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RESEARCH ARTICLE

Complete spacelike hypersurfaces in conformally stationary Lorentz manifolds

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Abstract We derive, for the square operator of Yau, an analogue of the Omori–Yau maximum principle for the Laplacian. We then apply it to obtain nonexistence results concerning complete noncompact spacelike hypersurfaces immersed with constant higher order mean curvature in a conformally stationary Lorentz manifold.

Keywords Lorentz manifolds · Timelike conformal vector field · Spacelike hypersurfaces · Higher order mean curvatures · Yau's square operator

1 Introduction

The interest in the study of spacelike hypersurfaces in Lorentz manifolds has increased very much in recent years, from both the physical and mathematical points of view. For example, Marsden and Tipler in [18], and Stumbles in [24], point out that spacelike hypersurfaces with constant mean curvature in an arbitrary Lorentz manifold play an important role in the general relativity, in that they serve as convenient initial data for the Cauchy problem for Einstein's equations.

From a mathematical point of view, a basic question related to this topic is the existence and uniqueness of spacelike hypersurfaces in Lorentz manifolds, with some reasonable geometric properties, like the vanishing of the mean curvature, for instance.

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A first relevant result in this direction was the proof of the Calabi-Bernstein conjecture for maximal hypersurfaces (that is, hypersurfaces with vanishing mean curvature) in Lorentz-Minkowski space, given by Cheng and Yau in [10]. As for the case of de Sitter space, Goddard in [13] conjectured that every complete spacelike hypersurface with constant mean curvature should be totally umbilical. Although the conjecture turned out to be false in its original form, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses (see, for example, [1,19]).

More recently, Alías, Brasil Jr. and Colares, in [2], developed general Minkowski-type formulae for compact spacelike hypersurfaces immersed into conformally stationary spacetimes, that is, spacetimes endowed with a timelike conformal vector field; then, they applied these formulae to the study of the umbilicity of compact spacelike hypersurfaces under certain conditions on their r-mean curvatures. Furthermore, the first author in [8] computed $L_r(S_r)$ for a spacelike hypersurface Σ^n immersed in a Lorentz ambient $\overline{M}^{n+1}$ of constant sectional curvature, applying the resulting formula to study both r-maximal spacelike hypersurfaces of $\overline{M}$, and, in the presence of a constant higher order mean curvature, constraints on the sectional curvature of Σ that also suffice to guarantee the umbilicity of it. Here, by L_r we mean the linearization of the second order differential operator associated to the rth elementary symmetric function on the eigenvalues of the second fundamental form of such immersion (cf. Sect. 2). Let us also remark that Alías and Colares in [3] studied the problem of uniqueness for spacelike hypersurfaces with constant higher order mean curvature in Generalized Robertson-Walker Lorentz manifolds (cf. Sect. 4). Their approach is based on the use of the Newton transformations P_r, together with their associated differential operators L_r and the above mentioned Minkowski formulae for spacelike hypersurfaces.

Back to the case of complete noncompact spacelike hypersurfaces, in [9] the first author applied the standard formula for the Laplacian of the squared norm of the second fundamental form and the Omori-Yau maximum principle to classify complete spacelike hypersurfaces with constant mean curvature in a Lorentz manifold of nonnegative constant sectional curvature, under appropriate bounds on the scalar curvature. For the de Sitter space, Brasil Jr., Colares and Palmas also used the Omori-Yau maximum principle in [6] to characterize the hyperbolic cylinders as the only complete hypersurfaces in the de Sitter space with constant mean curvature, nonnegative Ricci curvature and having at least two ends (see also [7] for the case of the scalar curvature).

The discussion of related questions involving higher order mean curvatures faces a first difficulty: there is no corresponding version of maximum principle for the appropriate second order partial differential operators. In this paper we overcome this obstacle by developing, for the square operator, an analogue of the Omori-Yau maximum principle for the Laplacian. The square operator is the metric contraction of the compound of a symmetric tensor with the Hessian of a smooth function on a manifold; it was introduced by Cheng and Yau in [11] (see also Sect. 3), in order to study constant scalar curvature hypersurfaces in constant sectional curvature Riemannian manifolds. As already made explicit in Cheng and Yau's paper, it is a natural object to work with, for an appropriate choice of the involved symmetric tensor allows one to capture the influence of several different kinds of curvatures on the manifold. The maximum principle for the square operator is the content of Corollary 3.4. With the aid of a suitable consequence of it (see Proposition 4.1), we obtain nonexistence results on complete noncompact spacelike hypersurfaces in the class of conformally stationary Lorentz manifolds (cf. Sect. 4). These are the content of our main results, i.e., Theorems 4.4, 4.6 and 4.7.

This paper is organized in the following way: in Sect. 2 we set notation and recall a few results which will be needed later; Sect. 3 is devoted to the statement and proof of the maximum principle and its corollaries; applications are collected in Sect. 4.

2 Preliminaries

In what follows, if $\overline{M}^{n+1}$ is a connected semi-Riemannian manifold with metric $\overline{g} = \langle\, , \rangle$, we let $\mathcal{D}(\overline{M})$ denote the ring of smooth functions $f : \overline{M} \to \mathbb{R}$ and $\mathfrak{X}(\overline{M})$ the algebra of smooth vector fields on $\overline{M}$. We also write $\overline{\nabla}$ for the Levi-Civita connection of $\overline{M}$.

In what follows, unless stated to the contrary Σ is a connected, $n-$dimensional, orientable differentiable manifold. If $\overline{M}$ is a Lorentz manifold, we recall (cf. [22]) that a vector field $X \in \mathfrak{X}(\overline{M})$ is said to be *timelike* if $\langle X, X\rangle < 0$ on $\overline{M}$; *spacelike* if $\langle X, X\rangle > 0$ on $\overline{M}$; a *unit* vector field if $\langle X, X\rangle = \pm 1$ on $\overline{M}$. We consider *spacelike immersions* $\psi : \Sigma^n \to \overline{M}^{n+1}$, namely, we assume that ψ induces a Riemannian metric on Σ, and furnish Σ with this metric. In this setting, we say that ψ (or Σ) is a *spacelike hypersurface* of $\overline{M}$, which is said to be *complete* if Σ is complete with the induced metric. Finally, we let ∇ denote the Levi-Civita connection of Σ with respect to the induced metric.

