

Tomographic representation of minisuperspace quantum cosmology and noether symmetries

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Abstract The probability representation, in which cosmological quantum states are described by a standard positive probability distribution, is constructed for minisuperspace models selected by Noether symmetries. In such a case, the tomographic probability distribution provides the classical evolution for the models and can be considered an approach to select “observable” universes. Some specific examples, derived from Extended Theories of Gravity, are worked out. We discuss also how to connect tomograms, symmetries and cosmological parameters.

Keywords Quantum cosmology · Tomograms · Alternative theories of gravity

1 Introduction

The main issue of Quantum Cosmology is to find out methods, derived from some quantum gravity approach, to achieve the so called Wave Function of the Universe (or the corresponding density matrix), a quantity related to the initial conditions from which dynamical systems (representing “classical universes”) can emerge [1,2].

Different representations of the density matrix such as the Wigner quasi-distribution were applied to study cosmological models in the minisuperspace [3,4]. Recently, a

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tomographic probability representation of quantum states was suggested: the quantum state can be associated to a standard positive probability distribution (tomogram). The tomographic probability of quantum states was introduced in Quantum Cosmology in [5]. In this framework, cosmological models described by the solutions of the Wheeler-de Witt equation can be mapped onto solutions of the Fokker–Planck-like equations for standard probability distributions. The tomographic probabilities are connected to the Wigner function of the Universe by the Radon integral transform [5] and by the wave function of the Universe with the wave related to the fractional Fourier transform [5,6] considered in quantum mechanics in [7–9].

Some important remarks are in order to point out why the tomographic approach is useful to improve the Wigner picture. As we said, the Wigner function [10] can be used to describe quantum states which are “universes”. It has been introduced in order to deal with mixed quantum states with some remarkable properties. In contrast to the standard wave function and density matrix representation, where complex functions are used, the Wigner function is a real one. It depends on two real variables, p and q , which label a point in the phase space. The marginals constructed by the Wigner function $\mathcal{W}(p, q)$ are integrals either over q or over p . They are positively defined probability densities in position and in momentum, respectively. These properties make the Wigner function to be very similar to the probability distribution density used to describe the classical states of a particle in the phase space as in classical statistical mechanics. The Wigner function was introduced, namely, to find a rigorous description of quantum states in terms of objects which have a classical probability distribution. However, such a function cannot play this role since it can take negative values and this feature does not allow to interpret the function as a fair probability distribution density. The necessity to find out such a kind of probability density, which rigorously describes the quantum state, is known as the *Pauli problem* [11]. Pauli supposed that the probability distribution in momentum, together with the one in position, determines completely the quantum state. But it was shown that different quantum states can exist having the same probability distribution density, so the Pauli problem was proved to have no solution, if one has only these two probability distributions (namely the one in momentum and the one in position). Nevertheless, this problem got a complete solution in the tomographic probability approach to quantum state description. In short, one needs a family of marginal probability densities in the phase space obtained from rotations in phase space of the Pauli distribution pair: such rotations can be parameterized by a given angle θ . This family, which is nothing else but the Radon integral transform [12] of the Wigner function, is sufficient to completely describe the quantum state, i.e. to find the density matrix of Wigner function if one knows the tomographic probability distribution. The tomographic probability contains the same information on the quantum state as the Wigner function of density matrix. On the other hand, like the Wigner function, being completely equivalent to density matrix in position and momentum representations, the tomographic probability density has its own merit: it is a fair “positive” probability distribution.

It is well known that the Wigner function has been intensively used in Quantum Cosmology [13–16]. Besides, some applications of tomographic probability, based on its property to be the fair positive measurable probability distribution, have been considered for cosmological problems [5,6]. For example, the notion of tomographic entropy,

using the Shannon construction of entropy and information [17], provides extra characteristics of cosmological quantum states [5,6]. The Shannon entropy cannot be used for the Wigner function since such a function is not a probability distribution.

Furthermore, using the properties of tomographic probability, related to its own features, it is possible to address several problems of Quantum Cosmology whose solutions cannot be achieved by the Wigner function. In particular, it has been shown that symmetries can play an important role to select viable cosmological models [18–21]. In the framework of the so-called *Noether Symmetry Approach* [21–23] it is possible, in principle, to find out cyclic variables related to conserved quantities and then to reduce cosmological dynamics. Besides, the existence of symmetries fixes the forms of couplings and self-interaction potential giving the relation between them in the interaction Lagrangians. The existence of Noether symmetries for minisuperspace cosmological models can be viewed as a sort of selection rule to recover classical behaviors in cosmic evolution [24]. In fact, it was shown in [1] that the form of the wave function with specific picks provides a classical regime in the evolution of the Universe. The picks (as oscillations) are characteristic properties of such regime and are selected in cosmological models where Noether symmetries are present [24].

In the framework of minisuperspace approach to Quantum Cosmology, the aim of this paper is to study cosmological models in order to seek for conditions to select classical (observable) behaviors by tomographic probability representation. We find out that, according to [24], if tomograms are related to extra symmetries (Noether symmetries), they allow oscillatory behaviours of the wave function and then give rise to classical solutions of dynamics. In the framework of the Hartle criterion, such solutions are observable universes. The approach is worked out for general classes of Extended Theories of Gravity (see for example [25–27] which have acquired a huge interest in recent years as possible schemes capable of explaining cosmological dynamics at all epochs, starting from inflation, through matter dominated era, up to the today observed accelerated behavior [28,30]).

The paper is organized as follows. In Sect. 2, the tomographic approach and its relation to the Wave Function of the Universe, considering the Wigner function and density is reviewed. Section 3 is devoted to the semiclassical limit of Quantum Cosmology and the Hartle criterion to select observable universes. In Sect. 4, the stationary phase method is discussed in view of the WKB approximation for the Wave Function of the Universe. Section 5 is a short summary of the minisuperspace cosmological models, coming from Extended Theories of Gravity, which we are going to analyze in the tomographic representation. Exact solutions (wave function of the universe) are obtained if Noether symmetries exist. Such solutions show oscillatory behaviors and then the possibility to implement the Hartle criterion. In Sect. 6, we give the related tomograms in stationary phase approximation for the previous selected solutions. Conclusions and a discussion of possible relations to the observed universe the classical regime evolution related to initial are presented in Sect. 7.

