Gen Relativ Gravit (2007) 39:2149–2203
DOI 10.1007/s10714-007-0507-2

RESEARCH ARTICLE

The York map as a Shanmugadhasan canonical transformation in tetrad gravity and the role of non-inertial frames in the geometrical view of the gravitational field

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Received: 16 October 2006 / Accepted: 14 August 2007 / Published online: 25 September 2007

Abstract A new parametrization of the 3-metric allows to find explicitly a York map by means of a partial Shanmugadhasan canonical transformation in canonical ADM tetrad gravity. This allows to identify the two pairs of physical tidal degrees of freedom (the Dirac observables of the gravitational field have to be built in term of them) and 14 gauge variables. These gauge quantities, whose role in describing generalized inertial effects is clarified, are all configurational except one, the York time, i.e. the trace ${}^3K(\tau,\vec{\sigma})$ of the extrinsic curvature of the instantaneous 3-spaces Σ_τ (corresponding to a clock synchronization convention) of a non-inertial frame centered on an arbitrary observer. In Σ_τ the Dirac Hamiltonian is the sum of the weak ADM energy $E_{\mathrm{ADM}} = \int d^3\sigma\, \mathcal{E}_{\mathrm{ADM}}(\tau,\vec{\sigma})$ (whose density $\mathcal{E}_{\mathrm{ADM}}(\tau,\vec{\sigma})$ is coordinate-dependent, containing the inertial potentials) and of the first-class constraints. The main results of the paper, deriving from a coherent use of constraint theory, are: (i) The explicit form of the Hamilton equations for the two tidal degrees of freedom of the gravitational field in an arbitrary gauge: a deterministic evolution can be defined only in a completely fixed gauge, i.e. in a non-inertial frame with its pattern of inertial forces. The simplest such gauge is the 3-orthogonal one, but other gauges are discussed and the Hamiltonian interpretation of the harmonic gauges is given. This frame-dependence derives from the geometrical view of the gravitational field and is lost when the theory is reduced to a linear spin 2 field on a background space-time. (ii) A general solution of the supermomentum constraints, which shows the existence of a generalized Gribov ambiguity

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associated to the 3-diffeomorphism gauge group. It influences: (a) the explicit form of the solution of the super-momentum constraint and then of the Dirac Hamiltonian; (b) the determination of the shift functions and then of the lapse one. (iii) The dependence of the Hamilton equations for the two pairs of dynamical gravitational degrees of freedom (the generalized tidal effects) and for the matter, written in a completely fixed 3-orthogonal Schwinger time gauge, upon the gauge variable ${}^3K(\tau, \vec{\sigma})$, determining the convention of clock synchronization. The associated relativistic inertial effects, absent in Newtonian gravity and implying inertial forces changing from attractive to repulsive in regions with different sign of ${}^3K(\tau, \vec{\sigma})$, are completely unexplored and may have astrophysical relevance in the interpretation of the dark side of the universe.

1 Introduction

In a series of papers a new formulation of canonical metric [1] and tetrad [2,3] gravity both based on the ADM action[1] was given with the aim [4–7] to identify the Dirac observables of the gravitational field (the generalized *tidal* effects) after having separated them from the gauge variables (the generalized *inertial* effects) by using the Shanmugadhasan canonical transformation [8,9] adapted to the first class constraints of the theory.

The formulation was given in a family of *non-compact* space-times M^4 with the following properties:

(i) *globally hyperbolic* and *topologically trivial*, so that they can be foliated with space-like hyper-surfaces Σ_τ diffeomorphic to R^3 ($3+1$ splitting of space-time with τ, the scalar parameter labeling the leaves, as a *mathematical time*);

(ii) *asymptotically flat at spatial infinity* and with boundary conditions at spatial infinity independent from the direction, so that the *spi group* of asymptotic symmetries is reduced to the Poincare' group with the ADM Poincare' charges as generators. In this way we can eliminate the *super-translations*, namely the obstruction to define angular momentum in general relativity, and we have the same type of boundary conditions which are needed to get well defined non-Abelian charges in Yang-Mills theory, opening the possibility of a unified description of the four interactions with all the fields belonging to same function space [1,10,11]. All these requirements imply that the *admissible foliations* of space-time must have the space-like hyper-surfaces tending in

[1] Tetrad gravity is more natural for the coupling to the fermions. This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer's gyroscopes. Tetrad gravity has ten primary first class constraints and four secondary first class ones. Six of the primary constraints describe the extra freedom in the choice of the tetrads. The other four primary (the vanishing of the momenta of the lapse and shift functions) and four secondary (the super-Hamiltonian and super-momentum constraints) constraints are the same as in metric gravity. In Ref. [3] 13 of the 14 constraints were solved: the super-Hamiltonian one can be solved only after linearization.

a direction-independent way to Minkowski space-like hyper-planes at spatial infinity, which moreover must be orthogonal there to the ADM 4-momentum. Therefore, M^4 is *asymptotically Minkowskian* [13]. Moreover the simultaneity 3-surfaces must admit an involution (Lichnerowicz 3-manifolds [14,15]) allowing the definition of a generalized Fourier transform with its associated concepts of positive and negative energy, so to avoid the claimed impossibility to define particles in curved space-times.

(iii) All the fields have to belong to suitable *weighted Sobolev spaces* so that; (i) the admissible space-like hyper-surfaces are Riemannian 3-manifolds without asymptotically vanishing Killing vectors [13,16,17] (we furthermore assume the absence of any Killing vector); (ii) the inclusion of particle physics leads to a formulation without Gribov ambiguity [18–20].

In absence of matter the class of Christodoulou–Klainermann space-times [12], admitting asymptotic ADM Poincare' charges and an asymptotic flat metric is selected.

This formulation, the *rest-frame instant form of metric and tetrad gravity*, emphasizes the role of *non-inertial frames* (the only ones existing in general relativity due to the global interpretation of the *equivalence principle*; see Ref. [4–7] for this viewpoint) and *deparametrizes* to the rest-frame instant form of dynamics in Minkowski space-time [4–7,10,11] when matter is present if the Newton constant is switched off. The non-inertial frames are the $3+1$ splittings admissible for the given space-time, after having chosen an arbitrary time-like observer as the origin of the 3-coordinates on the leaves Σ_τ, which are both [4–7] Cauchy surfaces and instantaneous 3-spaces corresponding to a convention for the synchronization of distant clocks.[2] As a consequence the $3+1$ splitting identifies a global non-inertial frame centered on the observer, namely a possible extended physical laboratory with its metrological conventions.

As shown in Refs. [1,3] in this way one gets the *rest-frame instant form of metric and tetrad gravity* with the weak ADM energy $E_{\mathrm{ADM}} = \int d^3\sigma\, \mathcal{E}_{\mathrm{ADM}}(\tau, \vec{\sigma})$ as the effective Hamiltonian (in accord with Refs. [13,24]).[3] The Γ-Γ term in the ADM energy density $\mathcal{E}_{\mathrm{ADM}}(\tau, \vec{\sigma})$ is coordinate-dependent (the problem of energy in general relativity) because it contains the *inertial potentials* giving rise to the generalized inertial effects in the non-inertial frame associated to the chosen $3+1$ splitting of the space-time.

In Ref. [25] there is the study of the Hamiltonian linearization of tetrad gravity without matter in these space-times, where the existence of an asymptotic flat metric at spatial infinity (*asymptotic background* with the presence of asymptotic inertial

[2] See Ref. [21] for the special relativistic case and Ref. [22,23] for the quantization of particles in non-inertial frames.

[3] Therefore the formulations with a frozen reduced phase space are avoided [4–7]. The super-Hamiltonian constraint generates *normal* deformations of the space-like hyper-surfaces, which are *not* interpreted as a time evolution (like in the Wheeler–DeWitt approach) but as the Hamiltonian gauge transformations ensuring that the description of gravity is independent from the $3+1$ splittings of space-time (i.e. from the clock synchronization convention) like it happens in parametrized Minkowski theories.

observers to be identified with the fixed stars) allows to avoid the splitting of the 4-metric into the flat one plus a perturbation. In this way we have obtained *post-Minkowskian background-independent gravitational waves* in a special non-harmonic 3-orthogonal gauge where the 3-metric is diagonal. As a consequence these space-times, after the inclusion of matter, are candidates for a general relativistic model of the solar system or of the galaxy. Maybe they can also be used in the cosmological context if the asymptotic inertial observers are identified with the preferred observers of the cosmic background radiation.

In Refs. [26–28] there is the description of relativistic fluids and of the Klein–Gordon field in the framework of parametrized Minkowski theories. This formalism allows to get the Lagrangian of these matter systems in the formulation of tetrad gravity of Refs. [2,3,25]. The resulting first-class constraints depend only on the mass density $\mathcal{M}(\tau, \vec{\sigma})$ (which is metric-dependent) and the mass-current density $\mathcal{M}_r(\tau, \vec{\sigma})$ (which is metric-independent) of the matter. For Dirac fields the situation is more complicated due to the presence of second class constraints (see Ref. [29] for the case of parametrized Minkowski theories with fermions). It turns out that the point Shanmugadhasan canonical transformation of Ref. [25], adapted to 13 of the 14 first class constraints is not suited for the inclusion of matter due to its *non-locality*. Therefore in this paper we will look for a local point Shanmugadhasan transformation adapted only to 10 of the 14 constraints.