Let us orient Σ by the choice of a unit, timelike normal vector field N on it, and A denote the corresponding shape operator. At each $p \in \Sigma$, A restricts to a self-adjoint linear map $A_p : T_p\Sigma \to T_p\Sigma$. For $1 \le r \le n$, let $S_r(p)$ denote the rth elementary symmetric function on the eigenvalues of A_p; this way one gets n smooth functions $S_r : \Sigma^n \to \mathbb{R}$, such that

$$\det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in \Sigma$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p, with corresponding eigenvalues $\{\lambda_k\}$, one immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$ is the rth elementary symmetric polynomial on the indeterminates $X_1, \dots, X_n$.

Now, let A^i_j be the components of the shape operator A with respect to some orthonormal frame; we use $|A|^2$ as a shorthand for the contraction $\sum_{i,j=1}^n A^i_j A^j_i$, i.e., the trace $\mathrm{tr}(A^2)$ of the operator A^2. Working with a basis with respect to which A is diagonal, it is immediate to check that

$$2S_2 + |A|^2 = S_1^2. \tag{2.1}$$

For $1 \leq r \leq n$, one defines the rth mean curvature H_r of ψ by

$$H_r = \frac{(-1)^r}{\binom{n}{r}} S_r = \frac{1}{\binom{n}{r}} \sigma_r(-\lambda_1, \ldots, -\lambda_n).$$

In particular, $H_1 = H$ is the mean curvature of x. It is a classical fact that such functions satisfy a very useful set of inequalities, usually refered to as Newton's inequalities (see [14]). It turns out, however, that such inequalities remain true for arbitrary real numbers. For future reference, we collect them here. A proof can be found in [8], Proposition 1.

Proposition 2.1 *Let $n > 1$ be an integer, and $\lambda_1, \ldots, \lambda_n$ be real numbers. Define, for $0 \leq r \leq n$, $S_r = S_r(\lambda_i)$ as above, and $H_r = H_r(\lambda_i) = \binom{n}{r}^{-1} S_r(\lambda_i)$.*

(a) *For $1 \leq r < n$, one has $H_r^2 \geq H_{r-1}H_{r+1}$. Moreover, if equality happens for $r = 1$ or for some $1 < r < n$, with $H_{r+1} \neq 0$ in this case, then $\lambda_1 = \cdots = \lambda_n$.*
(b) *If $H_1, H_2, \ldots, H_r > 0$ for some $1 < r \leq n$, then $H_1 \geq \sqrt{H_2} \geq \sqrt[3]{H_3} \geq \cdots \geq \sqrt[r]{H_r}$. Moreover, if equality happens for some $1 \leq j < r$, then $\lambda_1 = \cdots = \lambda_n$.*
(c) *If, for some $1 \leq r < n$, one has $H_r = H_{r+1} = 0$, then $H_j = 0$ for all $r \leq j \leq n$. In particular, at most $r - 1$ of the λ_i are different from zero.*

When the ambient space $\overline{M}$ has constant sectional curvature c, Gauss equation allows one to immediately check that the scalar curvature R of Σ relates to H_2 in the following manner:

$$R = n(n-1)(c - H_2). \tag{2.2}$$

For $0 \leq r \leq n$ one defines the rth Newton transformation P_r on Σ by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = (-1)^r S_r I + A P_{r-1}. \tag{2.3}$$

A trivial induction shows that

$$P_r = (-1)^r (S_r I - S_{r-1}A + S_{r-2}A^2 - \cdots + (-1)^r A^r),$$

so that Cayley-Hamilton Theorem gives $P_n = 0$. Moreover, since P_r is a polynomial in A for every r, it is also self-adjoint and commutes with A. Therefore, all bases of $T_p\Sigma$ diagonalizing A at $p \in \Sigma$ also diagonalize all of the P_r at p. Let $\{e_k\}$ be such a basis. Denoting by A_i the restriction of A to $\langle e_i \rangle^{\perp} \subset T_p\Sigma$, it is easy to see that

$$\det(tI - A_i) = \sum_{k=0}^{n-1} (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1\le j_1<\ldots<j_k\le n\\ j_1,\ldots,j_k\neq i}} \lambda_{j_1}\cdots\lambda_{j_k}.$$

With the above notations, it is also immediate to check that $P_r e_i = (-1)^r S_r(A_i)e_i$, and hence (Lemma 2.1 of [5])

(a) $S_r(A_i) = S_r - \lambda_i S_{r-1}(A_i)$;
(b) $\mathrm{tr}(P_r) = (-1)^r \sum_{i=1}^n S_r(A_i) = (-1)^r(n-r)S_r = b_r H_r$;
(c) $\mathrm{tr}(AP_r) = (-1)^r \sum_{i=1}^n \lambda_i S_r(A_i) = (-1)^r(r+1)S_{r+1} = -b_r H_{r+1}$;
(d) $\mathrm{tr}(A^2P_r) = (-1)^r \sum_{i=1}^n \lambda_i^2 S_r(A_i) = (-1)^r(S_1S_{r+1} - (r+2)S_{r+2})$,

where $b_r = (n-r)\binom{n}{r}$.

The next two results will be extremely useful in Sect. 4.

Proposition 2.2 (Proposition 1.5 of [16]) *With respect to a spacelike immersion* $\psi : \Sigma^n \to \overline{M}^{n+1}$,

(a) *if* $H_r = 0$ *on* Σ*, then* P_{r-1} *is semi-definite on* Σ.
(b) *if* $H_r = 0$ *and* $H_{r+1} \neq 0$ *on* Σ*, then* P_{r-1} *is definite on* Σ.

If $p \in \Sigma$ is such that all eigenvalues of A_p are either positive or negative, we say that p is an *elliptic point* of Σ.

Proposition 2.3 (Proposition 3.2 of [5]) *With respect to a spacelike immersion* $\psi : \Sigma^n \to \overline{M}^{n+1}$*, if* $H_r > 0$ *on* Σ *and* ψ *has an elliptic point, then* P_{r-1} *is positive definite on* Σ.

