Gravitational collapse: expanding and collapsing regions

M. Sharif · G. Abbas

Abstract　We investigate the expanding and collapsing regions by taking two well-known spherically symmetric spacetimes. For this purpose, the general formalism is developed by using Israel junction conditions for arbitrary spacetimes. This has been used to obtain the surface energy density and the tangential pressure. The minimal pressure provides the gateway to explore the expanding and collapsing regions. We take Minkowski and Kantowski-Sachs spacetimes and use the general formulation to investigate the expanding and collapsing regions of the shell.

Keywords　Junction conditions · Expanding and collapsing regions

1 Introduction

The final outcome of the gravitational collapse in general relativity (GR) is an issue of great importance from the perspective of black hole physics and its astrophysical implications. A continual gravitational collapse without any final equilibrium state, would end as a spacetime singularity. The singularity theorems indicate the existence of singularities in the form of either future or past incomplete timelike geodesics.

The main motivation for researchers working in the field of gravitational collapse is to determine the form of spacetime singularity (clothed or naked), resulting from

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the collapse of massive objects. For this purpose, Penrose [1] suggested a hypothesis known as *Cosmic Censorship Hypothesis* (CCH) which states that in generic situation all singularities arising from regular initial data are clothed by event horizon and hence invisible to distant observers. Over the past three decades, this conjecture remains un-proved and un-solved problem in the theory and applications of black hole physics.

After the failure of many attempts to establish CCH, Penrose [2] concluded that continual collapse of a massive object would end as black hole or naked singularity, depending on the initial conditions and equation of state. Virbhadra et al. [3] introduced a new theoretical tool using the gravitational lensing phenomena to discuss the spacetime singularity. Also, Virbhadra and Ellis [4] classified the naked singularity by the gravitational lensing into two kinds: weak naked singularity (those contained within at least one photon sphere [5]) and strong naked singularity (those not contained within any photon sphere). Virbhadra [6] explored the useful results to investigate the Seifert's conjecture for the naked singularity. He also found that naked singularity forming in the Vaidya null dust collapse supports the Seifert's conjecture [7]. The same author [8] used the gravitational lensing phenomena to find an improved form of the CCH.

Oppenheimer and Snyder [9] are the pioneers who studied the dust collapse by taking the static Schwarzschild spacetime as an exterior and Friedmann like solution as an interior spacetime. They found black hole as end state of the gravitational collapse. This work was generalized by Markovic and Shapiro [10] for positive cosmological constant.

To study the gravitational collapse, it is of vital importance to consider an appropriate geometry of the interior and exterior regions and junction conditions which allows the smooth matching of these regions [11,12]. Qadir with his collaborators [13–15] investigated the use of junction conditions for collapse of black hole in a closed Friedmann universe which was later extended with positive and negative cosmological constants. Lake [16] extended the work of Oppenheimer and Snyder for positive and negative cosmological constants. Debnath et al. [17] explored the quasi-spherical collapse with cosmological constant by using the Israel junction conditions modified by Santos. Ghosh and Deshkar [18] discussed the higher dimensional spherically symmetric dust collapse. Sharif and Ahmad [19–21] extended the spherically symmetric gravitational collapse with positive cosmological constant for perfect fluid. Nath et al. [22] studied the gravitational collapse of non-viscous, heat conducting fluid in the presence of electromagnetic field. In a recent paper [23], we have investigated the effect of electromagnetic field on the perfect fluid collapse by using the junction conditions.

Villas da Rocha et al. [24] discussed the self-similar gravitational collapse of perfect fluid using Israel's method. Pereira and Wang [25,26] studied the gravitational collapse of cylindrical shells made of counter rotating dust particles by using the same analysis. Sharif and Ahmad [27] extended this work to plane symmetric spacetime. Recently, Sharif and Iqbal [28] have investigated the spherically symmetric gravitational collapse for a class of metrics. In this paper, we generalize this work. We develop a general formulation and apply it to two well-known spherically symmetric spacetimes (Minkoski and Kantowski-Sachs) to study the gravitational collapse.

The main objective of this work is to investigate the expanding and collapsing regions in the framework of these well-known spacetimes. The plan of the paper is as follows: In the next section, the general formalism is presented. In Sect. 3, we discuss the application of this formalism to the known spacetimes. We conclude our discussion in the last section. All the Latin and Greek indices vary from 0 to 3, otherwise it will be mentioned.