The new insight comes from the so-called York-Lichnerowicz conformal approach [30–33] (see also the book [34] for a review and more bibliographical information) to metric gravity in globally hyperbolic (*but spatially compact*[4]) space-times. The starting point is the decomposition ${}^3g_{ij} = \phi^4\, {}^3\hat{g}_{ij}$ of the 3-metric on an instantaneous 3-space Σ_o of a 3 + 1 splitting of space-time in the product of a *conformal factor* $\phi = (\det {}^3g)^{1/12}$ and a *conformal 3-metric* ${}^3\hat{g}_{ij}$ with $\det {}^3\hat{g}_{ij} = 1$ (${}^3\hat{g}_{ij}$ contains 5 of the 6 degrees of freedom of ${}^3g_{ij}$). The extrinsic curvature 3-tensor ${}^3K_{ij}$ of Σ_o (determining the ADM momentum) is decomposed in its trace ${}^3K$ (the *York time*) plus the *distorsion tensor*, which is the sum of a TT[5] symmetric 2-tensor ${}^3A_{ij}$ (2 degrees of freedom) plus the 3-tensor ${}^3W_{i;j} + {}^3W_{j;i} - \frac{2}{3}\, {}^3g_{ij}\, {}^3W^k{}_{;k}$ depending on a covariant 3-vector ${}^3W_i$ (*York gravitomagnetic vector potential*; 3 degrees of freedom). Having fixed the lapse and shift functions of the 3 + 1 splitting and having put ${}^3K = const.$, one assigns ${}^3\hat{g}_{ij}$ and ${}^3A_{ij}$ on the Cauchy surface Σ_o. Then, ${}^3W_i$ is determined by the super-momentum constraints on Σ_o and ϕ is determined by the super-Hamiltonian constraint on Σ_o. Then, the remaining Einstein's equations (see Refs. [12,16,17,30,31] for the existence and unicity of solutions) determine the time derivatives of ${}^3g_{ij}$ and of ${}^3K_{ij}$, allowing to find the time development from the initial data on Σ_o.

[4] This is due to the influence of Mach principle, see for instance Chapter 5 of Ref. [34]. However, let us remark that the non-locality of the Dirac observables of the non-compact case (all the instantaneous 3-space is needed for their determination) has a Machian flavor.

[5] Traceless and transverse with respect to the conformal 3-metric.

However, a canonical basis adapted the the previous splittings was never found. The only result is contained in Ref. [35], where it was shown that, having fixed ${}^3K$, the transition from the non-canonical variables ${}^3\hat{g}_{ij}$, ${}^3A_{ij}$, ${}^3W_i$ to the space of the gravitational initial data satisfying the constraints is a canonical transformation, named *York map*.

In this paper we will show that a new parametrization of the original 3-metric ${}^3g_{ij}$ allows to find local point Shanmugadhasan canonical transformation adapted to 10 of the 14 constraints of tetrad gravity, which implements a York map. In particular one of the new momenta (a gauge variable) will be the York time ${}^3K$. The use of Dirac theory of constraints introduces a different point of view on the gauge-fixing and the Cauchy problem. While the gauge fixing to the extra six primary constraints fixes the tetrads (i.e. the spatial gyroscopes and their transport law), the gauge fixing to the four primary plus four secondary constraints follows a different scheme from the one used in the York-Lichnerowicz approach, which influenced contemporary numerical gravity. Firstly one adds the four gauge fixings to the secondary constraints (the super-Hamiltonian and super-momentum ones), i.e. one fixes ${}^3K$, i.e.the simultaneity 3-surface, and the 3-coordinates on it (namely 3 of the 5 degrees of freedom of the conformal 3-metric ${}^3\hat{g}_{ij}$). The preservation in time of these four gauge fixings generates other four gauge fixing constraints determining the lapse and shift functions consistently with the shape of the simultaneity 3-surface and with the choice of 3-coordinates on it (here is the main difference with the conformal approach and most of the approaches to numerical gravity). While the super-Hamiltonian constraint determines the conformal factor ϕ,[6] the super-momentum constraint determines 3 momenta (replacing the York gravitomagnetic potential ${}^3W_i$). The remaining 2 + 2 degrees of freedom (the genuine tidal effects) are the other two degrees of freedom in ${}^3\hat{g}_{ij}$ and the two ones inside ${}^3A_{ij}$. On the Cauchy surface the 2 + 2 tidal degrees of freedom are assigned and we have consistency with the initial data of the York-Lichnerowicz approach.

This is the natural procedure of fixing the gauge and of getting deterministic Hamilton equations for the tidal degrees of freedom according to Dirac theory of constraints. Since a completely fixed gauge is equivalent to give a non-inertial frame centered on some time-like observer, the gauge-fixed gauge variables will describe the inertial effects (the *appearances* of phenomena) present in this non-inertial frame, where the Dirac observables describe the tidal effects of the gravitational field. In particular, the gauge variable ${}^3K(\tau, \vec{\sigma})$ (York time) describes the freedom in the choice of the clock synchronization convention, i.e. in the definition of the instantaneous 3-spaces Σ_τ.

In Sect. 2 we will find the York map and we discuss some classes of Hamiltonian gauges. In the York canonical basis it is possible to express both the gauge variables (inertial effects) and the tidal degrees of freedom in terms of the original variables.

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[6] The only role of the conformal decomposition ${}^3g_{ij} = \phi^4\, {}^3\hat{g}_{ij}$ is to identify the conformal factor ϕ as the natural unknown in the super-Hamiltonian constraint, which becomes the *Lichnerowicz equation*. See Ref. [1] for a different justification of this result based on constraint theory and the two notions of strong and weak ADM energy.

In Sect. 3 we give the general solution of the super-momentum constraints in the York canonical basis: it is defined modulo the *zero modes* of the covariant derivative.

In Sect. 4 there is the form of the super-Hamiltonian constraint and the weak ADM energy in a family of completely fixed 3-orthogonal Schwinger time gauges parametrized by the gauge variable ${}^3K(\tau, \vec{\sigma})$ (it is a family of non-inertial frames with a fixed pattern of inertial effects) defined by suitable set of primary gauge-fixing constraints.

In Sect. 5 there are the equations determining the lapse and shift functions of the 3-orthogonal gauges gauges: they arise from the preservation in time of the primary gauge-fixing constraints. It is shown that, like in Yang-Mills theories, a generalized Gribov ambiguity arises in the determination of the shift functions and, as a consequence, also of the lapse function. It is induced by the zero modes of the covariant derivative.

In Sect. 6 there are the final Hamilton equations describing the deterministic evolution of the dynamical gravitational degrees of freedom (the generalized tidal effects) both in an arbitrary gauge and in the completely fixed ones. It is shown that only in a completely fixed gauge we can obtain a deterministic evolution, which, however depends upon the chosen non-inertial frame with its pattern of relativistic inertial forces. This frame-dependence derives from the geometrical view of the gravitational field and is lost when the theory is reduced to a linear spin 2 field on a background space-time.

In the Conclusions we make a summary of the results, emphasizing the methodological and interpretational insights induced by a correct use of constraint's theory. We also make some comments on the perspectives of technical developments (for instance the weak field limit but with relativistic motion) in connection with physical problems connected with space experiments in the solar system and with astrophysics.

Finally there are four Appendices: Appendix A, with the notations for tetrad gravity; Appendix B, with the calculations for the canonical transformation of Sect. 3; Appendix C, with the 3-geometry in 3-orthogonal gauges; Appendix D, with Green functions.

In Ref. [36] there is an expanded version of this paper, containing many Appendices with the explicit expression of many quantities in the York canonical basis and in the 3-orthogonal gauge. To have an exposition concentrated on the main theoretical aspects implied by a coherent and systematic use of constraint's theory and on the interpretational issues, we have not given the explicit expression of heavy calculations and of some cumbersome formulas. They can be found in Ref. [36] at the quoted positions.

2 The York map from a Shanmugadhasan canonical transformation adapted only to the rotation constraints

Let us look for a Shanmugadhasan canonical transformation interpretable as a York map. It can be obtained starting from a natural parametrization of the 3-metric and then by making an adaptation only to the rotation constraints (and not also the the super-momentum ones like in Refs. [3,25]).

2.1 Diagonalization of the 3-metric

The 3-metric ${}^3g_{rs}$ may be diagonalized with an *orthogonal* matrix $V(\theta^r)$, $V^{-1} = V^T$, $\det V = 1$, depending on 3 Euler angles θ^r [7]

$$
\begin{aligned}
{}^3g_{rs} &= \sum_{uv} V_{ru}(\theta^n)\,\lambda_u\,\delta_{uv}\,V^T_{vs}(\theta^n) = \sum_a \left(V_{ra}(\theta^n)\,\Lambda^a\right)\left(V_{sa}(\theta^n)\,\Lambda^a\right)\\
&= \sum_a {}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(a)s} \sum_a {}^3e_{(a)r}\,{}^3e_{(a)s} = \phi^4\,{}^3\hat{g}_{rs} \stackrel{\text{def}}{=} \phi^4 \sum_a Q_a^2\,V_{ra}(\theta^n)\,V_{sa}(\theta^n),\\
\Lambda_a(\tau,\vec{\sigma}) &\stackrel{\text{def}}{=} \sum_u \delta_{au}\sqrt{\lambda_u(\tau,\vec{\sigma})} \stackrel{\text{def}}{=} \phi^2(\tau,\vec{\sigma})\,Q_a(\tau,\vec{\sigma}),\\
Q_a &\stackrel{\text{def}}{=} e^{\sum_{\bar{a}}^{1,2}\gamma_{\bar{a}a}R_{\bar{a}}},\quad R_{\bar{a}} = \sum_b \gamma_{\bar{a}b}\,ln\,\frac{\Lambda_b}{(\Lambda_1\Lambda_2\Lambda_3)^{1/3}},\\
\phi &= (\det{}^3g)^{1/12} = ({}^3e)^{1/6} = (\lambda_1\lambda_2\lambda_3)^{1/12} = (\Lambda_1\Lambda_2\Lambda_3)^{1/6},
\end{aligned}
\tag{2.1}
$$

where the set of numerical parameters $\gamma_{\bar{a}a}$ satisfies [1] $\sum_u \gamma_{\bar{a}u} = 0$, $\sum_u \gamma_{\bar{a}u}\gamma_{\bar{b}u} = \delta_{\bar{a}\bar{b}}$, $\sum_{\bar{a}} \gamma_{\bar{a}u}\gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}$. The assumed boundary conditions imply $\Lambda_a(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} 1 + \frac{M}{4r} + \frac{a_a}{r^{3/2}} + O(r^{-3})$ and $\phi(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} 1 + O(r^{-1})$.

