_1\epsilon^2\left[1 + \left(1+\frac{r^2}{l^2}\right)\sin^2\theta + \left(1+\frac{r^2}{l^2}\right)\sin^4\theta\right] + O(\epsilon^3),\\
g_{03} &= g_{30} = \sqrt{k_1}\sin^2\theta\left(-\epsilon\frac{r^2}{l^2} - \epsilon^2\frac{1}{r}\right) + O(\epsilon^3).
\end{aligned} \tag{15}$$

Using the above metric coefficients in the geodesic equations given by (10), one can obtain the second-order perturbed geodesic equations for the KN-AdS spacetime. For $\epsilon \to 0$, the above perturbed spacetime reduces to the Ads spacetime. The AdS spacetime admits 10 dimensional isometry algebra $so(2,3)$ (see for example [41]).

To investigate the second-order approximate symmetries of the geodesic equations for this perturbed spacetime we apply the definition (4) of second-order approximate Lie symmetries of ODEs to them. For the exact case (when $\epsilon = 0$) the geodesic equations (for the AdS spacetime) admits the 10 KVs along with the dilatation algebra

$$d_2 = \left\langle \frac{\partial}{\partial s}, s\frac{\partial}{\partial s} \right\rangle, \tag{16}$$

generated by the reparametrization allowed for the geodetic parameter. In the first-order approximate case (i.e. when we consider terms containing only first power of ϵ) we recover these 12 symmetry generators as trivial approximate symmetry generators and there is no non-trivial approximate symmetry. In the second-order approximate case (i.e. when we consider terms containing first and second powers of ϵ) there is also no non-trivial approximate symmetry for the geodesic equations and only the 12 trivial approximate symmetry generators are recovered.

The definition of first-order approximate symmetries of DEs was used to calculate first-order approximate symmetries of the geodesic equations for the Schwarzschild metric [30]. The interesting result of energy rescaling in the RN [32], KN [33] and gravitational wave spacetimes [31] was forthcoming by application of the definition of the second-order approximate symmetries of DEs, wherein the perturbed system of geodesic equations was used.

The exact (i.e. when $\epsilon = 0$) Lie symmetry algebra of the geodesic equations involve d_2 corresponding to

$$\xi(s) = c_0 s + c_1. \tag{17}$$

In the set of determining equations for the second-order approximate symmetries of the geodesic equations for the KN-AdS spacetime terms involving $\xi_s = c_0$ *do* not cancel automatically but collects a factor which consists of three terms involving $\dot{t}$, $\dot{\phi}$ and $\dot{\theta}$. We consider the extreme effects on the energy; in the equatorial plane, i.e. in the regions where the rotational effect is maximum ($\theta = \pi/2$); and in the regions where the rotational effect is minimum, i.e. near the axis ($\theta = 0, \pi$) [42,43]. Beside, if the value of the invariant of motion (related to the total relative angular momentum of a test particle) arising from the integration of the geodesic equations is one, then the geodesic equations of the particle are consistently satisfied by a constant value of $\theta = \theta_0$. For this case the geodesic spirals inward on the surface of a hypercone, with its axis lying along the rotation axis of the black hole [42]. Thus we get $\dot{\theta} = 0$ and the scaling factor becomes

$$\frac{l^2}{(l^2+r^2)}\left[\left\{\frac{l^2}{(l^2+r^2)}\left(1-2k_2\left(\frac{(l^2+r^2)}{l^2}\right)\right)\right.\right.$$
$$\left. + 2k_1\frac{r^2}{l^2}\left(\left(\frac{(r^2-l^2)}{l^2}\right)+(1+\sin^2\theta)\cos^2\theta\right)\right.$$

$$
+\frac{2r^2}{l^2+r^2}\left(k_2-\frac{r^2}{l^2}\sin^2\theta+k_1\left(\frac{l^2+r^2}{l^2}\right)\left(2\sin^2\theta+\frac{l^2+r^2}{l^2}\right)\right)-1\Bigg\}
\times\frac{\dot{t}}{r^3}-\frac{2\sqrt{k_1}}{r^2}\sin^2\theta\dot{\phi}\Bigg]. \tag{18}
$$

