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RESEARCH ARTICLE

Fermions tunneling from Reissner–Nordström black hole

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Abstract Hawking tunneling radiation of spin $-1/2$ particles from the event horizon of the Reissner–Nordström black hole is studied. We introduce the Dirac equation of the charged particles. We further consider the gravitational interaction and back reaction of the emitted spin particles in the dynamical background space–time. The result shows that when the energy conservation and charge conservation are taken into account, the actual radiation spectrum of fermions also derivates from the thermal one and the tunneling rate is related to the change of Bekenstein–Hawking entropy.

Keywords Hawking radiation · Fermions tunneling · Reissner–Nordström black hole

1 Introduction

Recent years, Hawking tunneling radiation obtains great importance. Because it provides not only an alternative conceptual means for understanding the actual emission process of black hole but also a useful verification of black hole thermodynamics. Since the first paradigm of Parikh and Wilczek [1,2], which modeled Hawking radiation as a tunneling event from the classical forbidden path by the semi-classical WKB approximation, there have emerged several derivations of Hawking tunneling radiation [3–7]. Each of them until now has been extended to the charged and uncharged, massive and massless particles successfully [8–15]. Notably, all of them

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only involved scalar particles. In fact, as implied in reference [16], black holes can radiate all sorts of particles like a blackbody at the Hawking temperature. The emission spectrum contains particles of all spins. Therefore, investigation on Hawking tunneling radiation of fermions is also of great importance and necessary. Very recently, Kerner and Mann [17] brought forward a method to discuss the tunneling of fermions from the Rindler space–time and a general non-rotating black hole space–time. Their work, adopting the general Dirac equation but not the Newman–Penrose formalism to determine the action of the radiant spin particles, is mainly based on the idea of Padmanabhan et al. [18]. And the tunneling rate from inside to outside of the horizon is derived by the calculation of probability ratio of the outgoing modes and incoming modes. Nevertheless in their paper, they didn't consider the self-gravitational interaction and back reaction of the emission. The obtained radiation spectrum thus there only corresponds to the pure thermal one. Actually, Parikh et al. [1,2] have renewed the exact emission picture by the consideration of energy conservation in the dynamical background space–time several years ago. That is, after the materialized positive energy particle that stems from the vacuum fluctuation with energy ω tunnels out the horizon, due to energy conservation, the mass of the black hole would reduce to $M - \omega$ while the total ADM mass is fixed. The precise thermal spectrum, which is related to the tunneling rate, accordingly isn't pure thermal but has some corrections.

In this paper, we incorporate the self-gravitational interaction and back reaction of the emission to discuss the tunneling radiation of fermions from Reissner–Nordström black hole. Due to the electromagnetic field would couple with the matter field and gravity field, the Dirac equation here is modified as that of charged particles. Then while we discuss the correction spectrum, we consider not only the energy conservation but also the charge conservation. Namely, after the spin particle with charge q runs out, the total charge of the hole will reduce to $Q-q$ too. In addition, for the static, non-rotating Reissner–Nordström black hole, because the ADM mass is much larger than the Plank mass, we ignore the affect of the emitted spin particles on the angular momentum of the black hole. Then for the zero-angular momentum hole, because the number of the spin up and spin down particles statistically is the same, when the spin up particles tunnel out, we regard its partner also do at the same Hawking temperature.

The remainder of this paper is outlined as follows. In Sect. 2, the Dirac equation of charged particles is introduced and the pure thermal spectrum of fermions from Reissner–Nordström black hole is derived. Then in Sect. 3, we take the energy and charge conservations into account to discuss the correction spectrum of the charged fermions. Finally we present our conclusions in Sect. 4.