Associated to each Newton transformation P_r one has the second order linear differential operator $L_r : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$, given by

$$L_r(f) = \sum_{i,j=1}^n (P_r)_j^i\,(\mathrm{Hess}\,f)_i^j = \mathrm{tr}(P_r\,\mathrm{Hess}\,f),$$

where, as before, we write $(P_r)_j^i$ and $(\mathrm{Hess}\,f)_j^i$ for the components of the corresponding operators with respect to some orthonormal frame, and $\mathrm{tr}(\cdot)$ for the *trace* of the operator in parenthesis. Therefore, for $f, g \in \mathcal{D}(\Sigma)$, it follows from the properties of the Hessian of functions that

$$L_r(fg) = fL_r(g) + gL_r(f) + 2\langle P_r\nabla f, \nabla g\rangle. \tag{2.4}$$

3 The generalized maximum principle

Let Σ^n be a complete $n-$dimensional Riemannian manifold. Let also $\Phi : T\Sigma \to T\Sigma$ denote a field of self adjoint linear transformations on Σ. We consider the second order linear differential operator $\Box : \mathcal{D}(\Sigma) \to \mathcal{D}(\Sigma)$ by setting

$$\Box f = \sum_{i,j=1}^{n} \phi^i_j \,(\text{Hess } f)^j_i \;=\; \text{tr}\,(\phi\,\text{Hess } f). \tag{3.1}$$

For fixed $p \in \Sigma$, let $\rho(x) = \rho_p(x) = d(x, p)$ be the distance function from p and $C_m(p)$ denote the cut locus of p. Set also

$$K(x) = s_c'(\rho(x))s_c(\rho(x))\text{tr}(\Phi_x), \tag{3.2}$$

where $s_c : [0, +\infty] \to \mathbb{R}$ is defined by

$$s_c(t) = \begin{cases} \frac{\sinh(\sqrt{-c}t)}{\sqrt{-c}t}, & \text{if } c < 0 \\ t, & \text{if } c = 0 \\ \frac{\sin(\sqrt{c}t)}{\sqrt{c}t}, & \text{if } c > 0 \end{cases}.$$

Lemma 3.1 *If Φ is positive semi-definite on Σ and Σ has sectional curvature $K_\Sigma \geq c$ then, for all $x \in \Sigma \backslash C_m(p)$, one has $\Box\rho(x) \leq K(x)$.*

Proof Let $\gamma : [0, l] \to \Sigma$ be the only minimizing normalized geodesic joining p to x, with length $l = \rho(x)$. Decompose any unit vector $u \in T_x\Sigma$ as $u = v + w$, where u is collinear with $\gamma'(l)$ and $w \perp \gamma'(l)$. Then $|v|^2 + |w|^2 = 1$ and, at x,

$$\begin{aligned} \text{Hess}\,\rho(u, u) &= \text{Hess}\,\rho(v, v) + 2\text{Hess}\,\rho(v, w) + \text{Hess}\,\rho(w, w) \\ &= \langle \nabla_v \gamma', v\rangle + 2\langle \nabla_v \gamma', w\rangle + \text{Hess}\,\rho(w, w) \\ &= \text{Hess}\,\rho(w, w). \end{aligned}$$

It follows from the Hessian comparison theorem (cf. [23], chapter I) and from the characterization of Jacobi fields in spaces of constant sectional curvature (cf. [12], chapter 5) that if $K_\Sigma \geq c$ then, at x,

$$\text{Hess}\,\rho(w, w) \leq s_c'(\rho)s_c(\rho)|w|^2 \leq s_c'(\rho)s_c(\rho).$$

Now take a moving frame $\{e_1, \ldots, e_n\}$ on a neighborhood of x, diagonalizing Φ at x, with $\Phi(e_i) = \lambda_i e_i$. Then, one has at x

$$\begin{aligned} \Box\rho &= \text{tr}(\Phi\,\text{Hess}\,\rho) = \sum_i \lambda_i \,\text{Hess}\,\rho(e_i, e_i) \\ &\leq \sum_i \lambda_i s_c'(\rho)s_c(\rho) = s_c'(\rho)s_c(\rho)\text{tr}(\Phi). \end{aligned}$$

□

Theorem 3.2 *Let Σ be a complete Riemannian manifold with sectional curvature $K_\Sigma \geq c$, and $f \in \mathcal{D}(\Sigma)$ be a function bounded from above. If Φ is positive semi-definite at every $x \in \Sigma$ then, for every $p \in \Sigma$, there exists a sequence $(p_k)_{k\geq 1}$ in Σ such that*

$$\lim_{k\to+\infty} f(p_k) = \sup_\Sigma f, \tag{3.3}$$

$$|\nabla f(p_k)| = \frac{2(f(p_k) - f(p) + 1)\rho(p_k)}{k(\rho(p_k)^2 + 2)\log(\rho(p_k)^2 + 2)} \tag{3.4}$$

and

$$\Box f(p_k) \leq \frac{4\mathrm{tr}(\Phi_{p_k})\rho(p_k)^2(f(p_k) - f(p) + 1)}{k^2(\rho(p_k)^2 + 2)^2\log(\rho(p_k)^2 + 2)^2} + \frac{2(f(p_k) - f(p) + 1)}{k(\rho(p_k)^2 + 2)\log(\rho(p_k)^2 + 2)}\left\{\mathrm{tr}(\Phi_{p_k}) + \rho(p_k)K(p_k)\right\}, \tag{3.5}$$

where K is given as in (3.2).

Proof The proof parallels that of the classical Omori-Yau maximum principle in [26]. For positive integer k, let

$$g(x) = \frac{f(x) - f(p) + 1}{[\log(\rho(x)^2 + 2)]^{1/k}}.$$

One has that g is continuous, $g(p) = \frac{1}{(\log 2)^{1/k}} > 0$ and, since f is bounded above,

$$\limsup_{\rho(x)\to+\infty} g(x) \leq 0.$$

Therefore, g attains its maximum at some $p_k \in \Sigma$. In particular, $f(p_k) - f(p) + 1 > 0$. One now has to consider two cases separately: $p_k \notin C_m(p)$ and $p_k \in C_m(p)$. Here, we treat only the first case; for the second one and the conclusion of the proof of the theorem, copy the corresponding steps in [26].