Cotriads and triads are defined modulo rotations $R(\alpha_{(a)})$ on the flat 3-index (a)

$$
\begin{aligned}
{}^3e_{(a)r} &= R_{(a)(b)}(\alpha_{(c)})\,{}^3\bar{e}_{(b)r},\\
{}^3\bar{e}_{(a)r} &\stackrel{\text{def}}{=} \textstyle\sum_u \sqrt{\lambda_u}\,\delta_{u(a)}\,V^T_{ur}(\theta^n) = V_{ra}(\theta^n)\,\Lambda_a = \phi^2\,Q_a\,V_{ra}(\theta^n),\\
{}^3\bar{e}^r_{(a)} &= \textstyle\sum_u \frac{\delta_{u(a)}}{\sqrt{\lambda_u}}\,V_{ru} = \frac{V_{ra}(\theta^n)}{\Lambda_a} = \phi^{-2}\,Q_a^{-1}\,V_{ra}(\theta^n).
\end{aligned}
\tag{2.2}
$$

The gauge Euler angles θ^r give a description of the 3-coordinate systems on Σ_τ from a local point of view, because they give the orientation of the tangents to the 3 coordinate lines through each point (their conjugate momenta are determined by the super-momentum constraints), ϕ is the conformal factor of the 3-metric, i.e. the unknown in the super-Hamiltonian constraint (its conjugate momentum is a gauge variable, describing the form of the simultaneity surfaces Σ_τ), while the two independent eigenvalues of the conformal 3-metric ${}^3\hat{g}_{rs}$ (with determinant equal to 1) describe the genuine *tidal* effects of general relativity (the non-linear "graviton").

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[7] Due to the positive signature of the 3-metric, we define the matrix V with the following indices: V_{ru}. Since the choice of Shanmugadhasan canonical bases breaks manifest covariance, we will use the notation $V_{ua} = \sum_v V_{uv}\,\delta_{v(a)}$ instead of $V_{u(a)}$. We use the following types of indices: $a = 1, 2, 3$ and $\bar{a} = 1, 2$.

2.2 An intermediate point Shanmugadhasan canonical transformation

Let us consider the following point canonical transformation (realized in two steps)

$$\begin{array}{|c|c|c|c|}\hline \varphi_{(a)} & n & n_{(a)} & {}^3e_{(a)r} \\ \hline \approx 0 & \approx 0 & \approx 0 & {}^3\pi^r_{(a)} \\ \hline \end{array} \longrightarrow \begin{array}{|cc|cc|c|}\hline \varphi_{(a)} & \alpha_{(a)} & n & \bar{n}_{(a)} & {}^3\bar{e}_{(a)r} \\ \hline \approx 0 & \approx 0 & \approx 0 & \approx 0 & {}^3\tilde{\bar{\pi}}^r_{(a)} \\ \hline \end{array}$$

$$\longrightarrow \begin{array}{|cc|cc|cc|}\hline \varphi_{(a)} & \alpha_{(a)} & n & \bar{n}_{(a)} & \theta^r & \Lambda_r \\ \hline \approx 0 & \approx 0 & \approx 0 & \approx 0 & \pi_r^{(\theta)} & P^r \\ \hline \end{array} \tag{2.3}$$

where $\bar{n}_{(a)} = \sum_b n_{(b)}\, R_{(b)(a)}(\alpha_{(e)})$ are the shift functions at $\alpha_{(a)}(\tau,\vec{\sigma}) = 0$.

This is a *Shanmugadhasan canonical transformation adapted also to the rotation constraints*. It allows to separate the gauge variables $(\alpha_{(a)}, \varphi_{(a)})$ of the Lorentz gauge group acting on the tetrads.

Being a point transformation, we have

$$\begin{aligned} {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) &= \sum_b K^r_{(a)b}(\tau,\vec{\sigma})\, P^b(\tau,\vec{\sigma}) + \sum_i G^r_{(a)i}(\tau,\vec{\sigma})\, \pi_i^{(\theta)}(\tau,\vec{\sigma}) \\ &\quad + \sum_{(c)} F^r_{(a)(c)}(\tau,\vec{\sigma})\, \pi^{(\alpha)}_{(c)}(\tau,\vec{\sigma}) \\ &\approx \sum_b K^r_{(a)b}(\tau,\vec{\sigma})\, P^b(\tau,\vec{\sigma}) + \sum_i G^r_{(a)i}(\tau,\vec{\sigma})\, \pi_i^{(\theta)}(\tau,\vec{\sigma}). \end{aligned} \tag{2.4}$$

Here $\pi^{(\alpha)}_{(a)}(\tau,\vec{\sigma}) \approx 0$ are the Abelianized rotation constraints [1,3], canonically conjugate to $\alpha_{(a)}(\tau,\vec{\sigma})$.

Let us remark that the Shanmugadhasan canonical transformation identifying the York map is valid only in the configuration region where the 3-metric ${}^3g_{rs}(\tau,\vec{\sigma})$ has three distinct eigenvalues everywhere [i.e. $a_a \neq a_b$ for $a \neq b$ in the asymptotic behavior of Λ_a] except at spatial infinity, where they tend to the common value 1. The degenerate cases with two or three equal eigenvalues are *singular configurations* with less configurational degrees of freedom. To treat these cases we must add by hand extra first class constraints of the type $\Lambda_a(\tau,\vec{\sigma}) - \Lambda_b(\tau,\vec{\sigma}) \approx 0$, $a \neq b$, and apply the Dirac algorithm to the enlarged set of constraints.

The generating function of the canonical transformation is

$$\Phi = \int d^3\sigma \sum_{ar} {}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) \left[\sum_b R_{(a)(b)}(\alpha_{(e)})\, V_{rb}(\theta^n)\, \Lambda_b\right](\tau,\vec{\sigma}), \tag{2.5}$$

so that we get (see Ref. [3] for the O(3) Lie algebra-valued matrices $A_{(a)(b)}(\alpha_{(c)})$, $B(\alpha_{(c)}) = A^{-1}(\alpha_{(c)})$, such that $\frac{\partial\, R_{(b)(c)}(\alpha_{(e)})}{\partial\, \alpha_{(a)}} = \sum_{da} \epsilon_{(b)(d)(n)}\, R_{(d)(c)}(\alpha_{(e)})\, A_{(n)(a)}(\alpha_{(e)})$)

$$
\begin{aligned}
\pi^{(\alpha)}_{(c)}(\tau,\vec{\sigma}) &= \frac{\delta\,\Phi}{\delta\,\alpha_{(c)}(\tau,\vec{\sigma})} = -\sum_{krab}\Big[A_{(k)(c)}(\alpha_{(e)})\,\epsilon_{(k)(b)(a)}\,{}^3e_{(b)r}\,{}^3\tilde{\pi}^r_{(a)}\Big](\tau,\vec{\sigma})\\
&= -\sum_k\big[A_{(k)(c)}(\alpha_{(e)})\,M_{(k)}\big](\tau,\vec{\sigma})\approx 0,\\
P^b(\tau,\vec{\sigma}) &= \frac{\delta\,\Phi}{\delta\,\Lambda_b(\tau,\vec{\sigma})} = \sum_{ar}\Big[{}^3\tilde{\pi}^r_{(a)}\,R_{(a)(b)}(\alpha_{(e)})\,V_{rb}(\theta^n)\Big](\tau,\vec{\sigma})\\
&= \sum_{ar}\frac{{}^3\tilde{\pi}^r_{(a)}\,R_{(a)(b)}(\alpha_{(e)})\,{}^3\bar{e}_{(b)r}}{\Lambda_b}(\tau,\vec{\sigma}),\\
\pi^{(\theta)}_i(\tau,\vec{\sigma}) &= \frac{\delta\,\Phi}{\delta\,\theta^i(\tau,\vec{\sigma})} = -\sum_{lmra}\Big[A_{mi}(\theta^n)\,\epsilon_{mlr}\,{}^3e_{(a)l}\,{}^3\tilde{\pi}^r_{(a)}\Big](\tau,\vec{\sigma}).
\end{aligned}
\tag{2.6}
$$

As a consequence of the calculation of Appendix B we have [Eqs. (A10) are used; ${}^3\tilde{\bar{\pi}}^r_{(a)}$ and ${}^3\tilde{K}_{rs}$ are the cotriad momentum and the extrinsic curvature of Σ_τ after having used the rotation constraints $M_{(a)}(\tau,\vec{\sigma})\approx 0$]

$$
\begin{aligned}
{}^3\tilde{\pi}^r_{(a)} &\overset{\text{def}}{=} \sum_b R_{(a)(b)}(\alpha_{(e)})\,\bar{\pi}^r_{(b)},\\
{}^3\bar{\pi}^r_{(a)} &= \sum_b \pi^r_{(b)}\,R_{(b)(a)}(\alpha_{(e)})\\
&= V_{ra}(\theta^n)\,P^a + \sum_b^{b\neq a}\sum_{twi}\frac{V_{rb}(\theta^n)\,\epsilon_{abt}\,V_{tw}(\theta^n)}{\Lambda_b\left(\frac{\Lambda_b}{\Lambda_a}-\frac{\Lambda_a}{\Lambda_b}\right)}\,B_{iw}(\theta^n)\,\pi^{(\theta)}_i\\
&\quad -\sum_b^{b\neq a}\sum_{kti}\frac{V_{rb}(\theta^n)\,\epsilon_{bat}\,R_{(t)(k)}(\alpha_{(e)})}{\Lambda_b\left(\frac{\Lambda_b}{\Lambda_a}-\frac{\Lambda_a}{\Lambda_b}\right)}\,B_{(c)(k)}(\alpha_{(e)})\,\pi^{(\alpha)}_{(c)}\\
&\approx V_{ra}(\theta^n)\,P^a + \sum_b^{b\neq a}\sum_{twi}\frac{V_{rb}(\theta^n)\,\epsilon_{abt}\,V_{tw}(\theta^n)}{\Lambda_b\left(\frac{\Lambda_b}{\Lambda_a}-\frac{\Lambda_a}{\Lambda_b}\right)}\,B_{iw}(\theta^n)\,\pi^{(\theta)}_i\\
&\overset{\text{def}}{=} {}^3\tilde{\bar{\pi}}^r_{(a)} \quad \rightarrow_{\theta^n\rightarrow 0}\ \delta_{ra}\,P^a + \frac{\epsilon_{ari}}{\Lambda_r\left(\frac{\Lambda_r}{\Lambda_a}-\frac{\Lambda_a}{\Lambda_r}\right)}\,\pi^{(\theta)}_i.
\end{aligned}
$$

$$
\begin{aligned}
{}^3K_{rs} &\approx {}^3\tilde{K}_{rs} = \frac{\epsilon\,4\pi\,G}{c^3\,\Lambda_1\,\Lambda_2\,\Lambda_3}\Bigg[\sum_a \Lambda_a^2\,V_{ra}(\theta^n)\,V_{sa}(\theta^n)\,(2\,\Lambda_a\,P^a - \sum_b \Lambda_b\,P^b)\\
&\quad + \sum_{ab}^{a\neq b}\Lambda_a\,\Lambda_b\,\big(V_{ra}(\theta^n)\,V_{sb}(\theta^n) + V_{rb}(\theta^n)\,V_{sa}(\theta^n)\big)\\
&\quad \times\sum_{twi}\frac{\epsilon_{abt}\,V_{tw}(\theta^n)\,B_{iw}(\theta^n)\,\pi^{(\theta)}_i}{\frac{\Lambda_b}{\Lambda_a}-\frac{\Lambda_a}{\Lambda_b}}\Bigg],\\
{}^3K &\approx {}^3\tilde{K} = -\epsilon\,\frac{4\pi\,G}{c^3}\,\frac{\sum_b \Lambda_b\,P^b}{\Lambda_1\,\Lambda_2\,\Lambda_3}.
\end{aligned}
\tag{2.7}
$$