2 Tomograms of cosmological quantum states

We consider here the tomographic map of the wave function of the universe (or its density matrix) in the framework of the minisuperspace approach. Let us take into

account first the minisuperspace model in which the pure space of the universe is described by a vector $|\Psi\rangle$ in the Hilbert space of quantum states. The wave function $\Psi(x) = \langle x|\Psi\rangle$, $x \in \mathfrak{R}$ can be mapped onto a fair probability distribution $\mathcal{W}(X, \mu, \nu)$ called tomogram

$$\mathcal{W}(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \Psi(y) e^{\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y} dy \right|^2. \tag{1}$$

This probability distribution is positive and normalized for all the parameters $\mu, \nu \in \mathfrak{R}$

$$\int \mathcal{W}(X, \mu, \nu) dX = 1, \tag{2}$$

if the wave function is normalized, i.e.

$$\int |\Psi(y)|^2 dy = 1 \tag{3}$$

The tomographic map of the wave function of the universe onto the universe tomogram

$$\Psi(x) \mapsto \mathcal{W}(X, \mu, \nu) \tag{4}$$

is invertible (up to a constant phase factor). One has the inverse formula determining the density matrix of the universe pure state in the positive representation

$$\Psi(x)\Psi^*(x') = \frac{1}{2\pi} \int \mathcal{W}(y, \mu, x - x') e^{i(y - \mu(x+x')/2)} dy d\mu. \tag{5}$$

Thus one has the relation

$$|\Psi(0)|^2 = \frac{1}{2\pi} \int \mathcal{W}(y, \mu, 0) e^{iy} dy d\mu \tag{6}$$

In view of Eq. (5), one has the connection of the wave function of the universe with its tomogram

$$\Psi(x) = \frac{1}{2\pi|\Psi(0)|} \int \mathcal{W}(y, \mu, x) e^{i(y - \mu x/2)} dy d\mu \tag{7}$$

or in view of Eq. (6)

$$\Psi(x) = \frac{1}{2\pi} \int \mathcal{W}(y, \mu, x) e^{i(y - \mu x/2)} dy d\mu \left[\frac{1}{2\pi} \int \mathcal{W}(y', \mu', 0) e^{iy'} dy' d\mu' \right]^{-1/2} \tag{8}$$

Equation (8) determines the wave function of the universe in terms of the universe tomogram (up to a constant phase factor). This constant phase factor of the wave function is not essential. In fact, the wave function can be written in form of a modulus and a phase factor

$$\Psi(x) = |\Psi(x)|e^{iS(x)}. \quad (9)$$

In view of Eq. (8) one can obtain both the amplitude and the phase of the wave function in terms of the tomogram. For the amplitude, one has

$$|\Psi(x)|^2 = \frac{1}{2\pi} \int \mathcal{W}(y, \mu, x) e^{i(y-\mu x/2)} dy d\mu. \quad (10)$$

Since the amplitude squared is a real (positive) function, it is expressed in terms of the real part of the right hand side of Eq. (10), i.e.

$$|\Psi(x)|^2 = \frac{1}{2\pi} \int \mathcal{W}(y, \mu, x) \cos(y - \mu x/2) dy d\mu. \quad (11)$$

We point out that the tomogram $\mathcal{W}(y, \mu, x)$ is a real non negative function. On the other hand, the phase of the wave function $S(x)$ is also determined by the universe tomogram since

$$S(x) = -i \ln \left[\frac{\Psi(x)}{|\Psi(x)|} \right] \quad (12)$$

one has

$$S(x) = -i \ln \left[\frac{\int \mathcal{W}(y, \mu, x) e^{i(y-\mu x/2)} dy d\mu}{\left| \int \mathcal{W}(y, \mu, x) e^{i(y-\mu x/2)} dy d\mu \right|} \right]. \quad (13)$$

In quasi classical approximation (WKB) for the wave function of the universe, which corresponds to a stationary state with energy E , the amplitude and the phase are connected. In this case the phase $S(x)$ satisfies the Hamilton–Jacobi equation. One can consider the tomogram given by Eq. (1) as follows

$$\mathcal{W}(X, \mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int |\Psi(y)| e^{iS(y) + \frac{i\mu}{2\nu} y^2 - \frac{iX}{\nu} y} dy \right|^2. \quad (14)$$

The approach outlined above is in complete agreement with the semiclassical limit of Quantum Cosmology and it provides a physical interpretation of the wave function of the universe, as we are going to discuss below.

3 The semiclassical limit of Quantum Cosmology and the Hartle criterion

Semiclassical limit occupies an important role in Quantum Gravity due to the lack of a self-consistent and definitive theory. Only thanks to such a limit, we can obtain the solution of the Wheeler-De Witt (WDW) equation, the wave function of the universe Ψ , and search for interpretative criteria which allow physically motivated previsions. Furthermore, cosmological observations are based on a “classical universe” since we are not able, at least considering today’s facilities, to obtain data toward and before the

Planck epoch. Taking into account also the fact that the main goal of Quantum Cosmology is to give self-consistent laws for initial conditions, the correct identification of a semiclassical limit is crucial. In standard quantum mechanics, the wave function can be developed as a power law series in \hbar in the WKB approximation. Formally, the semiclassical limit corresponds to the limit $\hbar \rightarrow 0$. Analogously, we can write down the WDW equation as

$$\left(\frac{1}{m_P^2} \nabla^2 - m_P^2 U \right) \Psi[h_{ij}(x), \phi(x)] = 0 \quad (15)$$

where $U(h_{ij}, \phi)$ is the superpotential, h_{ij} are the components of the spatial three-metric, the geometrodynamical variables, and ϕ is a generic scalar field describing the matter content. The semiclassical limit can be achieved by using m_P^{-2} , the squared Planck mass, as an expansion parameter.