Suppose $p_k \notin C_m(p)$. Since (omitting x for clarity)

$$v(g) = \frac{v(f)}{[\log(\rho^2 + 2)]^{1/k}} - \frac{2(f - f(p) + 1)\rho v(\rho)}{k(\rho^2 + 2)[\log(\rho^2 + 2)]^{1/k+1}}, \tag{3.6}$$

one gets at p_k

$$0 = \nabla g = \frac{\nabla f}{[\log(\rho^2 + 2)]^{1/k}} - \frac{2(f - f(p) + 1)\rho\nabla\rho}{k(\rho^2 + 2)[\log(\rho^2 + 2)]^{1/k+1}},$$

from where (3.4) follows.

For the estimate on $\Box f$, it follows from (3.6) that

$$\begin{aligned} v(v(g)) = & \frac{v(v(f))}{[\log(\rho^2+2)]^{1/k}} - \frac{2\rho v(f) v(\rho)}{k(\rho^2+2)[\log(\rho^2+2)]^{1/k+1}} \\ & - \frac{2\left\{\rho v(f) v(\rho) + (f - f(p)+1)[v(\rho)^2 + \rho v(v(\rho))]\right\}}{k(\rho^2+2)[\log(\rho^2+2)]^{1/k+1}} \\ & + \frac{4(f-f(p)+1)\rho^2 v(\rho)^2}{k(\rho^2+2)^2[\log(\rho^2+2)]^{1/k+2}} \left(\frac{1}{k} + 1 + \log(\rho^2+2)\right). \end{aligned}$$

Now take a moving frame $\{e_1, \ldots, e_n\}$ on a neighborhood of p_k, geodesic at p_k and diagonalizing Φ at p_k, with $\Phi(e_i) = \lambda_i e_i$. Then, one has at p_k

$$\Box f = \sum_i \lambda_i e_i(e_i(f)).$$

On the other hand, since Hess $f_{p_k} \leq 0$ and $\Phi_{p_k} \geq 0$, one has $\Box g = \mathrm{tr}(\Phi \,\mathrm{Hess}\, g) \leq 0$ at p_k, and it follows at once from the above computations that

$$\begin{aligned} 0 \geq \Box g = & \frac{\Box f}{[\log(\rho^2+2)]^{1/k}} - \frac{4\rho\langle \Phi\nabla f, \nabla\rho\rangle}{k(\rho^2+2)[\log(\rho^2+2)]^{1/k+1}} \\ & - \frac{2(f-f(p)+1)(\langle \Phi\nabla\rho, \nabla\rho\rangle + \rho\Box\rho)}{k(\rho^2+2)[\log(\rho^2+2)]^{1/k+1}} \\ & + \frac{4(f-f(p)+1)\rho^2\langle \Phi\nabla\rho, \nabla\rho\rangle}{k(\rho^2+2)^2[\log(\rho^2+2)]^{1/k+2}} \left(\frac{1}{k} + 1 + \log(\rho^2+2)\right). \end{aligned}$$

One also has at p_k that

$$\langle \Phi\nabla f, \nabla\rho\rangle = \frac{2(f-f(p)+1)\rho\langle \Phi\nabla\rho, \nabla\rho\rangle}{k(\rho^2+2)\log(\rho^2+2)},$$

from where, substituting into the above and taking into account Lemma 3.1, we get at p_k

$$\begin{aligned} \Box f \leq & \frac{8(f-f(p)+1)\rho^2\langle \Phi\nabla\rho, \nabla\rho\rangle}{k^2(\rho^2+2)^2[\log(\rho^2+2)]^2} + \frac{2(f-f(p)+1)(\langle \Phi\nabla\rho, \nabla\rho\rangle + \rho K)}{k(\rho^2+2)\log(\rho^2+2)} \\ & - \frac{4(k+1)(f-f(p)+1)\rho^2\langle \Phi\nabla\rho, \nabla\rho\rangle}{k^2(\rho^2+2)^2[\log(\rho^2+2)]^2} - \frac{4(f-f(p)+1)\rho^2\langle \Phi\nabla\rho, \nabla\rho\rangle}{k(\rho^2+2)^2\log(\rho^2+2)} \\ = & \frac{2(f-f(p)+1)(\langle \Phi\nabla\rho, \nabla\rho\rangle + \rho K)}{k(\rho^2+2)\log(\rho^2+2)} \\ & + \frac{4(f-f(p)+1)\rho^2\langle \Phi\nabla\rho, \nabla\rho\rangle}{k^2(\rho^2+2)^2[\log(\rho^2+2)]^2}[2-(k+1)-k\log(\rho^2+2)] \\ \leq & \frac{2(f-f(p)+1)(\langle \Phi\nabla\rho, \nabla\rho\rangle + \rho K)}{k(\rho^2+2)\log(\rho^2+2)} + \frac{4(f-f(p)+1)\rho^2\langle \Phi\nabla\rho, \nabla\rho\rangle}{k^2(\rho^2+2)^2[\log(\rho^2+2)]^2}. \end{aligned}$$

Now, since $|\nabla\rho| = 1$ and Φ is positive semi-definite, one has $\langle \Phi\nabla\rho, \nabla\rho\rangle \leq \mathrm{tr}(\Phi)$, so that the desired estimate follows. □

Corollary 3.3 *Let Σ be a complete Riemannian manifold with sectional curvature $K_\Sigma \geq 0$, and $f \in \mathcal{D}(\Sigma)$ be a function bounded from above. If Φ is positive semi-definite and* $\mathrm{tr}(\Phi)$ *is bounded from above on Σ, then there exists a sequence $(p_k)_{k\geq 1}$ in Σ such that*

$$f(p_k) > \sup_M f - \frac{1}{k}, \quad |\nabla f(p_k)| < \frac{1}{k}, \quad \Box f(p_k) < \frac{1}{k}. \tag{3.7}$$