2 General formulation for surface energy-momentum tensor

A spacelike $3D$ hypersurface Σ is taken which separates the interior and exterior regions of a star represented by $4D$ manifolds M^- and M^+ respectively. For interior region M^-, we take spherically symmetric spacetime given by

$$ds_-^2 = W^- dt^2 - X^- dr^2 - Y^- (d\theta^2 + \sin\theta^2 d\phi^2), \tag{1}$$

where W^-, X^- and Y^- are functions of t and r. For the exterior manifold M^+, we take the line element of the form

$$ds_+^2 = W^+ dT^2 - X^+ dR^2 - Y^+ (d\theta^2 + \sin\theta^2 d\phi^2), \tag{2}$$

where W^+, X^+ and Y^+ are functions of T and R.

It is assumed that interior and exterior spacetimes are smoothly matched on the hypersurface Σ by the continuity of line elements over Σ following Israel junction conditions [11,12]. Thus we have

$$(ds_-^2)_\Sigma = (ds_+^2)_\Sigma = ds_\Sigma^2. \tag{3}$$

The equations of hypersurface in terms of interior and exterior coordinates are

$$f_-(r, t) = r - r_0(t) = 0, \tag{4}$$
$$f_+(R, T) = R - R_0(T) = 0. \tag{5}$$

When we make use of these equations, we obtain the following interior and exterior metrics

$$\begin{aligned}
(ds_-^2)_\Sigma &= [W^-(t, r_0(t)) - X^-(t, r_0(t))r_0'^2(t)]dt^2 \\
&\quad - Y^-(t, r_0(t))(d\theta^2 + \sin\theta^2 d\phi^2),
\end{aligned} \tag{6}$$
$$\begin{aligned}
(ds_+^2)_\Sigma &= [W^+(T, R_0(T)) - X^+(T, R_0(T))R_0'^2(T)]dT^2 \\
&\quad - Y^+(T, R_0(T))(d\theta^2 + \sin\theta^2 d\phi^2),
\end{aligned} \tag{7}$$

where prime means derivative with respect to the indicated variable.

We define the intrinsic metric on the hypersurface as follows

$$(ds^2)_\Sigma = d\tau^2 - Y(\tau)(d\theta^2 + \sin\theta^2 d\phi^2), \tag{8}$$

where τ is the proper time. Using Eqs. (6)–(8) in (3), we get

$$d\tau = [W^-(t, r_0(t)) - X^-(t, r_0(t))r_0'^2(t)]^{\frac{1}{2}} dt$$
$$= [W^+(T, R_0(T)) - X^+(T, R_0(T))R_0'^2(T)]^{\frac{1}{2}} dT, \tag{9}$$
$$Y(\tau) = Y^-(t, r_0(t)) = Y^+(T, R_0(T)). \tag{10}$$

Also, from Eqs. (4) and (5), the outward unit normals (timelike) in the coordinates of M^- and M^+, are given by

$$n_\mu^- = \left[\frac{W^- X^-}{W^- - X^- r_0'^2(t)} \right]^{\frac{1}{2}} (-r_0'(t), 1, 0, 0), \tag{11}$$

$$n_\mu^+ = \left[\frac{W^+ X^+}{W^+ - X^+ R_0'^2(T)} \right]^{\frac{1}{2}} (-R_0'(T), 1, 0, 0). \tag{12}$$

Now the extrinsic curvature K_{ij} is defined as

$$K_{ij}^\pm = n_\sigma^\pm \left(\frac{\partial^2 x_\pm^\sigma}{\partial \xi^i \partial \xi^j} + \Gamma_{\mu\nu}^\sigma \frac{\partial x_\pm^\mu \partial x_\pm^\nu}{\partial \xi^i \partial \xi^j} \right), \quad (i, j = 0, 2, 3). \tag{13}$$