The wave function can be represented as

$$\Psi[h_{ij}(x), \phi(x)] \sim e^{im_P^2 S} \quad (16)$$

where S is the an action. A state with classical correlations has to be a superposition of states of the form (16). The WKB approximation is achieved by the expansion

$$S = S_0 + m_P^{-2} S_1 + O(m_P^{-4}) \quad (17)$$

and inserting it into the WDW equation. Then equating terms of the same order in m_P , we obtain the Hamilton–Jacobi equation at any order in S . At the lowest order in S_0 , we get

$$\nabla S_0 \cdot \nabla S_0 + U = 0. \quad (18)$$

In the same way, we obtain equations at higher orders which can be solved taking into account results from the previous orders. It can be shown that we need only ∇S_0 to recover the semi-classical limit of Quantum Cosmology [30], that is the wave function of the Universe is given by

$$\Psi \sim e^{im_P^2 S_0}. \quad (19)$$

If S_0 is a real number, we have oscillating WKB modes and Ψ is peaked on a phase-space region defined by the equations

$$\Pi_{ij} = m_P^2 \frac{\delta S_0}{\delta h^{ij}}, \quad (20)$$

$$\Pi_\phi = m_P^2 \frac{\delta S_0}{\delta \phi} \quad (21)$$

where Π_{ij} and Π_ϕ are respectively the classical momenta conjugate to h^{ij} and ϕ .

Using the Hamilton–Jacobi Eq. (18), Eqs. (20) and (21) are first integrals of classical field equations defining a set of solutions. This fact, in principle, can solve the problem of initial conditions. In fact, for a given action S_0 , solutions (20) and (21) involve n arbitrary parameters, while the general solution of the field equations involves $\{2n - 1\}$ parameters. Then the wave function is peaked (or oscillates, being S_0 real) on a subset of the general solution. In other words, the boundary conditions on the wave function imply initial conditions for the classical solution. Besides, if the wave function Ψ is sufficiently peaked, oscillating about some region of the configuration space $Q = \{h^{ij}, \phi\}$, it is possible to find correlations among the observables that characterize this region. If it is not peaked, correlations among the observables are not present.

At this point a discussion on what we mean as “classical universe” and how the concept is related to an oscillating Ψ is due.

Let us take as an example a cosmological model characterized by a set of observable parameters as H_0 , the Hubble constant, q_0 , the deceleration parameter, Ω_M , the density parameter for matter, Ω_Λ the density parameter for the cosmological constant Λ (or dark energy) and Ω_k , the amount of spatial curvature.¹ All of them are related by some cosmological model, solution of the Einstein–Friedmann equations and can be measured with a certain accuracy, by observations (see e.g. [31] and references therein for a discussion). A “good” wave function of the universe is that one which is “peaked” about such a solution of the cosmological equations (which can be alternatively derived from the momenta (20) and (21) and then “correlates” the set of observables $\{H_0, q_0, \Omega_M, \Omega_\Lambda, \Omega_k\}$. It is worth noticing that Ψ does not predict any specific value of the set of observables, but guarantees the correlations among them. This is the so-called *Hartle criterion* [1]. On the other hand, such a correlation allows to define a parameter able to label the points along the classical trajectories, solutions of (20) and (21). Defining a tangent vector in the configuration space of classical trajectories, where Ψ is peaked, it is

$$\frac{d}{d\tau} = 2\nabla S_0 \cdot \nabla \quad (22)$$

where τ is the “proper time” along the classical trajectories. From these considerations, we can state that

1. time emerges as a parameter which labels points along trajectories where the wave function is peaked;
2. time parametrization invariance (a symmetry of classical theory) comes out as the freedom to choose such a parameter. For example, assuming the redshift z , in principle, every classical cosmological solution can be expressed in terms of z ;
3. classical time, and in general, classical spacetime are proper notions which emerge only in a superspace regions where the wave function is oscillating;
4. the existence of oscillating wave function, and then of classical spacetimes, depends on the boundary conditions of the theory, in particular on the shape of superpotential $U(h_{ij}, \phi)$.

¹ Actually, one should consider also the age of the universe t_0 , but, usually, it is referred as $t_0 \sim H_0^{-1}$.

At this point it is clear what is the semiclassical limit of the theory. It is the superspace region where the wave function oscillates with big values of the phase which indicates strong correlation among the dynamical variables of the form (20) and (21). Such variables describe the classical trajectories in the superspace and make the concept of classical spacetime emerge. The tangent vector to this classical paths defines the proper time. Formally, in analogy with standard quantum mechanics, the semiclassical limit is obtained for $m_P \rightarrow \infty$, which means low energies with respect to the Planck scale. Furthermore, the limiting conditions for the wave function select a particular set of classical universes and the “measure” defined by the Ψ itself indicates a “typical” universe. In principle, this is the way in which Quantum Cosmology approaches the problem of initial conditions.

The semiclassical region of superspace, defined by the oscillating structure of this wave function, is called the “Lorentzian” region. This can be identified as the “ensemble” of all three-geometries embedded in a classical spacetime. The region outside the Lorentzian one is called “Euclidean”. Here the action is imaginary, i.e. $S_0 = iI$ and the wave function is exponential

$$\Psi \sim e^{-m_P^{-2}I}. \quad (23)$$

The wave function of this form is not “classical” since it corresponds to an Euclidean spacetime.

If Ψ is a WKB solution, I is the action of Euclidean solutions of field equations called “instantons”. Unlike the Lorentzian case, the wave function (23) is not peaked on a set of instantons. It is not “classical” since it cannot predict classical correlations among Lorentzian momenta and their conjugate variables. With these considerations in mind, it is clear that any method by which conserved quantities, corresponding to conserved momenta, can be achieved is useful to define a semiclassical limit in Quantum Cosmology. In other words, conserved momenta like $\Pi_{ij} = \Sigma_0$ or $\Pi_\phi = \Sigma_1$ allow, in principle, to select classical trajectories corresponding to observable universes.