Since energy conservation comes from time translational invariance and ξ is the coefficient of $\partial/\partial s$ in the point symmetry transformations given by (1), where s is the proper time, the scaling factor (18) corresponds to a rescaling of energy. This scaling factor involves the derivatives of the coordinates t and ϕ and derivatives only apply to the paths of the particles. To get the energy in the spacetime field these derivatives can be replaced by the first integrals of the geodesic equations for the KN spacetime and involve a constant that is the mass. As such, we put it in as m and take $a \ll m$. Thus we get

$$
M_{KNA}=m\left(1-\frac{r^2}{l^2}+\frac{r^4}{l^4}-\cdots\right)\left[1-\left(1+\frac{r^2}{l^2}\right)\left(\frac{Q^2}{2m^2}-\frac{a}{2r}\right)\right.
\left.+\frac{2r^2}{l^2}\left\{\frac{Q^2}{2m^2}-\left(1-\frac{r^2}{l^2}\right)\right\}\right]. \tag{19}
$$

For $l\to\infty$ this scaling factor reduces to that of the KN spacetime [33]

$$
M_{KN}=m-\frac{Q^2}{2m}-\frac{ma}{2r}. \tag{20}
$$

Therefore the terms that are l-dependent in (19), i.e. with $l\to\infty$ vanish, giving extra contribution for the energy of the KN-AdS spacetime due to the cosmological constat. For $a\longrightarrow 0$, (19) gives the energy of the RN-AdS spacetime.

The energy of the KN-AdS spacetime has been discussed in [44], where the energy was obtained

$$
M_{KNA}=\frac{m}{\Xi^2}=m\left(1+2\frac{a^2}{l^2}+3\frac{a^4}{l^4}+\cdots\right). \tag{21}
$$

A comparison of this expression with that of our expression (19) is given in the next section.

4 Summary and discussion

We studied the second-order approximate symmetries of the geodesic equations for the KN-AdS spacetime. For this purpose the mass, angular momentum per unit mass and charge of the gravitating source were introduced as a small parameter ϵ. From the definition of the second-order approximate symmetries of ODEs, for this spacetime we obtained the scaling factor (19). This scaling factor depends on the mass, charge and spin of the gravitating source. Like the KN spacetime [33], also in this case the charge comes in quadratically compared to unity and the spin comes in linearly. It does not come with a constant term to compare. However, taken as a whole, it shows that the spin has an effectively lower order effect.

The perturbative approach has been introduced in GR by different authors. Stewart and Walker gave a definition of spacetime perturbation in GR which provides criteria for gauge invariance [45]. They gave a theorem which says that second and higher order perturbations are gauge dependent. Gleiser et al. studied the second-order perturbation of the Schwarzschild black hole by imposing some simplifying special conditions on the metric perturbation [46]. They showed that the Einstein Field Equations (EFEs) can be reduced to a single linear wave equation with a potential and a source term that is quadratic in terms of first-order perturbations. This formalism was applied to study first-order approximations in astrophysics. Bruni et al. considered the problem of gauge dependence that exists in second and higher order relativistic perturbation theory [47,48]. They defined gauge invariance to an arbitrary order n. As an example they studied second-order perturbation in cosmology, assuming a flat Friedmann-Robertson-Walker background. Higher than first-order gravitational perturbations in the Newman–Penrose formalism were investigated by Campanelli and Lousto by taking in account the tetrad invariant quantities [49]. They showed that the equations representing outgoing gravitational radiations, can be uncoupled into a single wave equation to any perturbative order. For second-order perturbation about the Kerr black hole, they proved the existence of a first and second-order gauge and tetrad invariant waveform.

Here approximate symmetry methods for DEs which appears to be a perturbative method, have been used. However, this gives an exact result as explained in the introduction. Therefore, though we used a second-order perturbation of the KN-AdS black hole on the AdS background, it does not violate the Stewart–Walker's theorem [45], as we get the energy scaling factor (19) independent of the perturbation parameter.

Due to the cosmological term an extra contribution in the expression (19) of energy for the KN-AdS spacetime has appeared. Since cosmological constant is a candidate for the dark energy therefore the relation of the l-dependent terms in this expression with the dark energy is worth exploring.

Though the expression (21) gives the energy of the KN-AdS spacetime but this do not involves the effects of charge. Since for $l \to \infty$, The KN-AdS black hole reduces to the KN black hole therefore in this limit the energy expression (21) should give the energy of the KN black hole. Unfortunately (21) gives the Schwarzschild mass in the desired limit. Our expression of energy (19), is more reasonable as it involves the effect of charge and reduces to the energy expression of the KN spacetime [33] in the required limit. Our expression of energy (19) is also reasonable as it gives the energy in the RN-AdS spacetime when $a \to 0$, which further reduces to that of the RN spacetime [32], when the cosmological constant vanishes.

Acknowledgments Thanks to National University of Sciences and Technology (NUST), Pakistan for supporting this work. Useful discussions with Prof. Asghar Qadir and Dr. Khalid Saifullah are gratefully acknowledged.

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