Since in Ref. [3] it was assumed ${}^3\tilde{\pi}^r_{(a)}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-5/2})$, from Eqs. (2.4) and Appendix B we get $P^a(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-5/2})$. However we must have $\pi^{(\theta)}_i(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-4})$, since the requirement $\Lambda_a(\tau,\vec{\sigma}) \neq \Lambda_b(\tau,\vec{\sigma})$ for $a \neq b$, needed to avoid singularities, implies $a_a \neq a_b$ for $a \neq b$ in their asymptotic behavior, so that we get $\left(\frac{\Lambda_b}{\Lambda_a} - \frac{\Lambda_a}{\Lambda_b}\right)^{-1}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} \frac{r^{3/2}}{2\,(a_b - a_a)}$. As a consequence, consistently with Eqs. (2.6), we have $\pi^{(\alpha)}_{(a)}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-5/2})$. Also the angles $\alpha_{(a)}(\tau,\vec{\sigma})$ and $\theta^i(\tau,\vec{\sigma})$ must tend to zero in a direction-independent way at spatial infinity.

2.3 The York map

Since from Eq. (2.7) we have ${}^3\tilde{K} = -\epsilon\, \frac{4\pi\, G}{c^3}\, \frac{\sum_b \Lambda_b\, P^b}{\Lambda_1\,\Lambda_2\,\Lambda_3}$, we can introduce the following pair $\tilde{\phi}$, $\pi_{\tilde{\phi}}$ of canonical variables

$$\tilde{\phi} = \phi^6 = \prod_a \Lambda_a, \quad \pi_{\tilde{\phi}} = -\epsilon\, \frac{c^3}{12\pi\, G}\, {}^3K = \frac{\sum_b \Lambda_b\, P^b}{3\,\Lambda_1\,\Lambda_2\,\Lambda_3},$$
$$\{\tilde{\phi}(\tau,\vec{\sigma}), \pi_{\tilde{\phi}}(\tau,\vec{\sigma}^{'})\} = \delta^3(\vec{\sigma},\vec{\sigma}^{'}), \tag{2.8}$$

with $\pi_{\tilde{\phi}}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-5/2})$ at spatial infinity.

Let us consider the following point canonical transformation (it is a family of canonical transformations depending on the set of numerical parameters $\gamma_{\bar{a}a}$)

$$\boxed{\begin{array}{c} \Lambda_r \\ \hline P^r \end{array}} \longrightarrow \boxed{\begin{array}{c|c} \tilde{\phi} & R_{\bar{a}} \\ \hline \pi_{\tilde{\phi}} & \Pi_{\bar{a}} \end{array}} \tag{2.9}$$

Since the generating function is $\Psi = \int d^3\sigma \left[\sum_b P^b\, \tilde{\phi}^{1/3}\, e^{\sum_{\bar{a}} \gamma_{\bar{a}b}\, R_{\bar{a}}}\right](\tau,\vec{\sigma})$, we get

$$\begin{aligned} \pi_{\tilde{\phi}}(\tau,\vec{\sigma}) &= \frac{\delta\,\Psi}{\delta\tilde{\phi}(\tau,\vec{\sigma})} = \frac{\sum_b \Lambda_b\, P^b}{3\,\Lambda_1\,\Lambda_2\,\Lambda_3}(\tau,\vec{\sigma}),\\ \Pi_{\bar{a}}(\tau,\vec{\sigma}) &= \frac{\delta\,\Psi}{\delta\, R_{\bar{a}}(\tau,\vec{\sigma})} = \left[(\Lambda_1\,\Lambda_2\,\Lambda_3)^{1/3} \sum_b \gamma_{\bar{a}b}\, P^b\, e^{\sum_{\bar{c}} \gamma_{\bar{c}b}\, R_{\bar{c}}}\right](\tau,\vec{\sigma}). \end{aligned} \tag{2.10}$$

Therefore, besides the definitions in Eqs. (2.1), we get

$$P^b = \tilde{\phi}^{-1/3}\, Q_b^{-1}\left[\tilde{\phi}\,\pi_{\tilde{\phi}} + \sum_{\bar{b}} \gamma_{\bar{b}b}\,\Pi_{\bar{b}}\right],$$
$$\Pi_{\bar{a}} = (\Lambda_1\,\Lambda_2\,\Lambda_3)^{1/3} \sum_b \gamma_{\bar{a}b}\, P^b\, Q_b = \sum_b \gamma_{\bar{a}b}\,\Lambda_b\, P^b,$$

$$\sum_b \Lambda_b\, P^b = 3\, \tilde{\phi}\, \pi_{\tilde{\phi}}, \quad \sum_{\bar{a}} \gamma_{\bar{a}u}\, \Pi_{\bar{a}} = \Lambda_u\, P^u - \tilde{\phi}\, \pi_{\tilde{\phi}},$$

$$\begin{aligned}
{}^3\tilde{\bar{\pi}}^r_{(a)} &= \tilde{\phi}^{-1/3} \Bigg[V_{ra}(\theta^n)\, Q_a^{-1}\, (\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar{b}} \gamma_{\bar{b}a}\, \Pi_{\bar{b}}) \\
&\quad + \sum_l^{l \neq a} \sum_{twi} Q_l^{-1}\, \frac{V_{rl}(\theta^n)\, \epsilon_{alt}\, V_{tw}(\theta^n)}{Q_l\, Q_a^{-1} - Q_a\, Q_l^{-1}}\, B_{iw}(\theta^n)\, \pi_i^{(\theta)} \Bigg],
\end{aligned}$$

$$\sum_r {}^3\bar{e}_{(b)r}\, {}^3\tilde{\bar{\pi}}^r_{(a)} = \delta_{ab}\, [\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar{a}} \gamma_{\bar{a}a}\, \Pi_{\bar{a}}] + \sum_{twi} \frac{\epsilon_{abt}\, V_{tw}(\theta^n)\, B_{iw}(\theta^n)\, \pi_i^{(\theta)}}{Q_b\, Q_a^{-1} - Q_a\, Q_b^{-1}},$$

$$\begin{aligned}
{}^3\tilde{K}_{rs} &= \epsilon\, \frac{4\pi\, G}{c^3}\, \tilde{\phi}^{-1/3} \Bigg(\sum_a Q_a^2\, V_{ra}(\theta^n)\, V_{sa}(\theta^n) \left[2 \sum_{\bar{b}} \gamma_{\bar{b}a}\, \Pi_{\bar{b}} - \tilde{\phi}\, \pi_{\tilde{\phi}} \right] \\
&\quad + \sum_{ab} Q_a\, Q_b\, [V_{ra}(\theta^n)\, V_{sb}(\theta^n) + V_{rb}(\theta^n)\, V_{sa}(\theta^n)] \\
&\quad \times \sum_{twi} \frac{\epsilon_{abt}\, V_{tw}(\theta^n)\, B_{iw}(\theta^n)\, \pi_i^{(\theta)}}{Q_b\, Q_a^{-1} - Q_a\, Q_b^{-1}} \Bigg).
\end{aligned} \tag{2.11}$$

See Appendix B of Ref. [36] for the explicit expression of the 3-Christoffel symbols ${}^3\Gamma^r_{uv}$ on Σ_τ and for the Γ-Γ potential $\mathcal{S}(\tau, \vec{\sigma})$ in the York canonical basis.

The sequence of canonical transformations (2.3) and (2.9) realize a *York map* because the gauge variable $\pi_{\tilde{\phi}}$ is proportional to *York internal extrinsic time* ${}^3K$. Its conjugate variable, to be determined by the super-Hamiltonian constraint, is $\tilde{\phi} = {}^3\bar{e}$, which is proportional to *Misner's internal intrinsic time*; moreover $\tilde{\phi}$ is the *volume density* on Σ_τ: $V_R = \int_R d^3\sigma\, \phi^6$, $R \subset \Sigma_\tau$.

Equations (2.3), (2.9) and (2.11) identify the two pairs of canonical variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$, $\bar{a} = 1, 2$, as those describing the generalized *tidal effects*, namely the independent degrees of freedom of the gravitational field In particular the configuration tidal variables $R_{\bar{a}}$ depend *only on the eigenvalues of the 3-metric*. They are Dirac observables *only* with respect to the gauge transformations generated by 10 of the 14 first class constraints. Let us remark that, if we fix completely the gauge and we go to Dirac brackets, then the only surviving dynamical variables $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ become two pairs of *non canonical* Dirac observables for that gauge: the two pairs of canonical Dirac observables have to be found as a Darboux basis of the copy of the reduced phase space identified by the gauge and they will be (in general non-local) functionals of the $R_{\bar{a}}$, $\Pi_{\bar{a}}$ variables. This shows the importance of canonical bases like the York one: the tidal effects are described by *local* functions of the 3-metric and its conjugate momenta.