In general, the Hamiltonian constraint gives the WDW equation, so that if $|\Psi\rangle$ is a *state* of the system (i.e. the wave function of the universe), dynamics is given by

$$\mathcal{H}|\Psi\rangle = 0. \quad (24)$$

In [24], it is shown that if Noether symmetries exist, a reduction procedure of dynamics can be implemented and then, we get

$$\begin{aligned} -i\partial_1|\Psi\rangle &= \Sigma_1|\Psi\rangle, \\ -i\partial_2|\Psi\rangle &= \Sigma_2|\Psi\rangle, \\ &\dots \quad \dots, \end{aligned} \quad (25)$$

which are translations along the q^j axes of configuration space (minisuperspace) singled out by the corresponding symmetry. Eqs. (25) can be immediately integrated and, being Σ_j real constants, we obtain oscillatory behaviors for $|\Psi\rangle$ in the directions of symmetries, i.e.

$$|\Psi\rangle = \sum_{j=1}^m e^{i\Sigma_j Q^j} |\chi(Q^l)\rangle, \quad m < l \leq n, \quad (26)$$

where m is the number of symmetries, l are the directions where symmetries do not exist, n is the total dimension of minisuperspace. Viceversa, dynamics given by (24) can be reduced by (25) if and only if it is possible to define constant conjugate momenta, that is oscillatory behaviors of a subset of solutions $|\Psi\rangle$ exist only if Noether symmetry exists for dynamics.

The m symmetries give first integrals of motion and then the possibility to select classical trajectories. In one and two-dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi-classical limit of Quantum Cosmology. As we shall see below, the tomographic probability representation gives results strictly related to this method since quantum tomograms become “classical” as soon as Noether symmetries exist. This occurrence allows, in principle, to trace back the cosmic evolution from quantum to classically observable states for several Extended Theories of Gravity.

It is worth noticing that the present approach has to be compared with previous results where minisuperspace method has been applied to similar models. For example, in [32], it is studied the quantum-to-classical transition in Jordan–Brans–Dicke quantum gravity showing the relevance of the method for inflation. In [33, 34] the scale factor duality and the cosmological constant problem are faced from the same minisuperspace point of view. As we are going to demonstrate, the tomographic approach is fully coherent with these results.

4 The stationary phase method

Let us now evaluate the integral (14) using the stationary phase approximation in view to select classical state by tomographic representation. The method is connected with the evaluation of an integral of the form

$$I = \oint A(z) e^{i\Phi(z)} dz \quad (27)$$

where z is a complex variable and a series decomposition of the function $\Phi(z)$ is used near its extremum

$$\Phi(z) \approx \Phi(z_0) + \frac{1}{2} \frac{\partial^2 \Phi}{\partial z^2} \Big|_{z=z_0} (z - z_0)^2. \quad (28)$$

The point(s) z_0 is(are) the solutions of the equation

$$\frac{\partial \Phi}{\partial z}(z_0) = 0. \quad (29)$$

The function $A(z)$ is considered slowly varying near the point z_0 . Then the integral (27) reads

$$I \approx A(z_0) \frac{\sqrt{\pi}}{\sqrt{-\frac{i}{2}\Phi''(z_0)}} e^{i\Phi(z_0)}. \tag{30}$$

In the case of N points $z_0 \rightarrow z_0^{(k)}$ (with $k = 1, \dots, N$), the integral assumes the form of a sum over these points.

Let us consider, in an explicit exact expression (14), the modulus $|\Psi(y)|$ to be a slowly varying amplitude function and the function

$$\Phi(q) = S(q) + \frac{\mu q^2}{2\nu} - \frac{Xq}{\nu}. \tag{31}$$

The point y_0 is determined by the expression

$$\frac{\partial S}{\partial q} + \frac{\mu}{\nu}q - \frac{X}{\nu} = 0. \tag{32}$$

One can see that using the notation

$$\frac{\partial S}{\partial q} = p \tag{33}$$

the relation (32) is equivalent to

$$\mu q + \nu p = X \tag{34}$$

One can remind that in classical mechanics the tomogram is defined as [35]

$$\mathcal{W}(X, \mu, \nu) = \int f(q, p) \delta(X - \mu q - \nu p) dq dp. \tag{35}$$

In quantum mechanics, the tomogram is determined via the Wigner function $W(q, p)$ as [36]

$$\mathcal{W}(X, \mu, \nu) = \int W(q, p) \delta(X - \mu q - \nu p) \frac{dq dp}{2\pi}. \tag{36}$$

Both relations mean the validity of (34). The Wigner function is determined by the wave function [10]

$$W(q, p) = \int \Psi\left(q + \frac{u}{2}\right) \Psi^*\left(q - \frac{u}{2}\right) e^{-iup} du. \tag{37}$$

In WKB approximation, the expression for the Wigner function is given just by the stationary phase method of evaluating the integral (37) that is

$$W(q, p) = \int \left| \Psi\left(q + \frac{u}{2}\right) \right| \left| \Psi^*\left(q - \frac{u}{2}\right) \right| e^{iS(q+\frac{u}{2})} e^{-iS(q-\frac{u}{2})} e^{-iup} du. \tag{38}$$

This expression was studied and applied in the cosmological context [3,4].