Here ξ^i correspond to the coordinates on Σ, $x_\pm^\sigma$ stand for coordinates in $M^\pm$, the Christoffel symbols $\Gamma_{\mu\nu}^\sigma$ are calculated from the interior or exterior spacetimes and $n_\sigma^\pm$ are components of outward unit normals to Σ in the coordinates $x_\pm^\sigma$. The non-vanishing components of $K^+{}_{ij}$ (exterior region) are given by

$$K_{\tau\tau}^+ = \frac{(W^+ X^+)^{\frac{1}{2}}}{[W^+ - X^+ R_0'^2(T)]^{\frac{3}{2}}} \left[R_0''(T) + \frac{W^+{}_{,R}}{W^+} + \frac{X^+{}_{,T}}{X^+} R_0'(T) \right.$$
$$\left. + \frac{X^+{}_{,R}}{2X^+} R_0'^2(T) - \frac{W^+{}_{,T}}{2W^+} R_0'(T) - \frac{X^+{}_{,T}}{2W^+} R_0'^3(T) - \frac{W^+{}_{,T}}{W^+} R_0'^2(T) \right], \tag{14}$$

$$K_{\theta\theta}^+ = \frac{1}{2} \left[\frac{(W^+ X^+)}{W^+ - X^+ R_0'^2(T)} \right]^{\frac{1}{2}} \left[-\frac{Y^+{}_{,R}}{X^+} - \frac{Y^+{}_{,T}}{W^+} R_0'(T) \right], \tag{15}$$

$$K_{\phi\phi}^+ = \sin^2 \theta K_{\theta\theta}^+. \tag{16}$$

The non-vanishing components of extrinsic curvature, $K^-{}_{ij}$, in terms of interior coordinates are found from the above expression by replacing

$$W^+, \ X^+, \ Y^+, \ R_0(T), \ T, \ R \ \to \ W^- \ X^-, \ Y^-, \ r_0(t), \ t, \ r. \tag{17}$$

The surface energy-momentum tensor in terms of $K_{ij}^{\pm}$ and γ_{ij} can be defined as [11,12]

$$S_{ij} = \frac{1}{\kappa}\{[K_{ij}] - \gamma_{ij}[K]\}, \tag{18}$$

where κ is coupling constant and γ_{ij} represents the metric coefficients of the hyper-surface Σ. Also, we have

$$[K_{ij}] = K^+{}_{ij} - K^-{}_{ij}, \quad [K] = \gamma^{ij}[K_{ij}]. \tag{19}$$

Using Eqs. (14)–(16) and the corresponding expression for $K^-{}_{ij}$, we can express S_{ij} in the form

$$S_{ij} = \rho\omega_i\omega_j + p(\theta_i\theta_j + \phi_i\phi_j), \quad (i, j = \tau, \theta, \phi), \tag{20}$$

where ρ is the surface energy density, p is the tangential pressure provided that they satisfy some energy conditions [29] and $\omega_i, \theta_i, \phi_i$ are unit vectors defined on the surface given by

$$\omega_i = \delta^\tau{}_i, \quad \theta_i = Y^{\frac{1}{2}}\delta^\theta{}_i, \quad \phi_i = Y^{\frac{1}{2}}\sin^2\theta\delta^\phi{}_i. \tag{21}$$

The energy density ρ and pressure p are given by

$$\rho = \frac{2}{\kappa Y}[K_{\theta\theta}], \quad p = \frac{1}{\kappa}\left\{[K_{\tau\tau}] - \frac{[\{K_{\theta\theta}\}]}{Y}\right\}. \tag{22}$$

3 Application

Now we use this formulation by taking particular interior and exterior spacetimes. We take Minkowski spacetime as an interior and Kantowski-Sach spacetime as an exterior region. The Minkowski spacetime is

$$ds_-^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{23}$$

The Kantowski-Sach spacetime is given by

$$ds_+^2 = dT^2 - A^2(T)dR^2 - B^2(T)(d\theta^2 + \sin\theta^2 d\phi^2). \tag{24}$$

The junction conditions (9) and (10) can be found from the above spacetimes as

$$d\tau = [1 - r_0'^2(t)]^{\frac{1}{2}}dt = [1 - A^2(T)R_0'^2(T)]^{\frac{1}{2}}dT, \tag{25}$$

$$r_0(t) = B(T). \tag{26}$$

These equations yield

$$\left(\frac{dT}{dt}\right)^2 = \frac{1}{[1 - A^2(T)R_0'^2(T) + B'^2(T)]} \equiv \frac{1}{\Delta^2}. \tag{27}$$