2 The pure thermal spectrum of fermions

The line element of the Reissner–Nordström black hole is given by

$$ds^2 = -f(r)dt_R^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where

$$f(r_{\pm}) = \left(1 - \frac{2M}{r_{\pm}} + \frac{Q^2}{r_{\pm}^2}\right) = 0, \tag{2}$$

in which, $r_{\pm}$, respectively, represent the outer horizon and inner horizon, M and Q denote the mass and charge of the black hole. And the corresponding non-vanishing component of the electromagnetic vector potential is

$$A_t = -\frac{Q}{r}. \tag{3}$$

For the robust tunneling paradigm, the core is the calculation of the imaginary part of the action for the process of s-wave from the inside to outside of the horizon, which in turn is related to the Boltzmann factor for radiation at Hawking temperature. As for the charged fermions, we finish it by the Dirac equation of charged particles

$$i\gamma^{\mu}\left(D_{\mu} - \frac{iqA_{\mu}}{\hbar}\right)\psi + \frac{m}{\hbar}\psi = 0, \tag{4}$$

where the Greek indices $\mu, \nu = 0, 1, 2, 3$, q and m are the charge and mass of the fermions and

$$D_{\mu} = \partial_{\mu} + \Omega_{\mu}, \quad \Omega_{\mu} = \frac{1}{2}i\Gamma_{\mu}^{\alpha\beta}\Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{1}{4}i[\gamma^{\alpha}, \gamma^{\beta}]. \tag{5}$$

For the case of this hole, according to the relation $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}I$, we pick for the component of the γ^{μ} matrixes as

$$\gamma^t = \frac{1}{\sqrt{f}}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^r = \sqrt{f}\begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix},$$
$$\gamma^{\theta} = \frac{1}{r}\begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^{\varphi} = \frac{1}{r\sin\theta}\begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \tag{6}$$

In which $\sigma^i (i = 1, 2, 3)$ is the Pauli Sigma matrixes that satisfy

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

The eigenvectors of σ^3 are

$$\xi_{\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{8}$$

And the matrix for γ^5 correspondingly takes the form as

$$\gamma^5 = i\gamma^t\gamma^r\gamma^\theta\gamma^\varphi = \frac{i}{r^2\sin\theta}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

In the Dirac field, the solution of the spin up and spin down particles can be expressed as

$$\psi_\uparrow(t,r,\theta,\varphi) = \begin{bmatrix} A(t,r,\theta,\varphi)\xi_\uparrow \\ B(t,r,\theta,\varphi)\xi_\uparrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\uparrow(t,r,\theta,\varphi)\right], \tag{10}$$

$$\psi_\downarrow(t,r,\theta,\varphi) = \begin{bmatrix} C(t,r,\theta,\varphi)\xi_\downarrow \\ D(t,r,\theta,\varphi)\xi_\downarrow \end{bmatrix} \exp\left[\frac{i}{\hbar} I_\downarrow(t,r,\theta,\varphi)\right], \tag{11}$$

where $I_{\uparrow/\downarrow}$ denote the action of the emitted spin up and spin down particles, respectively. In this paper, we are only interested in the spin up case since the spin down is fully similar to this other than some changes of the sign. Inserting Eq. (10) into the Dirac Equation, after dividing the exponential term and multiplying $\hbar$, we find to leading order in $\hbar$

$$\frac{iA(\partial_t I_\uparrow - qA_t)}{\sqrt{f(r)}} + B\sqrt{f(r)}\partial_r I_\uparrow - Am = 0, \tag{12}$$

$$-\frac{B}{r}\left(\partial_\theta I_\uparrow + \frac{i}{\sin\theta}\partial_\varphi I_\uparrow\right) = 0, \tag{13}$$

$$-\frac{iB(\partial_t I_\uparrow - qA_t)}{\sqrt{f(r)}} + A\sqrt{f(r)}\partial_r I_\uparrow - Bm = 0, \tag{14}$$

$$-\frac{A}{r}\left(\partial_\theta I_\uparrow + \frac{i}{\sin\theta}\partial_\varphi I_\uparrow\right) = 0. \tag{15}$$