Proof Letting $C_1 = \sup_\Sigma f$, it follows from (3.4) that

$$\begin{aligned}|\nabla f(p_k)| &\leq \frac{2(C_1 - f(p)+1)}{k}\cdot\frac{\rho(p_k)}{\rho(p_k)^2+2}\cdot\frac{1}{\log(\rho(p_k)^2+2)}\\ &\leq \frac{2(C_1 - f(p)+1)}{k}\cdot\frac{1}{2\sqrt{2}}\cdot\frac{1}{\log 2},\end{aligned}$$

so that

$$\lim_{k\to+\infty} |\nabla f(p_k)| = 0. \tag{3.8}$$

If f attains its maximum at some point of Σ, there is nothing to do. Otherwise, since (Σ, d) is a metric space, the sequence $(p_k)_{k\geq 1}$ whose existence is assured by the previous theorem is such that $\lim_{k\to+\infty}\rho(p_k) = +\infty$. Hence, since $K_\Sigma \geq 0$, it follows from Lemma 3.1 that, for sufficiently large k, one has $K(p_k) \leq \rho(p_k)\mathrm{tr}(\Phi_{p_k})$. Therefore, (3.5) gives

$$\begin{aligned}\Box f(p_k) &\leq \frac{2\mathrm{tr}(\Phi_{p_k})(C_1 - f(p)+1)}{k}\left(\frac{\rho(p_k)^2+1}{\rho(p_k)^2+2}\right)\frac{1}{\log(\rho(p_k)^2+2)}\\ &\quad+\frac{4\mathrm{tr}(\Phi_{p_k})(C_1 - f(p)+1)}{k^2}\left(\frac{\rho(p_k)}{\rho(p_k)^2+2}\right)^2\frac{1}{[\log(\rho(p_k)^2+2)]^2}.\\ &\leq \frac{2C_2(C_1 - f(p)+1)}{k\log 2}+\frac{C_2(C_1 - f(p)+1)}{2k^2\log^2 2},\end{aligned}$$

so that

$$\lim_{k\to+\infty} \Box f(p_k) = 0. \tag{3.9}$$

The statement of the corollary follows from (3.4), (3.8) and (3.9), passing to a subsequence, if necessary. □

Corollary 3.4 *Let Σ be a complete Riemannian manifold with sectional curvature $K_\Sigma \geq 0$, and $f \in \mathcal{D}(\Sigma)$ be a function bounded from below. If Φ is positive semi-definite and* $\mathrm{tr}(\Phi)$ *is bounded from above on Σ, then there exists a sequence $(p_k)_{k\geq 1}$*

in Σ *such that*

$$f(p_k) < \inf_{\Sigma} f + \frac{1}{k}, \quad |\nabla f(p_k)| < \frac{1}{k}, \quad \Box f(p_k) > -\frac{1}{k}. \tag{3.10}$$

Proof Apply the previous corollary to $-f$. □

4 Applications

Throughout this section, $\psi : \Sigma^n \to \overline{M}^{n+1}$ denotes, as before, a spacelike immersion into a Lorentz manifold $\overline{M}$. In all that follows we set $\Phi = H_{r-1}P_{r-1}$, where H_{r-1} and P_{r-1} are as in Sect. 2. If $H_r = 0$ on Σ, or else $H_r > 0$ on Σ and ψ has an elliptic point, then propositions 2.2 and 2.3 assure the semi-definiteness of P_{r-1} (actually, P_{r-1} is definite when $H_r > 0$). Moreover, since

$$\operatorname{tr}\Phi = b_{r-1}H_{r-1}^2 \geq 0, \tag{4.1}$$

Φ is *positive* semi-definite in each of the above cases. In addition, if H_{r-1} is bounded on Σ, then the same is true of $\operatorname{tr}\Phi$, so that we can apply Corollaries 3.3 and 3.4 to such a Φ.

The following proposition is the analogue, in our context, of Lemma 3 of [1] due to K. Akutagawa.

Proposition 4.1 *Let* $\overline{M}^{n+1}$ *be a Lorentz manifold and* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *a complete spacelike hypersurface of sectional curvature* $K_\Sigma \geq 0$. *Suppose that, for some* $0 < r \leq n$, H_{r-1} *is bounded on* Σ *and one of the following is true:*

(a) $H_r = 0$ *on* Σ.
(b) $H_r > 0$ *on* Σ *and* ψ *has an elliptic point.*

If $f \in \mathcal{D}(M)$ *is nonnegative and such that* $\Box f \geq af^\beta$, *for some* $a > 0$, $\beta > 1$, *then* $f \equiv 0$.

Proof Let $\phi : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be a smooth function to be chosen later, and $g = \phi \circ f$. Then $\nabla g = \phi'(f)\nabla f$, and it follows from (2.4) that

$$\begin{aligned}
\Box g &= \operatorname{tr}(\Phi \operatorname{Hess} g) = H_{r-1}L_{r-1}(g) \\
&= \phi'(f)H_{r-1}L_{r-1}(f) + \phi''(f)H_{r-1}\langle P_{r-1}\nabla f, \nabla f\rangle \\
&= \phi'(f)\Box f + \phi''(f)\langle \Phi\nabla f, \nabla f\rangle \\
&= \phi'(f)\Box f + \frac{\phi''(f)}{\phi'(f)^2}\langle \Phi\nabla g, \nabla g\rangle,
\end{aligned}$$

so that

$$-\frac{\phi''(f)}{\phi'(f)^2}\langle \Phi\nabla g, \nabla g\rangle + \Box g = \phi'(f)\Box f.$$

Letting $\phi(t) = \frac{1}{(1+t)^\alpha}$, $\alpha > 0$, one gets

$$\phi'(t) = -\alpha\phi(t)^{\frac{\alpha+1}{\alpha}}, \quad \frac{\phi''(f)}{\phi'(f)^2} = \left(\frac{\alpha+1}{\alpha}\right)\frac{1}{\phi(t)},$$

and hence

$$\left(\frac{\alpha+1}{\alpha}\right)\langle\Phi\nabla g, \nabla g\rangle - \phi(f)\Box g = \alpha\phi(f)^{\frac{2\alpha+1}{\alpha}}\Box f \geq a\alpha\frac{f^\beta}{(1+f)^{2\alpha+1}}.$$