From Eqs. (26) and (27), one gets

$$r_0''(t) = \frac{B''(T)}{\Delta^2} + \frac{1}{\Delta^4}\left[B'(T)(A^2(T)R_0'(T)R_0''(T) \right.$$
$$\left. + A(T)R_0'^2(T)A'(T) - B'(T)B''(T))\right]. \tag{28}$$

Using Eqs. (26)–(28) in the extrinsic curvature components, we get the expression for ρ and p in the following form

$$\rho = \frac{2}{\kappa} \frac{[\Delta - A(T)B'(T)R_0'(T)]}{(1 - A^2(T)R_0'^2(T))^{\frac{1}{2}}}, \tag{29}$$

$$p = \frac{1}{\kappa\Delta[1 - (A(T)R(T))^2]^{\frac{3}{2}}}[\Delta(A(T)R_0''(T) + 2R_0'(T)A'(T)$$
$$- A^2(T)R_0'^2(T)A'(T)) - B''(T)(1 - A^2(T)R_0'^2(T))$$
$$- B'(T)A(T)R_0'(T)(A(T)R_0''(T) + A'(T)R_0'(T))$$
$$- \Delta(1 - A^2(T)R_0'^2(T))(\Delta - A(T)B'(T)R_0'(T))]. \tag{30}$$

Now for the minimum effects of shell on the collapse, substitute $p = 0$ in the above equation which gives

$$R_0''(T) = \frac{R_0'(T)}{\Delta A(T) - B'(T)R'(T)A^2(T)}[(B''(T) - B'(T) - 1)$$
$$\times (1 - A^2(T)R_0'^2(T)) + \Delta(A(T)B'(T)$$
$$+ A^2(T)B^2(T)R_0'^2(T) + 2A(T)) - B'(T)A^2(T)]. \tag{31}$$

For the analysis of gravitational collapse, it is necessary to solve this equation for $R_0'(T)$. The exact solution of this equation is too complicated to provide any insight. However, it would be interesting to consider a particular case for which this equations reduces to the following form

$$R_0''(T) = R_0'(T)\alpha, \tag{32}$$

where we have assumed α to be an arbitrary constant given by

$$\alpha = \frac{1}{\Delta A(T) - B'(T)R'(T)A^2(T)}[(B''(T) - B'(T) - 1)(1 - A^2(T)R_0'^2(T))$$
$$+ \Delta(A(T)B'(T) + A^2(T)B^2(T)R_0'^2(T) + 2A(T)) - B'(T)A^2(T)]. \tag{33}$$

Integrating Eq. (32), it follows that

$$R_0'(T) = e^{\alpha T + c_1}, \tag{34}$$

where c_1 is constant of integration which may be positive or negative. Thus we can take

$$(1) \quad \alpha > 0 \quad (2) \quad \alpha < 0$$

3.1 Case 1

Here, we take $\alpha = 1$ and $c_1 = -1$, then Eq. (34) takes the form

$$R_0'(T) = e^{T-1}. \tag{35}$$

Integration of this equation yields

$$R_0(T) = e^{T-1} + c_2, \tag{36}$$

where c_2 is another constant of integration. When $c_2 = -1$, we obtain from Eqs. (35) and (36) as follows

$$R_0(T) = \begin{cases} 0, & T = 1, \\ +\infty, & T = +\infty, \end{cases}$$
$$R_0'(T) = \begin{cases} 1, & T = 1, \\ +\infty, & T = +\infty \end{cases} \tag{37}$$

We see from Figs. 1 and 2 that expansion starts at $T = 1$ (radial velocity is 1 and radial displacement is 0) and ends at $T = \infty$ (both radial velocity and displacement becomes infinite). In this case, acceleration is positive which goes on increasing with the passage of time and future directed spacelike geodesics exist in this region. It is mentioned here that we have determined the physical time interval by taking the integration constant as minus times α. All the positive values of α and negative values of integration constants will lead to the same range of radial velocity, displacement and time. This case gives the expanding process.
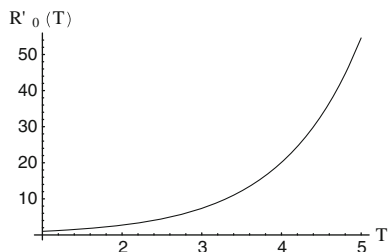
Fig. 1 Velocity–time graph
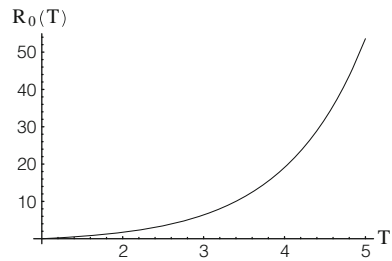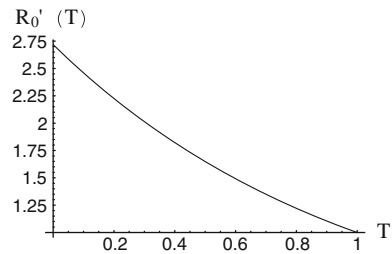
Fig. 2 Displacement–time graph