Since the variables $\tilde{\phi}$ [given in Eq. (2.8)] and $\pi_i^{(\theta)}$ [given in Eqs. (2.6)] are determined by the super-Hamiltonian and super-momentum constraints, the *arbitrary gauge variables* are $\alpha_{(a)}$, $\varphi_{(a)}$, θ^i, $\pi_{\tilde{\phi}}$, n and $\bar{n}_{(a)}$. As shown in Refs. [4–7], they describe the following generalized *inertial effects*:

(a) $\alpha_{(a)}(\tau, \vec{\sigma})$ and $\varphi_{(a)}(\tau, \vec{\sigma})$ describe the arbitrariness in the choice of a tetrad to be associated to a time-like observer, whose world-line goes through the point $(\tau, \vec{\sigma})$. They fix *the unit 4-velocity of the observer and the conventions for the gyroscopes and their transport along the world-line of the observer*.
(b) $\theta^i(\tau, \vec{\sigma})$ [depending only on the 3-metric, as shown in Eq. (2.1)] describe the arbitrariness in the choice of the 3-coordinates on the simultaneity surfaces Σ_τ of the chosen non-inertial frame centered on an arbitrary time-like observer. Their choice will induce a pattern of *relativistic standard inertial forces* (centrifugal, Coriolis, etc.), whose potentials are contained in the term $\mathcal{S}(\tau, \vec{\sigma})$ of the weak ADM energy E_{ADM} given in Eqs. (A8). These inertial effects are the relativistic counterpart of the non-relativistic ones (they are present also in the non-inertial frames of Minkowski space-time).
(c) $\bar{n}_{(a)}(\tau, \vec{\sigma})$, the shift functions appearing in the Dirac Hamiltonian, describe which points on different simultaneity surfaces have the same numerical value of the 3-coordinates. They are the inertial potentials describing the effects of the non-vanishing off-diagonal components ${}^4g_{\tau r}(\tau, \vec{\sigma})$ of the 4-metric, namely they are the *gravito-magnetic potentials*[8] responsible of effects like the dragging of inertial frames (Lens-Thirring effect) [34] in the post-Newtonian approximation.
(d) $\pi_{\tilde{\phi}}(\tau, \vec{\sigma})$, i.e. the York time ${}^3K(\tau, \vec{\sigma})$, describes the arbitrariness in the shape of the simultaneity surfaces Σ_τ of the non-inertial frame, namely the arbitrariness in the choice of the convention for the synchronization of distant clocks. Since this variable is present in the Dirac Hamiltonian,[9] it is a *new inertial potential* connected to the problem of the relativistic freedom in the choice of the *instantaneous 3-space*, which has no non-relativistic analogue (in Galilei space-time time is absolute and there is an absolute notion of Euclidean 3-space). Its effects are completely unexplored. For instance, since the sign of the trace of the extrinsic curvature may change from a region to another one on the simultaneity surface Σ_τ, *the associated inertial force in the Hamilton equations may change from attractive to repulsive in different regions*.
(e) $n(\tau, \vec{\sigma})$, the lapse function appearing in the Dirac Hamiltonian, describes the arbitrariness in the choice of the unit of proper time in each point of the simultaneity surfaces Σ_τ, namely how these surfaces are packed in the $3+1$ splitting.

From Eqs. (A4), (A8) and Eq. (B4) of Ref. [36], where Eq. (B1) gives the expression of the Γ-Γ term $\mathcal{S}$, we get the following expression of the super-Hamiltonian constraint

---

[8] In the post-Newtonian approximation in harmonic gauges they are the counterpart of the electro-magnetic vector potentials describing magnetic fields [25,34]: (a) $N = 1 + n$, $n \stackrel{\text{def}}{=} -\frac{4\epsilon}{c^2}\,\Phi_G$ with Φ_G the *gravito-electric potential*; (b) $n_r \stackrel{\text{def}}{=} \frac{2\epsilon}{c^2}\,A_{G\,r}$ with $A_{G\,r}$ the *gravito-magnetic* potential; (c) $E_{G\,r} = \partial_r\,\Phi_G - \partial_\tau\,(\frac{1}{2}\,A_{G\,r})$ (the *gravito-electric field*) and $B_{G\,r} = \epsilon_{ruv}\,\partial_u\,A_{G\,v} = c\,\Omega_{G\,r}$ (the *gravito-magnetic field*). Let us remark that in arbitrary gauges the analogy with electro-magnetism [34] breaks down.

[9] See Eqs. (2.12) for its presence in the super-Hamiltonian constraint and in the weak ADM energy, and Eqs. (3.1) for its presence in the super-momentum constraints.

and of weak ADM energy in the York canonical basis ($^3\hat{R}$ and $\hat{\triangle}$ are the 3-curvature of Σ_τ and the Laplace-Beltrami operator for the conformal 3-metric $^3\hat{g}_{rs}$, respectively)

$$
\begin{aligned}
\mathcal{H}(\tau,\vec{\sigma}) &= \epsilon\,\frac{c^3}{16\pi\, G}\,\tilde{\phi}^{1/6}(\tau,\vec{\sigma})\,[-8\,\hat{\triangle}\,\tilde{\phi}^{1/6} + {}^3\hat{R}\,\tilde{\phi}^{1/6}](\tau,\vec{\sigma}) - \frac{\epsilon}{c}\,\mathcal{M}(\tau,\vec{\sigma})\\
&\quad -\epsilon\,\frac{2\pi\, G}{c^3}\,\tilde{\phi}^{-1}\Bigg[-3\,(\tilde{\phi}\,\pi_{\tilde{\phi}})^2 + 2\sum_{\bar{b}}\Pi^2_{\bar{b}}\\
&\quad +2\sum_{abtwiuvj}\frac{\epsilon_{abt}\,\epsilon_{abu}\,V_{tw}(\theta^n)\,B_{iw}(\theta^n)\,V_{uv}(\theta^n)\,B_{jv}(\theta^n)\,\pi_i^{(\theta)}\,\pi_j^{(\theta)}}{\left[Q_a\,Q_b^{-1} - Q_b\,Q_a^{-1}\right]^2}\Bigg](\tau,\vec{\sigma}) \approx 0,\\
E_{\mathrm{ADM}} &= \int d^3\sigma\,\Bigg[\mathcal{M} - \frac{c^4}{16\pi\, G}\,\mathcal{S} + \frac{2\pi\, G}{c^2}\,\tilde{\phi}^{-1}\Bigg(-3\,(\tilde{\phi}\,\pi_{\tilde{\phi}})^2 + 2\sum_{\bar{b}}\Pi^2_{\bar{b}}\\
&\quad +2\sum_{abtwiuvj}\frac{\epsilon_{abt}\,\epsilon_{abu}\,V_{tw}(\theta^n)\,B_{iw}(\theta^n)\,V_{uv}(\theta^n)\,B_{jv}(\theta^n)\,\pi_i^{(\theta)}\,\pi_j^{(\theta)}}{\left[Q_a\,Q_b^{-1} - Q_b\,Q_a^{-1}\right]^2}\Bigg)\Bigg](\tau,\vec{\sigma}). \qquad (2.12)
\end{aligned}
$$

2.4 Gauges

Once we are in the York canonical basis, it is useful to restrict ourselves to the *Schwinger time gauges* implied by the gauge fixing constraints $\varphi_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0$, which imply $\lambda_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\lambda_{\vec{\varphi}(a)}(\tau,\vec{\sigma}) \approx 0$ in Eq. (A7). In this way we can go to Dirac brackets with respect to the primary 6 constraints $\pi_{\vec{\varphi}\,(a)}(\tau,\vec{\sigma}) \approx 0$, $\pi^{(\alpha)}_{(a)}(\tau,\vec{\sigma}) \approx 0$ [the Abelianized rotation constraints of Eq. (2.6)] and these gauge fixings (in total there are six pairs of second class constraints). In this reduced phase space the York canonical basis is formed by the pairs: $n(\tau,\vec{\sigma})$, $\pi_n(\tau,\vec{\sigma}) \approx 0$, $\bar{n}_{(a)}(\tau,\vec{\sigma})$, $\pi_{\vec{n}\,(a)}(\tau,\vec{\sigma}) \approx 0$, $\theta^i(\tau,\vec{\sigma})$, $\pi_i^{(\theta)}(\tau,\vec{\sigma})$, $\tilde{\phi}(\tau,\vec{\sigma})$, $\pi_{\tilde{\phi}}(\tau,\vec{\sigma})$, $R_{\bar{a}}(\tau,\vec{\sigma})$, $\Pi_{\bar{a}}(\tau,\vec{\sigma})$. We shall ignore global problems about the validity of the gauge fixing constraints everywhere in M^4: our results will in general be valid only locally.

The *CMC gauges* (Σ_τ has constant mean curvature $^3\tilde{K}(\tau,\vec{\sigma}) = const.$) [34] are those associated to the gauge fixing $\pi_{\tilde{\phi}}(\tau,\vec{\sigma}) \approx -\epsilon\,\frac{c^3}{12\pi\, G} \times const$. See Ref. [37] for the existence of surfaces of prescribed mean curvature in asymptotically flat space-times.

The CMC gauge fixing $\pi_{\tilde{\phi}}(\tau,\vec{\sigma}) = -\epsilon\,\frac{c^3}{12\pi\, G}\,{}^3\tilde{K}(\tau,\vec{\sigma}) \approx 0$ identifies the special gauge in which the simultaneity and Cauchy hyper-surfaces Σ_τ are the CMC hyper-surfaces with $^3\tilde{K}(\tau,\vec{\sigma}) \approx 0$.

We shall not use the special CMC gauge $\pi_{\tilde{\phi}}(\tau,\vec{\sigma}) \approx 0$, but we shall consider the class of gauges with given trace of the extrinsic curvature $^3\tilde{K}(\tau,\vec{\sigma}) \approx \epsilon\,K(\tau,\vec{\sigma})$, so that $\pi_\phi(\tau,\vec{\sigma}) \approx -\frac{c^3}{12\pi\, G}\,K(\tau,\vec{\sigma})$, to see the dependance of the dynamics on the shape of the simultaneity surfaces Σ_τ, namely on the convention chosen for clock synchronization.

Let us remember that the gauge fixings determining the lapse and shift functions are obtained by requiring the τ -constancy of the gauge fixings determining $\pi_{\tilde{\phi}}$ and θ^n.