If one now takes $\alpha = \frac{\beta-1}{2} > 0$, we arrive at

$$\left(\frac{\alpha+1}{\alpha}\right)\langle\Phi\nabla g, \nabla g\rangle - g\Box g \geq a\alpha\left(\frac{f}{1+f}\right)^\beta. \tag{4.2}$$

Since g is bounded from below, by corollary 3.4 we get a sequence (p_k) of points in M such that

$$g(p_k) < \inf_M g + \frac{1}{k}, \quad |\nabla g|(p_k) < \frac{1}{k}, \quad \Box g(p_k) > -\frac{1}{k}.$$

Therefore, $f(p_k) \to \sup_M f$, and taking into account that

$$\langle\Phi\nabla g, \nabla g\rangle \leq (\mathrm{tr}\,\Phi)|\nabla g|^2 = b_{r-1}H_{r-1}^2|\nabla g|^2,$$

we get from (4.2) that

$$b_{r-1}H_{r-1}^2\left(\frac{\alpha+1}{\alpha k^2}\right) - \frac{1}{k}\left(\inf_M g + \frac{1}{k}\right) \geq a\alpha\left(\frac{f(p_k)}{1+f(p_k)}\right)^\beta.$$

Making $k \to +\infty$, we get $\sup_M f = 0$, and since $f \geq 0$ this gives $f \equiv 0$. □

For what follows, a vector field V on $\overline{M}^{n+1}$ is said to be *conformal* if

$$\mathcal{L}_V\langle\ ,\ \rangle = 2\phi\langle\ ,\ \rangle \tag{4.3}$$

for some function $\phi \in \mathcal{D}(\overline{M})$, where $\mathcal{L}$ stands for the Lie derivative of the metric of $\overline{M}$. The function ϕ is called the *conformal factor* of V; also, V is called a *Killing* vector field if $\phi \equiv 0$.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathfrak{X}(\overline{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathfrak{X}(\overline{M})$ is conformal if and only if

$$\langle\overline{\nabla}_X V, Y\rangle + \langle X, \overline{\nabla}_Y V\rangle = 2\phi\langle X, Y\rangle, \tag{4.4}$$

for all $X, Y \in \mathfrak{X}(\overline{M})$. A particular class of conformal vector fields is that of the *closed* ones, i.e., those vector fields V on $\overline{M}$ satisfying the relation

$$\overline{\nabla}_X V = \phi X,$$

for some $\phi \in \mathcal{D}(\overline{M})$ and all $X \in \mathfrak{X}(\overline{M})$.

Following [25], Chap. 6, we say that a Lorentz manifold is stationary if there exists a one-parameter group of isometries whose orbits are timelike curves; for spacetimes, this group of isometries express time translation symmetry. Mathematically, a stationary Lorentz manifold is furnished with a timelike Killing vector field. A *conformally stationary* Lorentz manifold is one furnished with a timelike conformal vector field. Our interest in conformally stationary Lorentz manifolds is due to the fact that, under an appropriate conformal change of metric, the conformal vector field turns into a Killing one, so that the new Lorentz manifold is now stationary.

A particular class of conformally stationary Lorentz manifolds is that of Generalized Robertson-Walker Lorentz manifolds, a member of which we shall refer to as a *GRW*. More precisely, let M^n be a connected, n-dimensional oriented Riemannian manifold and $I \subset \mathbb{R}$ an interval. In the product manifold $\overline{M}^{n+1} = I \times M^n$, let π_I and π_M denote the projections onto the I and M factors, respectively. If $f : I \to \mathbb{R}$ is a positive smooth function, we obtain a particular class of Lorentz metrics in $\overline{M}^{n+1}$ by setting

$$\langle v, w\rangle_p = -\langle(\pi_I)_* v, (\pi_I)_* w\rangle + (f \circ \pi_I)(p)^2 \langle(\pi_M)_* v, (\pi_M)_* w\rangle,$$

for all $p \in \overline{M}$ and all $v, w \in T_p\overline{M}$. Furnished with such a metric, $\overline{M}$ is a GRW, and will be denoted by writing $\overline{M}^{n+1} = -I \times_f M^n$; one calls f the *warping* function of $\overline{M}$. In Cosmology, GRW spacetimes give simple, yet physically plausible, static relativistic models (cf. [22]), so are natural spaces to work with.

In a GRW $\overline{M}^{n+1} = -I \times_f M^n$ one has the globally defined conformal vector field $V = f\partial_t$, which is even closed; moreover, one can easily prove that $\operatorname{div} V = (n+1)f'$. Conversely, S. Montiel proved in [20] that every conformally stationary Lorentz manifold whose timelike conformal vector field V is closed is locally a GRW; obviously, the warping function f can be recovered from V as $f \circ \pi_I = \sqrt{-\langle V, V\rangle}$. Due to this fact, from now on we shall restrict our attention to a GRW.

If $\psi : \Sigma^n \to \overline{M}^{n+1}$ is a spacelike immersion, we orient Σ by choosing a timelike unit normal vector field N. For future use, we quote Lemma 5.4 of [2].

Lemma 4.2 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW, and $\psi : \Sigma^n \to \overline{M}^{n+1}$ a spacelike immersion. If the restriction of $f \circ \pi_I$ to $\psi(\Sigma)$ attains a local minimum at some $p \in \psi(\Sigma)$, such that $f'(\pi_I(p)) \neq 0$, then p is an elliptic point for Σ.*

The following proposition is due to L. J. Alías and A. G. Colares, as Lemma 4.1 of [3]. Here, and for the sake of completeness, we present a more direct proof.