Owing to the action is the function of coordinate components, it is not easy to solve it directly now. However, taking into account the existence of time-like killing vector $(\frac{\partial}{\partial t})^a$, we can carry out the following separation variable

$$I_\uparrow = -\omega t + W(r) + J(\varphi,\theta), \tag{16}$$

then we find

$$-\frac{iA(\omega + qA_t)}{\sqrt{f(r)}} + B\sqrt{f(r)}W'(r) - Am = 0, \tag{17}$$

$$-\frac{B}{r}\left(J_\theta + \frac{i}{\sin\theta}J_\varphi\right) = 0, \tag{18}$$

$$\frac{iB(\omega + qA_t)}{\sqrt{f(r)}} + A\sqrt{f(r)}W'(r) - Bm = 0, \tag{19}$$

$$-\frac{A}{r}\left(J_\theta + \frac{i}{\sin\theta}J_\varphi\right) = 0. \tag{20}$$

Obviously, Eqs. (18) and (20) both yield $J_\theta + \frac{i}{\sin\theta} J_\varphi = 0$ regardless the explicit value of A or B, implying $J(\theta, \varphi)$ must be a complex function that would contribute to the imaginary part of the action. While in Eqs. (17) and (19), for the case of massless, there exist two possible solutions, namely

$$A = -iB, \quad W'(r) = W'_+(r) = \frac{\omega + qA_t}{f(r)}, \tag{21}$$

$$A = iB, \quad W'(r) = W'_-(r) = -\frac{(\omega + qA_t)}{f(r)}, \tag{22}$$

where $W_\pm(r)$, respectively, stand for the outgoing and incoming solutions. Meanwhile, solving Eqs. (17) and (19) with $m \neq 0$ leads to

$$\left(\frac{A}{B}\right)^2 = \frac{-i(\omega + qA_t) + \sqrt{f(r)}m}{i(\omega + qA_t) + \sqrt{f(r)}m}. \tag{23}$$

Taking the near-horizon approximation, which result in $A^2 = -B^2$, we find the same result as the case of massless. This is true and reasonable, because all massive spinless or spin particles near the black hole horizon become ultra-relativistic and thus behave massless.

Since the event horizon and infinite red-shift surface of this hole are coincident with each other, the geometrical optics limit hence is reliable. In the semi-classical limit, adopting the Wentzel–Kramers–Brilloin (WKB) approximation, the relationship between the tunneling rate and the action's imaginary part of the radiation can be expressed as

$$\Gamma \sim \mathrm{e}^{-2\,\mathrm{Im}\,W}. \tag{24}$$

Considering the contribution of the ingoing modes, the total tunneling probability of the spin particles that cross the horizon should be

$$\Gamma \sim \frac{P(out)}{P(in)} = \frac{\exp[-2(\mathrm{Im}\,W_+ + \mathrm{Im}\,J)]}{\exp[-2(\mathrm{Im}\,W_- + \mathrm{Im}\,J)]} = \exp(-4\,\mathrm{Im}\,W_+), \tag{25}$$

where

$$W_+ = \int \frac{(\omega + qA_t)}{f(r)} dr = \frac{i\pi(\omega - \omega_0)\left(m^2 + m\sqrt{m^2 - Q^2} - \frac{1}{2}Q^2\right)}{\sqrt{m^2 - Q^2}}, \tag{26}$$

in which $\omega_0 = qV_0 = qQ/r_+$. Based on this action, we can easily obtain the Hawking temperature

$$T = \frac{\sqrt{m^2 - Q^2}}{4\pi\left(m^2 + m\sqrt{m^2 - Q^2} - \frac{1}{2}Q^2\right)}. \tag{27}$$

This is consistent with that in [15]. However, we find the derived radiation spectrum from Eq. (27) is pure thermal. The reason for this arises from the background space–time is fixed. To precisely picture the Hawking radiation, we have to take the self-gravitational interaction and back reaction of the emitted charged spin particles into account.