The *3-orthogonal gauges* correspond to the gauge fixings $\theta^n(\tau, \vec{\sigma}) \approx 0$ and imply

$$
\begin{aligned}
{}^3\bar{e}_{(a)r} &= \tilde{\phi}^{1/3}\, \delta_{ra}\, Q_a, \quad {}^3\bar{e}^r_{(a)} = \tilde{\phi}^{-1/3}\, \delta_{ra}\, Q_a^{-1},\\
{}^3g_{rs} &= \tilde{\phi}^{2/3}\, Q_a^2\, \delta_{rs}, \quad {}^3g^{rs} = \tilde{\phi}^{-2/3}\, Q_a^{-2}\, \delta_{rs},\\
{}^3\bar{\tilde{\pi}}^r_{(a)} &= \tilde{\phi}^{-1/3} \left[\delta_{ra}\, Q_a\, (\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar{b}} \gamma_{\bar{b}a}\, \Pi_{\bar{b}}) - \sum_i Q_r^{-1}\, \frac{\epsilon_{ari}\, \pi_i^{(\theta)}}{Q_r\, Q_a^{-1} - Q_a\, Q_r^{-1}} \right],\\
{}^3\tilde{K}_{rs} &= \epsilon\, \frac{4\pi\, G}{c^3}\, \tilde{\phi}^{-1/3}\\
&\quad \times \left(\delta_{rs}\, Q_r^2\, [2 \sum_{\bar{b}} \gamma_{\bar{b}r}\, \Pi_{\bar{b}} - \tilde{\phi}\, \pi_{\tilde{\phi}}] + 2\, Q_r\, Q_s\, \frac{\sum_i \epsilon_{rsi}\, \pi_i^{(\theta)}}{Q_s\, Q_r^{-1} - Q_r\, Q_s^{-1}} \right).
\end{aligned}
\tag{2.13}
$$

The expression of the super-Hamiltonian constraint and of the weak ADM energy in the 3-orthogonal gauges is given in Eqs. (4.2) and (4.3).

In Ref. [36] there is the expression for the *3-normal gauges with respect to the origin of 3-coordinates* and for the completely fixed *ADM 4-coordinate gauge* used for the ADM post-Newtonian limit in Refs. [38,39] (it is a CMC gauge). In the York canonical basis these gauge fixings are *algebraic equations* for $\pi_{\tilde{\phi}}(\tau, \vec{\sigma})$ but *first-order elliptic partial differential equations* for the three Euler angles $\theta^n(\tau, \vec{\sigma})$.

Instead the family of *harmonic gauges*, defined by adding the 4 gauge-fixing constraints $\chi^A = \sum_B \partial_B \left(N\, {}^3e\, g^{AB}\right) = 0$ to the secondary first-class constraints and used both in theoretical studies [40] and in the post-Newtonian approximation [41,42], belongs to a different class of gauges at the Hamiltonian level. Their gauge fixings are neither algebraic conditions nor elliptic equations defined on a single instantaneous 3-space Σ_τ.

By using the first half (A10) of the Hamilton equations (the kinematical connection between velocities and phase space variables) associated with the Dirac Hamiltonian (A7), the gauge-fixing constraints $\chi_A(\tau, \vec{\sigma}) \approx 0$ can be rewritten as four Hamiltonian gauge fixings *explicitly depending upon the four Dirac multipliers* $\lambda_n = \partial_\tau\, n$ and $\lambda_{\vec{n}\,(a)} = \partial_\tau\, \bar{n}_{(a)}$ (see Eqs. (6.4) of Ref. [36] for their expression in the York canonical basis). These unconventional Hamiltonian constraints ($\chi^\tau \approx 0$ does not define a CMC gauge) are four coupled equations for π_ϕ and θ^i in terms of ϕ, $R_{\bar{a}}$, $\Pi_{\bar{a}}$, n, $\lambda_n = \partial_\tau\, n$, $\bar{n}_{(a)}$, $\lambda_{\vec{n}(a)} = \partial_\tau\, \bar{n}_{(a)}$.

The stability of these gauge fixings requires to impose $\partial_\tau\, \tilde{\chi}_a(\tau, \vec{\sigma}) \approx 0$ and $\partial_\tau\, \tilde{\chi}_\tau(\tau, \vec{\sigma}) \approx 0$. In this way we get four equations for the determination of n and $\bar{n}_{(a)}$. But these are not equations of the "elliptic" type like with ordinary gauge fixings. They are coupled equations depending upon n, $\partial_r\, n$, $\partial_\tau\, n$, $\partial_\tau^2\, n$ and $\bar{n}_{(a)}$, $\partial_r\, \bar{n}_{(a)}$, $\partial_\tau\, \bar{n}_{(a)}$, $\partial_\tau^2\, \bar{n}_{(a)}$, namely *hyperbolic* equations like Eq. (6.3). As a consequence there is a *problem of initial conditions not only for $R_{\bar{a}}$ but also for the lapse and shift functions of the harmonic gauge*. Each possible set of initial values should correspond to a different completely fixed harmonic gauge, since once we have a solution for n and $\bar{n}_{(a)}$ the corresponding Dirac multipliers are determined by taking their τ-derivative.

3 The super-momentum constraints and their solution

3.1 The super-momentum constraints

By using the results of Ref. [3] for the transformation property ${}^3\omega_{r(a)(b)} = [R\, {}^3\bar{\omega}_r\, R^T + R\, \partial_r\, R^T]_{(a)(b)}$ of the spin connection (B17) under O(3)-rotations and Eqs. (2.7) and (2.11), the super-momentum constraints (A4) in presence of matter, to be solved in $\pi_i^{(\theta)}(\tau, \vec{\sigma})$, take the form ($\bar{D}_{r(a)(b)}$ is the covariant derivative for $\alpha_{(a)}(\tau, \vec{\sigma}) = 0$)

$$
\begin{aligned}
\mathcal{H}_{(a)} &= \mathcal{H}^{(o)}_{(a)} - {}^3e^v_{(a)}\, \mathcal{M}_v = \sum_c R_{(a)(c)}\, \bar{\mathcal{H}}_{(c)} \approx \sum_c R_{(a)(c)}\, \tilde{\bar{H}}_{(a)} \approx 0,\\
\mathcal{H}^{(o)}_{(a)} &\overset{\text{def}}{=} \sum_{rb} D_{r(a)(b)}\, {}^3\tilde{\pi}^r_{(b)} = \sum_r \partial_r\, {}^3\tilde{\pi}^r_{(a)} - \sum_{rbc} \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(b)}\, {}^3\tilde{\pi}^r_{(c)}\\
&= \sum_{rc} \partial_r\, [R_{(a)(c)}\, {}^3\bar{\pi}^r_{(c)}] + \sum_{rbc} [R\, {}^3\bar{\omega}_r\, R^T + R\, \partial_r\, R^T]_{(a)(b)}\, R_{(b)(c)}\, \bar{\pi}^r_{(c)}\\
&= \sum_{rc} R_{(a)(c)} \left[\partial_r\, \bar{\pi}^r_{(c)} + \sum_d {}^3\bar{\omega}_{r(c)(d)}\, \bar{\pi}^r_{(d)} \right] \approx \sum_{vc} R_{(a)(c)}\, {}^3\bar{e}^v_{(c)}\, \mathcal{M}_v,\\
\bar{\mathcal{H}}_{(a)} &\overset{\text{def}}{=} \sum_r \partial_r\, {}^3\bar{\pi}^r_{(a)} + \sum_{rb} {}^3\bar{\omega}_{r(a)(b)}\, {}^3\bar{\pi}^r_{(b)} - \sum_v {}^3\bar{e}^v_{(a)}\, \mathcal{M}_v\\
&\approx \tilde{\bar{\mathcal{H}}}_{(a)} \overset{\text{def}}{=} \sum_{rb} \bar{D}_{r(a)(b)}\, {}^3\tilde{\bar{\pi}}^r_{(b)} - \sum_v {}^3\bar{e}^v_{(a)}\, \mathcal{M}_v\\
&= \sum_{rb} \Bigg[\delta_{ab}\, \partial_r + \frac{1}{2} \sum_{ucd} [\delta_{ac}\, \delta_{bd} - \delta_{ad}\, \delta_{bc}]\, V_{ud}(\theta^n)\\
&\quad \times \Bigg[Q_c\, Q_d^{-1} \Bigg(\frac{1}{3} [V_{uc}(\theta^n)\, \partial_r\, ln\, \tilde{\phi} - V_{rc}(\theta^n)\, \partial_u\, ln\, \tilde{\phi}]\\
&\quad + \sum_{\bar{b}} \gamma_{\bar{b}c}\, [V_{uc}(\theta^n)\, \partial_r\, R_{\bar{b}} - V_{rc}(\theta^n)\, \partial_u\, R_{\bar{b}}] + \partial_r\, V_{uc}(\theta^n) - \partial_u\, V_{rc}(\theta^n) \Bigg)\\
&\quad + \frac{1}{2} \sum_{ve} Q_e^2\, Q_d^{-1}\, Q_c^{-1}\, V_{vc}(\theta^n)\, V_{re}(\theta^n)\\
&\quad \times \Bigg(\frac{1}{3} [V_{ue}(\theta^n)\, \partial_v\, ln\, \tilde{\phi} - V_{ve}(\theta^n)\, \partial_u\, ln\, \tilde{\phi}]\\
&\quad + \sum_{\bar{b}} \gamma_{\bar{b}e}\, [V_{ue}(\theta^n)\, \partial_v\, R_{\bar{b}} - V_{ve}(\theta^n)\, \partial_u\, R_{\bar{b}}] + \partial_v\, V_{ue}(\theta^n) - \partial_u\, V_{ve}(\theta^n) \Bigg) \Bigg] \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\Bigg[\tilde{\phi}^{-1/3}\Bigg(V_{rb}(\theta^n)\,Q_b^{-1}\,(\tilde{\phi}\,\pi_{\tilde{\phi}}+\sum_{\bar{c}}\gamma_{\bar{c}b}\,\Pi_{\bar{c}})\\
&\quad+\sum_f^{f\neq b}\sum_{twi}Q_f^{-1}\,\frac{V_{rf}(\theta^n)\,\epsilon_{bft}\,V_{tw}(\theta^n)}{Q_f\,Q_b^{-1}-Q_b\,Q_f^{-1}}\,B_{iw}(\theta^n)\,\pi_i^{(\theta)}\Bigg)\Bigg]\\
&\quad-\tilde{\phi}^{-1/3}\sum_v V_{va}(\theta^n)\,Q_v^{-1}\,\mathcal{M}_v\quad\approx 0.
\end{aligned}
\tag{3.1}
$$