Fig. 3 Velocity–time graph

3.2 Case 2

In this case, we take $\alpha = -1$ and $c_1 = 1$, then Eq. (34) becomes

$$R_0'(T) = e^{-T+1}. \tag{38}$$

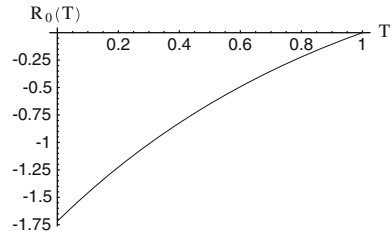Integrating this equation, it follows that

$$R_0(T) = -e^{-T+1} + c_3, \tag{39}$$

where c_3 is another constant of integration. For $c_3 = 1$, we obtain from Eqs. (38) and (39)

$$R_0(T) = \begin{cases} -1.7182, & T = 0, \\ 0, & T = 1, \end{cases}$$
$$R_0'(T) = \begin{cases} +2.7182, & T = 0, \\ 1, & T = 1. \end{cases} \tag{40}$$

From Figs. 3 and 4, it is obvious that shell starts collapsing at $T = 0$ with radial velocity 2.7182 and magnitude of displacement 1.7182 and ends with radial velocity 1 and displacement 0 at $T = 1$. Thus the time interval for the collapsing region is $0 \leq T \leq 1$. The acceleration decreases positively in this time interval and past directed spacelike geodesics exist in this region. The relation between α and integration constants enable us to determine the physical time interval. All the negative values of α and positive values of integration constants lead to the same range of radial velocity, displacement and time. This case represents the collapsing process.

Fig. 4 Displacement–time graph

4 Summary and conclusion

In this paper, we investigate the expanding and collapsing regions in the spherically symmetric background. For this purpose, we have formulated a general formalism of surface energy-momentum tensor using Israel junction conditions with arbitrary spacetimes. Further, we have taken two particular spacetimes representing the interior and exterior regions of a star. The surface density and pressure are calculated by applying the general formulation with particular spacetimes. To examine the minimal effects of shell on the collapse, we assume that the tangential pressure $p = 0$, which provides the gateway for studying the expanding and collapsing regions.

After applying the assumption of minimal effect of shell on the collapse, we obtain a simplest form given in Eq. (32). The arbitrary constant α in Eq. (32) is either positive or negative. If we take the constants of integration $c_1, c_2 < 0$ in Eqs. (34) and (36) respectively then for all $\alpha > 0$, we obtain the same time interval, radial velocity and displacement for the expanding regions. It is found that $\forall\ \alpha > 0$ and $c_1 = c_2 = -\alpha$, the expansion starts at $T = 1$ and ends at $T = +\infty$. In the expanding region, radial velocity ranges from 1 to $+\infty$ and displacement varies from 0 to $+\infty$ while future directed spacelike geodesics exist in this region.

On the other hand, all $\alpha < 0$ and the constants of integration $c_1, c_2 > 0$ in Eqs. (34) and (39) give the same time interval, radial velocity and displacement in the collapsing regions. It is found that $\forall\ \alpha < 0$ and $c_1 = c_3 = -\alpha$, the collapsing starts at $T = 0$ and ends at $T = 1$. In the collapsing region, radial velocity decreases from 2.7182 to 1 and magnitude of displacement decreases 1.7182 to 0. Further, past directed spacelike geodesics exist in this region. We find that in the region, where collapse occurs, density remains finite which means that collapse does not end as singularity and hence known as non-singular collapse. One can extend this analysis by excluding the assumption of minimal effects of shell on the collapse (i.e., $p \neq 0$).

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