Proposition 4.3 *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW, and $\psi : \Sigma^n \to \overline{M}^{n+1}$ a spacelike immersion. If $h = \pi_{I_{|\Sigma}} : \Sigma^n \to I$ is the height function of Σ, then*

$$L_r(h) = -(\log f)'\{b_r H_r + \langle P_r(\nabla h), \nabla h\rangle\} - b_r H_{r+1}\langle N, \partial_t\rangle. \tag{4.5}$$

Proof One has

$$\begin{aligned}\nabla h &= \nabla(\pi_{I_{|\Sigma}}) = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top \\ &= -\partial_t - \langle N, \partial_t\rangle N,\end{aligned}$$

where $\overline{\nabla}$ denotes the gradient with respect to the metric of the ambient space and $X^\top$ the tangential component of a vector field $X \in \mathfrak{X}(\overline{M})$ in Σ. Now fix $p \in M$, $v \in T_pM$ and let A denote the shape operator with respect to N. Write $v = w - \langle v, \partial_t\rangle\partial_t$, so that $w \in T_p\overline{M}$ is tangent to the fiber of $\overline{M}$ passing through p. By repeated use of the formulae of item (2) of Proposition 7.35 of [22], we get

$$\begin{aligned}\overline{\nabla}_v\partial_t &= \overline{\nabla}_w\partial_t - \langle v, \partial_t\rangle\overline{\nabla}_{\partial_t}\partial_t = \overline{\nabla}_w\partial_t \\ &= (\log f)'w = (\log f)'(v + \langle v, \partial_t\rangle\partial_t).\end{aligned}$$

Thus,

$$\begin{aligned}\nabla_v\nabla h &= \overline{\nabla}_v\nabla h + \langle Av, \nabla h\rangle N \\ &= \overline{\nabla}_v(-\partial_t - \langle N, \partial_t\rangle N) + \langle Av, \nabla h\rangle N \\ &= -(\log f)'w - v(\langle N, \partial_t\rangle)N + \langle N, \partial_t\rangle Av + \langle Av, \nabla h\rangle N \\ &= -(\log f)'w + (\langle Av, \partial_t\rangle - \langle N, \overline{\nabla}_v\partial_t\rangle)N + \langle N, \partial_t\rangle Av + \langle Av, \nabla h\rangle N \\ &= -(\log f)'w + (\langle Av, \partial_t^\top\rangle - \langle N, (\log f)'w\rangle)N + \langle N, \partial_t\rangle Av + \langle Av, \nabla h\rangle N \\ &= -(\log f)'w - (\log f)'\langle v, \partial_t\rangle\langle N, \partial_t\rangle N + \langle N, \partial_t\rangle Av \\ &= -(\log f)'\{v - \langle v, \partial_t\rangle(-\partial_t - \langle N, \partial_t\rangle N)\} + \langle N, \partial_t\rangle Av \\ &= (\log f)'(-v + \langle v, \partial_t^\top\rangle\nabla h) + \langle N, \partial_t\rangle Av \\ &= -(\log f)'(v + \langle v, \nabla h\rangle\nabla h) + \langle N, \partial_t\rangle Av.\end{aligned}$$

Now, by fixing $p \in \Sigma$ and an orthonormal frame $\{e_i\}$ at $T_p\Sigma$, one gets

$$\begin{aligned}L_r h = \mathrm{tr}(P_r\,\mathrm{Hess}\,h) &= \sum_{i=1}^n\langle\nabla_{e_i}\nabla h, P_r e_i\rangle \\ &= \sum_{i=1}^n\langle -(\log f)'(e_i + \langle e_i, \nabla h\rangle\nabla h) + \langle N, \partial_t\rangle Ae_i, P_r e_i\rangle \\ &= -(\log f)'\{\mathrm{tr}(P_r) + \langle P_r(\nabla h), \nabla h\rangle\} + \langle N, \partial_t\rangle\mathrm{tr}(AP_r).\end{aligned}$$

The result follows from the formulae for the traces of P_r and AP_r. □

As before, let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW. For a fixed $t_0 \in I$, we say that $M_{t_0} = \{t_0\} \times M$ is a *slice* of $\overline{M}^{n+1}$. It was proved by S. Montiel in [20] that M_{t_0} has constant mean curvature, equal to $\frac{f'(t_0)}{f(t_0)}$ with respect to ∂_t. Whenever we talk about the mean curvature of the slices of a GRW, we shall assume that it is computed with respect to ∂_t. A spacelike immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is said to be *r-maximal* if $H_r = 0$ on Σ. Also, if the height function $h : \Sigma \to I$ is such that $h \geq t_0$ for some $t_0 \in \mathbb{R}$, then we say that ψ is a spacelike hypersurface *over* the slice M_{t_0}. With such notations and conventions we have the following

Theorem 4.4 *If the mean curvature of the slices of $\overline{M}^{n+1} = -I \times_f M^n$ is always greater than or equal to one, then there exists no r-maximal complete spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ over a slice of $\overline{M}$, with sectional curvature $K_\Sigma \geq 0$ and such that $C_1 \leq |H_{r-1}| \leq C_2$, for some positive constants C_1, C_2.*

Proof Suppose, by contradiction, the existence of such a hypersurface. Given a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, a straightforward computation shows that

$$L_{r-1}(\varphi \circ h) = \varphi''(h)\,\langle P_{r-1}\nabla h, \nabla h\rangle + \varphi'(h) L_{r-1}(h),$$

so that Eq. (4.5) gives

$$L_{r-1}\left(e^{-h+t_0}\right) = e^{-h+t_0}\left\{\langle P_{r-1}\nabla h, \nabla h\rangle + (\log f)'(b_{r-1}H_{r-1} + \langle P_{r-1}\nabla h, \nabla h\rangle)\right\}.$$

Consequently, since $\Phi = H_{r-1}P_{r-1}$, we have that

$$\begin{aligned}\Box(e^{-h+t_0}) &= \mathrm{tr}(\Phi\,\mathrm{Hess}(e^{-h+t_0})) = H_{r-1}L_{r-1}(e^{-h+t_0})\\ &= e^{-h+t_0}\left\{\langle \Phi(\nabla h), \nabla h\rangle + \frac{f'}{f}(b_{r-1}H_{r-1}^2 + \langle \Phi(\nabla h), \nabla h\rangle)\right\}\\ &\geq e^{-h+t_0}\left\{2\langle \Phi(\nabla h), \nabla h\rangle + b_{r-1}H_{r-1}^2\right\},\end{aligned}$$

where the above inequality uses the hypothesis on the mean curvature of the slices. Therefore, since Φ is positive semi-definite and $h - t_0 \geq 0$, we get

$$\Box(e^{-h+t_0}) \geq C_1^2 b_{r-1} e^{\beta(-h+t_0)},\ \forall\,\beta > 1,$$

and Proposition 4.1 allows us to conclude that $e^{-h+t_0} \equiv 0$, which is in turn an absurd. □