3 The correction spectrum of fermions

According to the quantum theory, Hawking radiation is triggered by vacuum fluctuations. When a pair of virtual particle that creates spontaneously near the inside of the horizon, the negative energy particle will be absorbed, which effectively lower the mass of the black hole, while the positive energy virtual particle tunnels out the horizon and turn into true particle. In the opinion of Parikh and Wilczek, when the ADM mass and charge of black hole are given and admit to fluctuate, as the particle with a shell of energy ω and charge q runs out, the total mass and charge of the hole would reduce to $M-\omega$ and $Q-q$ accordingly and the radius of the horizon will shrink. So the imaginary part of the actual action should be rewritten as

$$\operatorname{Im} W' = -\int_{0,0}^{\omega,q} \frac{\pi[d(M-\omega)-d(Q-q)]\left[(M-\omega)^2+(M-\omega)\Xi-\frac{1}{2}(Q-q)^2\right]}{\Xi}, \tag{28}$$

where $V_0' = (Q-q)/r_+'$, $r_+' = r_+(M-\omega, Q-q)$ and

$$\Xi = \sqrt{(M-\omega)^2-(Q-q)^2}. \tag{29}$$

To get the final result, one can integrate it directly. But recalling the differential form of the first law of black hole thermodynamics

$$dM = TdS + VdQ + \Omega dJ, \tag{30}$$

and the Hawking temperature in Eq. (27), one can easily get

$$\operatorname{Im} W' = -\int_{s_i}^{s_f} \frac{1}{4} ds = -\frac{\Delta S}{4}, \tag{31}$$

in which

$$\Delta S = S(r_f) - S(r_i) = 2\pi\left[(M-\omega)^2 - M^2 + (M-\omega)\sqrt{(M-\omega)^2-(Q-q)^2} - M\sqrt{M^2-Q^2} + \frac{1}{2}(Q-q)^2 - \frac{1}{2}Q^2\right], \tag{32}$$

is the change of Bekenstein–Hawking entropy. Where we have invoked the formalism of entropy of the Reissner–Nordström black hole

$$S = \frac{1}{4}A = \pi r_+^2 = 2\pi \left[M^2 + M\sqrt{M^2 - Q^2} - \frac{1}{2}Q^2 \right]. \tag{33}$$

Thus the total tunneling probability of the emitted spin particles is

$$\Gamma \sim \exp(\Delta S). \tag{34}$$

This result is consistent with the initial viewpoint of Parikh and Wilczek that the tunneling rate is related to the change of Bekenstein–Hawking entropy. The difference of the entropy of the black hole before and after the radiation can be expanded as

$$\Delta S = \frac{dS}{dr_+}\Delta r_+ + \frac{1}{2!}\frac{d^2S}{dr_+^2}(\Delta r_+)^2 + \frac{1}{3!}\frac{d^3S}{dr_+^3}(\Delta r_+)^3 + \cdots \tag{35}$$

Employing Eq. (33) and the expression of event horizon, we find

$$\frac{dS}{dr_+} = 2\pi r_+, \quad \Delta r_+ = \left(\frac{M + \sqrt{M^2 - Q^2}}{\sqrt{M^2 - Q^2}} \right) \Delta M - \frac{Q\Delta Q}{\sqrt{M^2 - Q^2}}. \tag{36}$$

Subtitling Eq. (36) into Eq. (34) and considering the changes of mass and charge of the hole $\Delta M = -\omega$ and $\Delta Q = -q$, one would get

$$\Gamma \sim \exp(\Delta S) = \exp\left[-\beta(\omega - \omega_0)\left(1 - \frac{1}{2!\beta(\omega - \omega_0)}\frac{d^2S}{dr_+^2}(\Delta r_+)^2 \right) \right], \tag{37}$$

where $\beta = 1/T$ is the inverse temperature. When the higher order term of $\omega - \omega_0$ that may include more information is ignored, one can obtain the pure thermal spectrum of the Reissner–Nordström black hole. That is, after the energy conservation and charge conservation are considered in the unfixed background space–time, the actual radiation spectrum of fermions derivates from the pure thermal one.