3.2 Their solution

The solution of Eqs. (3.1) in terms of the matter mass-current $\mathcal{M}_r$ is

$$
\begin{aligned}
{}^3\tilde{\bar{\pi}}^r_{(a)}(\tau,\vec{\sigma}) &\approx g^r_{(a)}(\tau,\vec{\sigma})-\int d^3\sigma_1\,\sum_c\bar{\zeta}^r_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\,J_{(c)}(\tau,\vec{\sigma}_1)\\
&\stackrel{\text{def}}{=}\sum_b{}^3\bar{e}^r_{(b)}(\tau,\vec{\sigma})\,\big[g_{(a)(b)}+j_{(a)(b)}\big](\tau,\vec{\sigma}),\\
J_{(a)} &\stackrel{\text{def}}{=}\sum_r{}^3\bar{e}^r_{(a)}\,\mathcal{M}_r=\tilde{\phi}^{-1/3}\sum_r V_{ra}(\theta^n)\,Q_a^{-1}\,\mathcal{M}_r\\
&=\sum_{sd}\bar{D}_{s(a)(d)}\sum_b{}^3\bar{e}^s_{(b)}\,j_{(d)(b)},\\
j_{(a)(b)}(\tau,\vec{\sigma}) &= -\sum_{rc}{}^3\bar{e}_{(b)r}(\tau,\vec{\sigma})\int d^3\sigma_1\,\bar{\zeta}^r_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau)\,J_{(c)}(\tau,\vec{\sigma}_1)\\
&=-\sum_{rc}\Big[\tilde{\phi}^{1/3}\,V_{rb}(\theta^n)\,Q_b\Big](\tau,\vec{\sigma})\\
&\quad\times\int d^3\sigma_1\,\bar{\zeta}^r_{(a)(c)}(\vec{\sigma},\vec{\sigma}_1;\tau|\theta^n,\phi,R_{\bar{a}}]\\
&\quad\times\sum_s\Big[\tilde{\phi}^{-1/3}\,V_{sc}(\theta^n)\,Q_c^{-1}\,\mathcal{M}_s\Big](\tau,\vec{\sigma}_1),\\
g^r_{(a)} &= \sum_b g_{(a)(b)}\,{}^3e^r_{(b)},\quad g_{(a)(b)}=\sum_r g^r_{(a)}\,{}^3\bar{e}_{(b)r},\\
g_{((a)(b))} &= \frac{1}{2}\,(g_{(a)(b)}+g_{(b)(a)}),\quad g_{[(a)(b)]}=\frac{1}{2}\,(g_{(a)(b)}-g_{(b)(a)}),\\
\Rightarrow\ \sum_r{}^3\bar{e}_{(b)r}\,{}^3\tilde{\bar{\pi}}^r_{(a)} &= g_{(a)(b)}+j_{(a)(b)}=g_{((a)(b))}+j_{((a)(b))}+g_{[(a)(b)]}+j_{[(a)(b)]}\\
&=\delta_{ab}\,(\tilde{\phi}\,\pi_{\tilde{\phi}}+\sum_{\bar{b}}\gamma_{\bar{b}a}\,\Pi_{\bar{b}})+\sum_{twi}\frac{\epsilon_{abt}\,V_{tw}(\theta^n)\,B_{iw}(\theta^n)\,\pi_i^{(\theta)}}{Q_b\,Q_a^{-1}-Q_a\,Q_b^{-1}}
\end{aligned}
$$

$$= \sum_r {}^3\bar{e}_{((a)r}\, \tilde{\bar{\pi}}^r_{(b))},$$

$$\Rightarrow \qquad g_{[(a)(b)]} = -j_{[(a)(b)]},$$

$$\sum_{rb} \bar{D}_{r(a)(b)}\, g^r_{(b)}(\tau, \vec{\sigma}) = \left[\sum_r \partial_r\, g^r_{(a)} + \sum_{rb} {}^3\bar{\omega}_{r(a)(b)}\, g^r_{(b)}\right](\tau, \vec{\sigma}) = 0, \tag{3.2}$$

where the Green function $\bar{\zeta}^r_{(a)(b)}$ of Ref. [2] is given in Eq. (D1) of Appendix D and $g^r_{(a)}$ are *zero modes of the covariant divergence with the covariant derivative* $\bar{D}_{r(a)(b)}$.

Then Eq. (2.6), evaluated in the reduced phase space of the Schwinger time gauge $\varphi_{(a)}(\tau, \vec{\sigma}) \approx 0$, $\alpha_{(a)}(\tau, \vec{\sigma}) \approx 0$, implies

$$\begin{aligned}
\pi^{(\theta)}_i(\tau, \vec{\sigma}) &= -\sum_{lmra} \left[A_{mi}(\theta^n)\, \epsilon_{mlr}\, {}^3\bar{e}_{(a)l}\, {}^3\tilde{\bar{\pi}}^r_{(a)}\right](\tau, \vec{\sigma}) \\
&\approx -\sum_{lmra} \left[A_{mi}(\theta^n)\, \epsilon_{mlr}\, \tilde{\phi}^{1/3}\, V_{la}(\theta^n)\, Q_a\right](\tau, \vec{\sigma}) \Bigg[g^r_{(a)}(\tau, \vec{\sigma}) \\
&\quad - \int d^3\sigma_1 \sum_c \bar{\zeta}^r_{(a)(c)}(\vec{\sigma}, \vec{\sigma}_1; \tau)\, J_{(c)}(\tau, \vec{\sigma}_1)\Bigg] \\
&= -\sum_{lmrab} \left[A_{mi}(\theta^n)\, \epsilon_{mlr}\, V_{la}(\theta^n)\, V_{rb}(\theta^n)\, Q_a\, Q_b^{-1}\right](\tau, \vec{\sigma}) \\
&\quad \times \left[g_{(a)(b)} + j_{(a)(b)}\right](\tau, \vec{\sigma}).
\end{aligned} \tag{3.3}$$

Let us remark that, in absence of matter ($J_{(a)}(\tau, \vec{\sigma}) = 0$) and with the choice $g^r_{(a)}(\tau, \vec{\sigma}) = 0$ for the homogeneous solution, we get $\pi^{(\theta)}_i(\tau, \vec{\sigma}) \approx 0$.

3.3 The zero modes of the covariant divergence

We have now to see whether we can find the zero modes $g^r_{(a)}(\tau, \vec{\sigma})$ of the operator $\bar{D}_{r(a)(b)}$.

If we put $g^r_{(a)} = \sum_c {}^3\bar{e}^r_{(c)}\, g_{(a)(c)}$ in the second of Eqs. (2.6), we can determine $g_{(a)(a)}$ since, by using Eqs. (3.2), we have

$$\Lambda_a\, P^a = \sum_r {}^3\bar{e}_{(a)r} \sum_c {}^3\bar{e}^r_{(c)}\, [g_{(a)(c)} + j_{(a)(c)}] = g_{(a)(a)} + j_{(a)(a)},$$

$$\begin{aligned}
g_{(a)(a)}(\tau, \vec{\sigma}) &= \left[\Lambda_a\, P^a - j_{(a)(a)}\right](\tau, \vec{\sigma}) = \left[\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar{b}} \gamma_{\bar{b}a}\, \Pi_{\bar{b}}\right](\tau, \vec{\sigma}) \\
&\quad + \sum_{sc} \Bigg[\tilde{\phi}^{1/3}\, V_{sa}(\theta^n)\, Q_a \int d^3\sigma_1\, \bar{\zeta}^s_{(a)(c)}(\vec{\sigma}, \vec{\sigma}_1; \tau)
\end{aligned}$$

$$\times \sum_w \Big[\tilde{\phi}^{-1/3}\, V_{wc}(\theta^n)\, Q_c^{-1}\, \mathcal{M}_w \Big] (\tau, \vec{\sigma}_1) \Big]$$

and for $a \neq b$,

$$\begin{aligned}
g_{(a)(b)} &= g_{((a)(b))} + g_{[(a)(b)]} = g_{((a)(b))} - j_{[(a)(b)]} \\
&= g_{((a)(b))} + \frac{1}{2}\, \tilde{\phi}^{1/3}(\tau, \vec{\sigma}) \sum_{rc} \int d^3\sigma_1 \\
&\quad \times \Big(\big[V_{rb}(\theta^n)\, Q_b \big] (\tau, \vec{\sigma})\, \bar{\zeta}^r_{(a)(c)} - \big[V_{ra}(\theta^n)\, Q_a \big] (\tau, \vec{\sigma})\, \bar{\zeta}^r_{(b)(c)} \Big) \\
&\quad \times (\vec{\sigma}, \vec{\sigma}_1; \tau) \sum_s \Big[\tilde{\phi}^{-1/3}\, V_{sc}(\theta^n)\, Q_c^{-1}\, \mathcal{M}_s \Big] (\tau, \vec{\sigma}_1).
\end{aligned} \tag{3.4}$$

As a consequence, the homogeneous equation for $g^r_{(a)}$ in Eq. (3.2) gives rise to an inhomogeneous equation for $g_{((a)(b))}$ with $a \neq b$

$$\begin{aligned}
&\sum_{rb} \bar{D}_{r(a)(b)}(\tau, \vec{\sigma}) \sum_c \Big[{}^3\bar{e}^r_{(c)}\, g_{(b)(c)} \Big] (\tau, \vec{\sigma}) = 0, \\
&\sum_{rbc}^{b\neq c} \bar{D}_{r(a)(b)}(\tau, \vec{\sigma}) \Big[{}^3\bar{e}^r_{(c)}\, g_{((b)(c))} \Big] (\tau, \vec{\sigma}) \\
&\quad = - \sum_{rb} \bar{D}_{r(a)(b)}(\tau, \vec{\sigma}) \left[{}^3\bar{e}^r_{(b)}\, g_{(b)(b)} + \sum_c^{c\neq b} {}^3\bar{e}^r_{(c)}\, g_{[(b)(c)]} \right] (\tau, \vec{\sigma}), \\
&\quad \Downarrow \\
&g_{((a)(b))}(\tau, \vec{\sigma}) = g^{\text{hom}}{}_{((a)(b))}(\tau, \vec{\sigma}) + \int d^3\sigma_1 \sum_d \bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau) \\
&\sum_{re} \bar{D}_{r(d)(e)}(\tau, \vec{\sigma}_1) \left[{}^3\bar{e}^r_{(e)}\, g_{(e)(e)} + \sum_f^{f\neq e} {}^3\bar{e}^r_{(f)}\, g_{[(e)(f)]} \right] (\tau, \vec{\sigma}_1), \\
&\sum_{rbc}^{b\neq c} \Big[\bar{D}_{r(a)((b)}\ {}^3\bar{e}^r_{(c))} \Big] (\tau, \vec{\sigma})\, \bar{\mathcal{G}}_{((b)(c))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau) = -\delta_{ad}\, \delta^3(\vec{\sigma}, \vec{\sigma}_1), \\
&\sum_{rbc}^{b\neq c} \Big[\bar{D}_{r(a)((b)}\ {}^3\bar{e}^r_{(c))} \Big] (\tau, \vec{\sigma})\, g^{\text{hom}}_{((b)(c))}(\tau, \vec{\sigma}) = 0,
\end{aligned} \tag{3.5}$$

where $\bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau)$ is the Green function of the operator $\sum_r \Big[\bar{D}_{r(a)((b)}\,{}^3\bar{e}^r_{(c))} \Big]|_{b\neq c}$ [see Eq. (D2) and its Minkowski limit given in Eq. (D3)]. In Eq. (3.5) $g^{\text{hom}}_{((a)(b))}(\tau, \vec{\sigma})$ is an arbitrary zero mode of the operator $\sum_r \Big[\bar{D}_{r(a)((b)}\ {}^3\bar{e}^r_{(c))} \Big]|_{b\neq c}$. There are as many independent such zero modes as independent zero modes $g^r_{(a)}$ of $\bar{D}_{r(a)(b)}$.