Since the Wigner function determines the tomograms by (36), one can get the quasi classical approximation for the tomogram inserting into (36) the Berry's expression for the Wigner function. But as it is clear, one can get the approximation directly using in (30) expression the function $\Phi(q)$ given by (31). Then the point q_0 is defined by (32) and $\Phi''(q_0)$ reads

$$\Phi''(q_0) = \frac{\partial^2 S(q)}{\partial q^2} \Big|_{q=q_0} + \frac{\mu}{\nu}. \quad (39)$$

Thus one has

$$\mathcal{W}(X, \mu, \nu) \approx \frac{1}{|\nu|} |\Psi(q_0)|^2 \left| \frac{\partial^2 S}{\partial q^2}(q_0) + \frac{\mu}{\nu} \right|^{-1} = \frac{|\Psi(q_0)|^2}{\left| \frac{\partial^2 S}{\partial q^2}(q_0)\nu + \mu \right|} \quad (40)$$

More generally, if Eq. (32) has more solutions $q_0^{(1)} \dots q_0^{(N)}$, the tomogram takes the form

$$\mathcal{W}(X, \mu, \nu) \approx \left| \sum_{i=1}^N \frac{\Psi(q_0^{(i)})}{\sqrt{\left| \frac{\partial^2 S}{\partial q^2} \Big|_{q_0^{(i)}} \nu + \mu \right|}} \right|^2. \quad (41)$$

The relation with the semiclassical limit of quantum cosmology is straightforward. In fact, the existence of such stationary points means that the wave function of the universe is peaked on conserved momenta and then classical trajectories in the minisuperspace can be found out.

5 Minisuperspace models from extended theories of gravity

The previous discussion can be applied to several classes of minisuperspaces, in particular one can take in to account Extended Theories of Gravity as the scalar–tensor theories or higher-order theories of gravity. Such theories are interesting since they are directly related to the issue of recovering suitable effective actions in quantum gravity [37]. Starting from pioneering works of Sakharov [38], the effects of vacuum polarization on the gravitational constant, i.e. the fact that gravitational constant can be induced by vacuum polarization, have been extensively investigated. All these attempts led to take into account gravitational actions extended beyond the simple Hilbert–Einstein action of General Relativity (GR) which is linear in the Ricci scalar R . The motivation is to investigate alternative theories in order to cure the shortcomings of GR, essentially due to the emergence of singularities in high-curvature regimes.

The Brans–Dicke approach is one of this attempt which, asking for dynamically inducing the gravitational coupling by a scalar field, is more coherent with the Mach principle requests [39].

Besides, it has been realized that corrective terms are inescapable if we want to obtain the effective action of quantum gravity at scales close to the Planck length (see e.g. [40]). In other words, it seems that, in order to construct a renormalizable theory of gravity, we need higher-order terms of curvature invariants such as R^2 , $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$, $R\Box R$, $R\Box^k R$ or nonminimally coupled terms between scalar fields and geometry as $\varphi^2 R$ (see [41–43] for a review).

Stelle [44,45] constructed a renormalizable theory of gravity by introducing quadratic terms in curvature invariants. Barth and Christensen gave a detailed analysis of the one-loop divergences of fourth-order gravity theories providing the first general scheme of quantization of higher-order theories [46,47]. Several results followed and today it is well known that a renormalizable theory of gravity is obtained, at least at one-loop level, if quadratic terms in the Riemann curvature tensor and its contractions are introduced [37]. Any action, where a finite number of terms involving power laws of curvature tensor or its derivatives appears, is a low-energy approximation to some fundamental theory of gravity which, up to now, is not available. For example, String Theory or Supergravity present low-energy effective actions where higher-order or nonminimally coupled terms appear [48,49].

However, if Lagrangians with higher-order terms or arbitrary derivatives in curvature invariants are considered, they are expected to be non-local and give rise to some characteristic length l_0 of the order of Planck length. The expansion in terms of R and $\Box R$, for example, at scales larger than l_0 produces infinite series which should break near l_0 [50]. With these facts in mind, taking into account such Lagrangians, means to make further steps toward a complete renormalizable theory of gravity. Cosmological models coming from such effective theories have been extensively studied by the *Noether Symmetry Approach*. In particular, nonminimally coupled theories of the form

$$\mathcal{L} = \sqrt{-g} \left[F(\varphi)R + \frac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi) \right], \quad (42)$$

where $F(\varphi)$ and $V(\varphi)$ are respectively the coupling and the potential of the scalar field, and fourth-order theories like

$$\mathcal{L} = \sqrt{-g} f(R), \quad (43)$$

where $f(R)$ is a generic function of the scalar curvature.² In [18–20,23,51], it was shown that asking for the existence of a Noether symmetry

$$L_X\mathcal{L} = 0 \rightarrow X\mathcal{L} = 0, \quad (44)$$

where L_X is the Lie derivative with respect to the Noether vector X , it is possible to select physically interesting forms of the interaction potential $V(\varphi)$, the gravitational coupling $F(\varphi)$ and the function $f(R)$.

² The field equations of $f(R)$ theories are of fourth-order in metric derivatives. They reduce to the standard second order Einstein equations for $f(R) = R$.

As we said above, the existence of Noether symmetries allows to select constants of motion so that the dynamics results simplified. Often such a dynamics is exactly solvable by a straightforward change of variables where a cyclic one is present. This occurrence reveals extremely useful in Quantum Cosmology since allows to formulate minisuperspace models exactly solvable. Considering the above cases, i.e. scalar–tensor and fourth-order gravity, suitable 2-dimensional configuration spaces (minisuperspace) can be achieved adopting a Friedmann–Robertson–Walker (FRW) metric which gives the pointlike Lagrangians.

In the case of scalar–tensor theories, we have

$$\mathcal{L} = 6a\dot{a}^2 F + 6a^2\dot{a}\dot{F} - 6kaF + a^3 \left[\frac{\dot{\varphi}}{2} - V \right], \quad (45)$$

in terms of the scale factor a .