4 Conclusions

We have extended Kerner–Mann's work on fermions tunneling to the charged black hole. Due to the coupling among mater field, gravity field and electromagnetic field, we introduced the Dirac equation of charged particles. As an example, we discussed the tunneling radiation of fermions with and without mass from the Reissner–Nordström black hole. To picture the tunneling event of fermions precisely, we further considered the self-gravitational interaction and back reaction of the emitted spin particles. Our result shows that when the unfixed background space–time is taken into account, the tunneling probability of spin $1/2$ particle is related to the change of the Bekenstein–Hawking entropy. This result agrees with that of the scalar particles.

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References

1. Parikh, M.K., Wiclzek, F.: Hawking radiation as tunneling. Phys. Rev. Lett. **85**, 5042 (2000)
2. Parikh, M.K.: New Coordinates for de Sitter Space and de Sitter Radiation. Phys. Lett. B **546**, 189 (2000)
3. Angheben, M., Nadalini, M., Vanzo, L., Zerbini, S.: Hawking radiation as tunneling for extremal and rotating Black holes. JHEP **05**, 014 (2005)
4. Medved, A.J.M., Vagenas, E.: On Hawking radiation as tunneling with back-reaction. Mod. Phys. Lett. A **20**, 2449 (2005)
5. Kim, S.P.: Hawking radiation as quantum tunneling in rindler coordinate. arXiv:0710.0915[hep-th]
6. Kim, S.P.: Schwinger mechanism and Hawking radiation as quantum tunneling. arXiv:0709.4313 [hep-th]
7. Liu, W.B.: Damour–Ruffini method and the non-thermal Hawking radiation of Reissner–Nordström black hole. Acta. Phys. Sin. **56**, 6164 (2007) (in Chinese)
8. Zhang, J.Y., Zhao, Z.: New coordinates for Kerr–Newman black hole radiation. Phys. Lett. B **618**, 14 (2005)
9. Zhang, J.Y., Zhao, Z.: Massive particles black hole tunneling and de Sitter tunneling. Nucl. Phys. B **725**, 173 (2005)
10. Yang, S.Z.: Kerr–Newman–Kasuya black hole tunneling radiation. Chin. Phys. Lett. **22**, 2492 (2005)
11. Han, Y.W.: Massive particles tunneling radiation from the black hole with a mass-quadruple moment. Chin. Phys. **16**, 923 (2007)
12. Liu, W.B.: New coordinates for BTZ black hole and Hawking radiation via tunneling. Phys. Lett. B **99**, 379 (2006)
13. Yang, S.Z., Jiang, Q.Q., Li, H.L.: Quantum tunnelling radiation of Einstein–Maxwell-dolation-Axion black hole. Chin. Phys. **14**, 2411 (2005)
14. Chen, D.Y., Yang, S.Z.: Charged particle tunnels from the stationary and non-stationary Kerr–Newman black holes. Gen. Relat. Grav. **39**, 1503 (2007)
15. Li, H.L., Qi, D.J., Yang, S.Z.: Massive particle's tunneling radiation from higher dimensional Schwarzschild de sitter and Anti-de sitter black holes. Gen. Relat. Grav. **39**, 179 (2007)
16. Page, D.: Particle emission rates from a black hole: massless particles from an uncharged, non-rotating hole. Phys. Rev. D **13**, 198 (1976)
17. Kerner, R., Mann, R.B.: Fermions tunneling from black holes. arXiv:0710.0612 [hep-th]
18. Srinivasan, K., Padmanabhan, T.: Particle production and complex path analysis. Phys. Rev. D **60**, 24007 (1999)