The above result can be applied to two interesting particular models of GRW spacetimes, which we now describe. The first one is the *steady state* model of the universe, as proposed by Bondi, Gold and Hoyle (cf. [15], Chap. 5; see also [17,21]), namely

$$\mathcal{H}^{n+1} = -\mathbb{R} \times_{e^t} \mathbb{R}^n.$$

Such a space appears naturally in physical context as an exact solution to the Einstein's equations, and is a cosmological model where matter is supposed to travel along geodesics normal to horizontal hyperplanes (i.e., to the slices); these, in turn, serve as the initial data for the Cauchy problem associated to those equations. For $\mathcal{H}^{n+1}$, all slices have mean curvature one with respect to ∂_t.

Remark 4.5 As a consequence of Bonnet-Myers Theorem (cf. [12], Theorem 9.3.1), a complete spacelike hypersurface $\psi : \Sigma^n \to \mathcal{H}^{n+1}$ having (not necessarily constant) mean curvature H satisfying $|H| \leq \rho < 2\sqrt{n-1}/n$ (ρ a real constant), has to be compact; in fact, for such a bound on H, Gauss' equation would give

$$\mathrm{Ric}_\Sigma \geq (n-1) - n^2\rho^2/4 > 0,$$

where Ric_Σ denotes the Ricci curvature of Σ. However, since $\mathcal{H}^{n+1}$ is not spatially closed (i.e., since its Riemannian fiber is not compact), such a hypersurface does not exist (cf. [4], Proposition 3.2(i)). As a special case of this reasoning, we see that there are no complete maximal spacelike hypersurfaces in $\mathcal{H}^{n+1}$. Thus, in the case of $\mathcal{H}^{n+1}$, Theorem 4.4 can be seen as a sort of generalization of this situation for higher order mean curvatures.

As a second particular case we have a suitable open subset of de Sitter space, so let us first of all describe such a space. If we modify Einstein's equations by setting the cosmological constant to be positive and the stress-energy tensor to be identically zero, de Sitter space appears as a particular exact solution (cf. [25], Chap. 5). The subset of it we are interested in has the GRW model

$$-(0, +\infty) \times_{\sinh t} \mathbb{H}^n,$$

where $\mathbb{H}^n$ stands for the $n-$dimensional hyperbolic space (cf. [20], Example 2). In this case, all slices have mean curvature $\frac{\cosh t}{\sinh t} > 1$.

Back to the general setting, if $\overline{M}^{n+1} = -I \times_f M^n$, we say that a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ has the same *time-orientation* of ∂_t if Σ is oriented by the choice of a timelike unit normal vector field N, such that $\langle N, \partial_t \rangle \leq -1$; otherwise we say that Σ has time-orientation *opposite* to that of ∂_t.

Theorem 4.6 *Suppose that all slices of $\overline{M}^{n+1} = -I \times_f M^n$ have mean curvature greater than or equal to one, and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface over a particular slice M_{t_0} of $\overline{M}^{n+1}$, with sectional curvature $K_\Sigma \geq 0$ and time-orientation opposite to that of ∂_t. If $H_r > 0$ and $C_1 \leq H_{r-1} \leq C_2$ for some positive constants C_1 and C_2, then the height function $h = \pi_{I_{|\Sigma}}$ does not attain a local minimum on Σ.*

Proof Suppose, for some such hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$, that the height function do attains a local minimum, at $p \in \psi(\Sigma)$, say. Since the warping function f satisfies $f' \geq f > 0$, we conclude that p is also a local minimum for $(f \circ \pi_I)_{|\Sigma}$, and hence Lemma 4.2 assures the existence of an elliptic point for $\psi(\Sigma)$; therefore, by Proposition 2.3, P_{r-1} is positive definite. Now, Eq. (4.5) gives

$$\begin{aligned} L_{r-1}\left(e^{-h+t_0}\right) = {} & e^{-h+t_0}\{\langle P_{r-1}\nabla h, \nabla h\rangle + (\log f)'(b_{r-1}H_{r-1} + \langle P_{r-1}\nabla h, \nabla h\rangle)\} \\ & + e^{-h+t_0} b_{r-1} H_r \langle N, \partial_t\rangle. \end{aligned}$$

Substituting $\Phi = H_{r-1}P_{r-1}$ and taking into account that Φ is positive definite, $\langle N, \partial_t \rangle \geq 1$ and $h - t_0 \geq 0$ we obtain, as in the previous theorem, the inequality

$$\Box(e^{-h+t_0}) \geq C_1^2 b_{r-1} e^{\beta(-h+t_0)}, \ \forall \beta > 1.$$

Proposition 4.1 gives, again as in our previous application, an absurd. □

When $r = 2$ and Σ has time-orientation opposite to that of ∂_t, Lemma 3.2 of [3] assures the ellipticity of L_1 whenever $H_2 > 0$. Since, by Gauss' equation, this is the same as asking that Σ has scalar curvature $R < n(n-1)$. One can then reason as in the previous result to obtain the following

Theorem 4.7 *If the mean curvature of the slices of* $\overline{M}^{n+1} = -I \times_f M^n$ *is always greater than or equal to one, then there exists no complete spacelike hypersurface* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *over a slice of* $\overline{M}^{n+1}$, *with sectional curvature* $K_\Sigma \geq 0$ *and satisfying the following conditions*:

(a) Σ *has scalar curvature* $R < n(n-1)$;
(b) *If the time-orientation of* Σ *is opposite to that of* ∂_t, *then its mean curvature* H *is such that* $C_1 \leq H \leq C_2$, *for some positive constants* C_1 *and* C_2.

Finally, we point out that, as remarked right after Theorem 4.4, the last two results can be immediately applied to the particular cases of the spacetime $\mathcal{H}^{n+1}$ and the GRW $-(0, +\infty) \times_{\sinh t} \mathbb{H}^n$.

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