The configuration space of such a Lagrangian is $\mathcal{Q} \equiv \{a, \varphi\}$, i.e. a two-dimensional minisuperspace. A Noether symmetry exists if (44) holds. The related Noether vector has to be

$$X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \varphi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\varphi}}, \quad (46)$$

where α, β depend on a, φ . The condition (44) gives rise to a set of partial differential equations whose solutions are $\alpha, \beta, F(\varphi)$ and $V(\varphi)$ (see [24] for details). For example, a solution is

$$\alpha = -\frac{2}{3}p(s)\beta_0 a^{s+1} \varphi^{m(s)-1}, \quad \beta = \beta_0 a^s \varphi^{m(s)}, \quad (47)$$

$$F(\varphi) = D(s)\varphi^2, \quad V(\varphi) = \lambda\varphi^{2p(s)}, \quad (48)$$

where

$$D(s) = \frac{(2s+3)^2}{48(s+1)(s+2)}, \quad p(s) = \frac{3(s+1)}{2s+3}, \quad m(s) = \frac{2s^2+6s+3}{2s+3}, \quad (49)$$

and s, λ are free parameters. A suitable change of variables gives

$$w = \sigma_0 a^3 \varphi^{2p(s)}, \quad z = \frac{3}{\beta_0 \chi(s)} a^{-s} \varphi^{1-m(s)}, \quad (50)$$

where σ_0 is an integration constant and

$$\chi(s) = -\frac{6s}{2s+3}. \quad (51)$$

Lagrangian (45) becomes, for $k = 0$,

$$\mathcal{L} = \gamma(s)w^{s/3}\dot{z}\dot{w} - \lambda w, \quad (52)$$

where z is cyclic and

$$\gamma(s) = \frac{2s + 3}{12\sigma_0^2(s + 2)(s + 1)}. \tag{53}$$

The conjugate momenta are

$$\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \gamma(s)w^{s/3}\dot{w} = \Sigma_0, \quad \pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \gamma(s)w^{s/3}\dot{z}, \tag{54}$$

and then the WDW equation is

$$\left[(i\partial_z)(i\partial_w) + \tilde{\lambda}w^{1+s/3} \right] |\Psi \rangle = 0, \tag{55}$$

where $\tilde{\lambda} = \gamma(s)\lambda$.

The quantum version of the first of momenta (54) is

$$-i\partial_z|\Psi \rangle = \Sigma_0|\Psi \rangle, \tag{56}$$

so that dynamics results reduced. A straightforward integration of Eqs. (55) and (56) gives

$$|\Psi \rangle = |\Omega(w) \rangle |\chi(z) \rangle \propto e^{i\Sigma_0 z} e^{-i\tilde{\lambda}w^{2+s/3}}, \tag{57}$$

which is an oscillating wave function.

Analogously, in fourth-order gravity case, the pointlike FRW Lagrangian is

$$\mathcal{L} = 6a\dot{a}^2 p + 6a^2\dot{a}\dot{p} - 6kap - a^3W(p), \tag{58}$$

which is of the same form of (45) a part the kinetic term., being $p = f'(R)$. This is an Helmholtz-like Lagrangian [52] and the configuration space is now $\mathcal{Q} \equiv \{a, p\}$; p has the same role of the above φ . Condition (44) is now realized by the vector field

$$X = \alpha(a, p) \frac{\partial}{\partial a} + \beta(a, p) \frac{\partial}{\partial p} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{p}} \tag{59}$$

The solution of this system, i.e. the existence of a Noether symmetry, gives α , β and $W(p)$. It is satisfied for

$$\alpha = \alpha(a), \quad \beta(a, p) = \beta_0 a^s p, \tag{60}$$

where s is a parameter and β_0 is an integration constant. In particular,

$$s = 0 \longrightarrow \alpha(a) = -\frac{\beta_0}{3} a, \quad \beta(p) = \beta_0 p, \quad W(p) = W_0 p, \quad k = 0, \tag{61}$$

$$s = -2 \longrightarrow \alpha(a) = -\frac{\beta_0}{a}, \quad \beta(a, p) = \beta_0 \frac{p}{a^2}, \quad W(p) = W_1 p^3, \quad \forall k, \tag{62}$$

Table 1 Summary of the changes of variables induced by the Noether symmetry for scalar–tensor and fourth-order minisuperspace models

Gravitational theory	p	z	w
Scalar–Tensor	$\frac{(2s+3)^2}{48(s+1)(s+2)}\varphi^2$	$\frac{3}{\beta_0\chi(s)}a^{-5}\varphi^{1-m(s)}$	$\sigma_0a^3\varphi^{2l(s)}$
Fourth order gravity $s = -2$	$f'(R)$	$\ln p$	ap
Fourth order gravity $s = 0$	$f'(R)$	a^2	a^3p

The parameters as functions of s are $\chi(s) = -6s/(2s + 3)$, $m(s) = (2s^2 + 6s + 3)/(2s + 3)$ and $l(s) = 3(s + 1)/(2s + 3)$

Table 2 Summary of the solutions of the WDW equation for the minisuperspace models

Gravitational theory	$\Psi(z, w)$
Scalar–Tensor	$e^{i\Sigma_0z}e^{-i\tilde{\lambda}w^{2+s/3}}$
Fourth order gravity $s = -2$	$e^{i[\Sigma_1z+9kw^2+(3w_1/4)w^4]}$
Fourth order gravity $s = 0$	$e^{i\Sigma_0[z-(1/4)\ln w]}w^{1/2}Z_{\tilde{\nu}}(\lambda, w)$
Fourth order gravity $s = 0$	$e^{i[\Sigma_0z-(\Sigma_0/4)\ln w\pm\lambda w]}$

The $Z_{\tilde{\nu}}(\lambda, w)$ are Bessel function and $w \gg 0$

where W_0 and W_1 are constants. As above, the new set of variables $\mathcal{Q} \equiv \{z, w\}$ adapted to the foliation induced by X can be achieved. The procedure to achieve the solution for the WDW equation is exactly the same. The results are summarized in Table 1. The solutions for the respective WDW equations are given in Table 2.

By a rapid inspection of the wave functions in Table 2, it is clear that the oscillatory behavior, i.e. the presence of peaks, is strictly related to the Noether constant Σ_0 . In other words, the Hartle criterion can be immediately implemented when conserved quantities are found out.

6 Minisuperspace tomograms in stationary phase approximation

The above results can be immediately translated in the tomographic representation. If Noether symmetries are present, the solutions of the WDW equation, take the following form

$$\Psi(Q_1, \dots, Q_n) = \sum_{j=1}^m e^{i\Sigma_j Q^j} \chi(Q^l) \quad m < l \leq n \tag{63}$$

then we find, according to the results of Sect. 4, that the corresponding tomograms are

$$\begin{aligned} &\mathcal{W}(X_1 \dots X_n, \mu_1, \dots, \mu_n, \nu_1, \dots, \nu_n) \\ &= \sum_{j=1}^m \frac{1}{\mu_j} \left| \int \chi(Q^l) \left(\prod_{l=m+1}^n e^{\frac{i\mu}{2\nu}(Q^l)^2 - \frac{iX}{\nu}(Q^l)} d(Q^l) \right) \right|^2. \end{aligned} \tag{64}$$

It is then straightforward to study the form of the tomograms satisfying the Hartle criterion.

Let us consider first the tomograms resulting from the scalar–tensor theories. From Table 2, the wave function takes the form

$$\Psi(z, w) \propto e^{i\Sigma_0 z} e^{-i\tilde{\lambda} w^{2+s/3}}. \tag{65}$$

If we consider the case $s = 0$, the corresponding two variable tomogram is

$$\mathcal{W}(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2) \propto \frac{1}{|\mu_1|} \frac{1}{|\mu_2 - 2\tilde{\lambda}\nu_2|}. \tag{66}$$

For the fourth order gravity models, we have considered the two cases $s = -2$ and $s = 0$. In the first case, the wave function is

$$\Psi(z, w) \propto e^{i[\Sigma_1 z + 9kw^2 + (3w_1/4)w^4]} \tag{67}$$

and the corresponding tomogram is

$$\begin{aligned} &\mathcal{W}(X_1, X_2, \mu_1, \mu_2, \nu_1, \nu_2) \\ &\propto \frac{1}{|\mu_1|} \left| \sum_{j=1}^3 \frac{\exp(i(9k\alpha_j^2(X_2, \mu_2, \nu_2) + \frac{3w_1}{4}\alpha_j^4(X_2, \mu_2, \nu_2)))}{\sqrt{9w_1\nu_2 + 18k\nu_2 + \mu_2}} \right|^2 \end{aligned} \tag{68}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the solutions of equation

$$3w_1w^3 + \left(\frac{\mu_2}{\nu_2} + 18k\right)w - \frac{X_2}{\nu_2} = 0 \tag{69}$$

$$\begin{aligned} \alpha_1 &= \frac{-3 \cdot 2^{1/3} (\mu + 18k\nu)}{\left(-81 X \nu^2 + \sqrt{6561 X^2 \nu^4 + 2916 \nu^3 (\mu + 18k\nu)^3}\right)^{1/3}} \\ &\quad + \frac{\left(-81 X \nu^2 + \sqrt{6561 X^2 \nu^4 + 2916 \nu^3 (\mu + 18k\nu)^3}\right)^{1/3}}{3 \cdot 2^{1/3} \nu}, \\ \alpha_2 &= \frac{3 \left(1 + i\sqrt{3}\right) (\mu + 18k\nu)}{2^{2/3} \left(-81 X \nu^2 + \sqrt{6561 X^2 \nu^4 + 2916 \nu^3 (\mu + 18k\nu)^3}\right)^{1/3}} \\ &\quad - \frac{\left(1 - i\sqrt{3}\right) \left(-81 X \nu^2 + \sqrt{6561 X^2 \nu^4 + 2916 \nu^3 (\mu + 18k\nu)^3}\right)^{1/3}}{6 \cdot 2^{1/3} \nu}, \end{aligned}$$

$$\alpha_3 = \frac{3 \left(1 - i \sqrt{3}\right) (\mu + 18k v)}{2^{2/3} \left(-81 X v^2 + \sqrt{6561 X^2 v^4 + 2916 v^3 (\mu + 18k v)^3}\right)^{1/3}} - \frac{\left(1 + i \sqrt{3}\right) \left(-81 X v^2 + \sqrt{6561 X^2 v^4 + 2916 v^3 (\mu + 18k v)^3}\right)^{1/3}}{6 \cdot 2^{1/3} v}.$$

The second interesting case is for $s = o$. When $w \gg 0$ the wave function takes the form

$$\Psi(z, w) = e^{i[\Sigma_0 z - (\Sigma_0/4) \ln w \pm \lambda w]} \tag{70}$$

and the corresponding tomogram is

$$\mathcal{W}(X_1, X_2, \mu_1, \mu_2, v_1, v_2) \propto \frac{1}{|\mu_1|} \left| \frac{\exp\left(i\left(-\frac{\Sigma_0}{4} \ln \beta_1 \pm \tilde{\lambda} \beta_1\right)\right)}{\left((\Sigma_0 v_2/4\beta_1^2) + \mu_2\right)^{1/2}} + \frac{\exp\left(i\left(-\frac{\Sigma_0}{4} \ln \beta_2 \pm \tilde{\lambda} \beta_2\right)\right)}{\left((\Sigma_0 v_2/4\beta_2^2) + \mu_2\right)^{1/2}} \right|^2 \tag{71}$$

which can be rewritten in the form

$$\mathcal{W}(X_1, X_2, \mu_1, \mu_2, v_1, v_2) \propto \frac{1}{|\mu_1|} \left(\frac{1}{\left((\Sigma_0 v_2/4\beta_1^2) + \mu_2\right)} + \frac{1}{\left((\Sigma_0 v_2/4\beta_2^2) + \mu_2\right)} + 2 \frac{\cos(\beta_1 - \beta_2)}{\left((\Sigma_0 v_2/4\beta_1^2) + \mu_2\right)^{1/2} \left((\Sigma_0 v_2/4\beta_2^2) + \mu_2\right)^{1/2}} \right) \tag{72}$$

where β_1 and β_2 are the solutions of equation

$$\frac{\partial}{\partial w} \left[-(\Sigma_0/4) \ln w \pm \lambda w + \frac{\mu_2}{2v_2} w^2 - \frac{X_2}{v_2} w \right] = -(\Sigma_0/4) \frac{1}{w} \pm \lambda + \frac{\mu_2}{v_2} w - \frac{X_2}{v_2} = 0 \tag{73}$$

$$\beta_1 = -\frac{(\pm \tilde{\lambda} - X_2) - \sqrt{(\pm \tilde{\lambda} v_2 - X_2)^2 - \Sigma_0 \mu_2 v_2}}{2\mu_2},$$

$$\beta_2 = -\frac{(\pm \tilde{\lambda} - X_2) + \sqrt{(\pm \tilde{\lambda} v_2 - X_2)^2 - \Sigma_0 \mu_2 v_2}}{2\mu_2}.$$

In general, if we are able to achieve the WDW wave function in an explicit form it is always possible to construct the tomographic counterpart.

At this point, we have to make a remark. If one takes the wave function $\Psi(x)$ in the form of the De Broglie wave $\frac{1}{\sqrt{\pi}}e^{ikx}$, the probability density $|\Psi(x)|^2$ is $\frac{1}{2\pi}$. Also the tomogram calculated for this wave function is proportional to the value $\frac{1}{|\mu|}$. Using the tomogram for reconstructing the wave function, we get divergent integrals. Due to this fact, one can use a “regularized” wave packet $\tilde{\Psi}(x) = A(x)e^{ikx}$ with the amplitude $A(x)$, e.g., of the Gaussian form $\sim e^{-gx^2}$. In this case, the tomogram assumes the form of a Gaussian distribution and the density matrix for the pure state $\tilde{\psi}(x)\tilde{\psi}^*(x)$ can be reconstructed since all the integrals do diverge. The “regularized” wave packet, having a highly oscillatory behavior for $g \rightarrow 0$, corresponds to the Hartle criterion and then to the presence of Noether symmetries. In summary, when we have the tomograms as in the above examples containing $|\mu|$ in the denominator, the reconstruction formula can be explained in the described sense.

7 Discussion and conclusions

In this paper, we discussed the realization of minisuperspace Quantum Cosmology adopting a tomographic probability representation for the wave function of the universe. The physical meaning of the results is recovered if, in semiclassical WKB approximation, it is possible to select conserved quantities (stationary phases) where the wave function is peaked. In this case, the interpretative Hartle criterion can be applied and classical trajectories, corresponding to observable universes, are recovered. As a matter of fact the existence of Noether symmetries lead to oscillating components of the wave function, which present the picks required for a transition from the quantum cosmological states to the classical universes. In our approach, we found that a Noether symmetry Σ_0 implies a factor of the form $1/|\mu|$ in the tomogram. Noteworthy the tomograms assume a form which is very close to classical ones, showing in a natural way the transition from the quantum initial stages of the universe to its classical evolution. This fact is crucial in our discussion since tomograms can allow, in principle, a full description of the universe from its initial quantum states up to the today observed. Specifically, considering the above considered models, it is easy to find out, either from tomograms or from Noether constants, exact classical solutions. In the case of scalar–tensor models, we get

$$w(t) = [k_1t + k_2]^{3/(s+3)}, \quad (74)$$

$$z(t) = [k_1t + k_2]^{(s+6)/(s+3)} + z_0, \quad (75)$$

which can be immediately translated into the original configuration space $\mathcal{Q} \equiv \{a, \varphi\}$, that is

$$a(t) = a_0(t - t_0)^{b(s)}, \quad (76)$$

$$\varphi(t) = \varphi_0(t - t_0)^{q(s)}, \quad (77)$$

where

$$b(s) = \frac{2s^2 + 9s + 6}{s(s + 3)}, \quad q(s) = -\frac{2s + 3}{s}. \quad (78)$$

Depending on the value of s , we get Friedman, power-law, or pole-like behaviors, that is all the standard classical cosmological behaviors. Analogously, for the fourth-order models, we have, for $s = 0$,

$$a(t) = a_0 e^{(\lambda/6)t} \exp\left\{-\frac{z_1}{3} e^{-(2\lambda/3)t}\right\}, \quad (79)$$

$$p(t) = p_0 e^{(\lambda/6)t} \exp\{z_1 e^{-(2\lambda/3)t}\}, \quad (80)$$

where a_0 , p_0 and z_1 are integration constants. It is clear that λ plays the role of a cosmological constant and inflationary behavior is asymptotically recovered. For $s = -2$, we get power-law behaviors for $a(t)$ and $p(t)$.

Such solutions, in principle, give rise to “observable universes” since the set of parameters $\{H_0, q_0, \Omega_M, \Omega_\Lambda, \Omega_k\}$ can be obtained, in a standard way, from the solutions $a(t)$, $\varphi(t)$ and $p(t)$ [53]. This fact could be extremely relevant, also in view of the recent observational trends which have given rise to the so called *Precision Cosmology* (see for example [54–56]). In fact, the possibility to trace back the dynamics, via tomograms, from the today observed parameters up to the initial quantum conditions could be an interesting approach to formulate comprehensive cosmological models enclosing early and late evolution. In other words, due to this feature, the tomographic approach could be, with respect to other approaches included the Wigner function one, the most suitable to allow a connection between inflation and dark energy cosmology.

To conclude, we point out that all the main results of Quantum Cosmology can be obtained using, arbitrarily, some well known formulation of Quantum Mechanics [57] including the density matrix and the Wigner function. This means that, in principle, also other approaches could be successfully used with significant results [58–60]. However, as emphasized here, the formulation based on tomographic probability distribution has some interesting properties which are convenient to be used as soon as the quantum-classical transitions or the comparison between classical and quantum pictures are relevant: in this framework, tomograms assume a fundamental role being objects which fairly describe states both in classical and quantum domains.

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