The presence of this second Green function is a consequence of using the Darboux canonical basis identified by the York map in the solution of the super-momentum constraints (so that the Green function (D1) is no more sufficient).

Equations (3.5) imply

$$
\begin{aligned}
&\sum_{ca}^{c\neq a}\left[{}^3\bar{e}^r_{(c)}\, g_{((a)(c))}\right](\tau,\vec{\sigma})\\
&\quad = \int d^3\sigma_1 \sum_d \bar{\eta}^r_{(a)(d)}(\vec{\sigma},\vec{\sigma}_1;\tau)\sum_{se}\Bigg[\bar{D}_{s(d)(e)}\Bigg({}^3\bar{e}^s_{(e)}\, g_{(e)(e)}\\
&\qquad + \sum_c^{c\neq e} {}^3\bar{e}^s_{(c)}\, g_{[(e)(c)]}\Bigg)\Bigg](\tau,\vec{\sigma}_1) + \sum_c^{c\neq a}\left[{}^3\bar{e}^r_{(c)}\, g^{\text{hom}}_{((a)(c))}\right](\tau,\vec{\sigma}),\quad \text{with}\\
&\bar{\eta}^r_{(a)(d)}(\vec{\sigma},\vec{\sigma}_1;\tau) \stackrel{\text{def}}{=} \sum_c^{c\neq a} {}^3\bar{e}^r_{(c)}(\tau,\vec{\sigma})\,\bar{\mathcal{G}}_{((a)(c))(d)}(\vec{\sigma},\vec{\sigma}_1;\tau),
\end{aligned}
\tag{3.6}
$$

so that a lengthy calculation, given in Eqs. (C1)–(C4) of Appendix C of Ref. [36], and the use of Eq. (3.2) for $j_{(a)(b)}$ and of Eq. (3.3) for $\pi_i^{(\theta)}$ lead to the following form for the solution of the super-momentum constraints, which explicitly shows its non-uniqueness being defined modulo the zero modes of the covariant divergence

$$
\begin{aligned}
\pi_i^{(\theta)}(\tau,\vec{\sigma}) &\approx -\sum_{lmrab}\left[A_{mi}(\theta^n)\,\epsilon_{mlr}\,{}^3\bar{e}_{(a)l}\,{}^3\bar{e}^r_{(b)}\right](\tau,\vec{\sigma})\,\left[g_{(a)(b)} + j_{(a)(b)}\right](\tau,\vec{\sigma})\\
&= -\sum_{lmrab}\left[A_{mi}(\theta^n)\,\epsilon_{mlr}\,V_{la}(\theta^n)\,V_{rb}(\theta^n)\,Q_a\,Q_b^{-1}\right](\tau,\vec{\sigma})\\
&\quad\times\Bigg[g^{\text{hom}}{}_{((a)(b))}(\tau,\vec{\sigma}) - \sum_d \int d^3\sigma_1\,\bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma},\vec{\sigma}_1;\tau)\\
&\quad\times\Bigg[\tilde{\phi}^{-1/3}\sum_w V_{wd}(\theta^n)\,Q_d^{-1}\,\mathcal{M}_w - \sum_{s,e}\bar{D}_{s(d)(e)}\,V_{se}(\theta^n)\,\tilde{\phi}^{-1/3}\,Q_e^{-1}\\
&\quad\times\Bigg(\tilde{\phi}\,\pi_{\tilde{\phi}} + \sum_{\bar{b}}\gamma_{\bar{b}e}\,\Pi_{\bar{b}}\Bigg)\Bigg](\tau,\vec{\sigma}_1)\\
&\quad-\sum_{ec}^{c\neq e}\int d^3\sigma_1\,\Bigg(\delta_{ac}\,\delta_{be}\,\delta^3(\vec{\sigma},\vec{\sigma}_1) + \sum_{d,s}\bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma},\vec{\sigma}_1;\tau)\\
&\quad\times\left[\bar{D}_{s(d)(e)}\,V_{sc}(\theta^n)\,\tilde{\phi}^{-1/3}\,Q_c^{-1}\right](\tau,\vec{\sigma}_1)\Bigg)
\end{aligned}
$$

$$\times \sum_{r,f} \int d^3\sigma_2 \, \frac{1}{2} \Big[\Big(V_{re}(\theta^n)\, \tilde{\phi}^{1/3}\, Q_e \Big)(\tau, \vec{\sigma}_1)\, \bar{\zeta}^r_{(c)(f)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)$$
$$+ \Big(V_{rc}(\theta^n)\, \tilde{\phi}^{1/3}\, Q_c \Big)(\tau, \vec{\sigma}_1)\, \bar{\zeta}^r_{(e)(f)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau) \Big]$$
$$\times \left(\tilde{\phi}^{-1/3} \sum_w V_{wf}(\theta^n)\, Q_f^{-1}\, \mathcal{M}_w \right)(\tau, \vec{\sigma}_2) \Bigg], \tag{3.7}$$

Since we have $Q_a(\tau, \vec{\sigma})\, Q_b^{-1}(\tau, \vec{\sigma}) = \Lambda_a(\tau, \vec{\sigma})\, \Lambda_b^{-1}(\tau, \vec{\sigma}) \to_{r\to\infty} 1$, the leading order of Eq. (3.7) is $\epsilon_{iab}\, f_{(ab)} = 0$, consistently with the vanishing of $\pi_i^{(\theta)}(\tau, \vec{\sigma})$ at spatial infinity.

In the final expression of Eq. (3.7) we made explicit the symmetry in e and c.

Let us remark that the zero modes $g^{\mathrm{hom}}_{((a)(b))}(\tau, \vec{\sigma})$ may be written in the following form

$$g^{\mathrm{hom}}_{((a)(b))}(\tau, \vec{\sigma}) = \sum_{ec}^{c\neq e} \int d^3\sigma_1 \, \Big(\delta_{c(a}\, \delta_{b)e}\, \delta^3(\vec{\sigma}, \vec{\sigma}_1)$$
$$+ \sum_{ds} \bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau)\, \frac{1}{2} \Big[\bar{D}_{s(d)(e)}\, V_{sc}(\theta^n)\, \tilde{\phi}^{-1/3}\, Q_c^{-1}$$
$$+ \bar{D}_{s(d)(c)}\, V_{se}(\theta^n)\, \tilde{\phi}^{-1/3}\, Q_e^{-1} \Big](\tau, \vec{\sigma}_1) \Big)\, \tilde{g}_{(ec)}(\tau, \vec{\sigma}_1), \tag{3.8}$$

with an arbitrary $\tilde{g}_{(ec)}(\tau, \vec{\sigma})$ symmetric in e and c.

If we put Eq. (3.7) into Eq. (2.13), we get the expression of ${}^3\tilde{\bar{\pi}}^r_{(a)}(\tau, \vec{\sigma}) = \Big[\sum_b {}^3\bar{e}^r_{(b)} \big(g_{(a)(b)} + j_{(a)(b)} \big) \Big](\tau, \vec{\sigma})$ when restricted to the solution of the super-momentum constraints (see Eq. (3.10) of Ref. [36]).

Let us remark that both $\pi_i^{(\theta)}(\tau, \vec{\sigma})$ and ${}^3\tilde{\bar{\pi}}^r_{(a)}(\tau, \vec{\sigma})$ are defined modulo homogeneous solutions

(i) of Eq. (D1): $\bar{\zeta}^r_{(a)(b)} \mapsto \bar{\zeta}^r_{(a)(b)} + \bar{\zeta}^{(\mathrm{hom})r}_{(a)(b)}$ with $\sum_{rb} \bar{D}_{r(a)(b)}(\tau, \vec{\sigma})\, \bar{\zeta}^{(\mathrm{hom})r}_{(b)(c)}(\vec{\sigma}, \vec{\sigma}_1; \tau) = 0$;

(ii) of Eq. (D2): $\bar{\mathcal{G}}_{((a)(b))(d)} \mapsto \bar{\mathcal{G}}_{((a)(b))(d)} + \bar{\mathcal{G}}^{(\mathrm{hom})}_{((a)(b))(d)}$ with $\sum_{rbc}^{b\neq c} \Big[\bar{D}_{r(a)(b)}\, {}^3\bar{e}^r_{(c))} \Big](\tau, \vec{\sigma})\ \bar{\mathcal{G}}^{(\mathrm{hom})}_{((a)(b))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau) = 0$.

While these freedoms are connected to the *choice of the initial data*,[10] Eq. (3.8) connects the freedom $\tilde{g}_{(ab)}(\tau, \vec{\sigma})$ to the *existence in general relativity of the zero modes* $g^{\mathrm{hom}}_{((a)(b))}(\tau, \vec{\sigma})$ of the operators $\sum_r \Big[\bar{D}_{r(a)((b)}\, {}^3\bar{e}^r_{(c))} \Big]|_{b\neq c}$, see Eq. (3.5), and $g^r_{(a)}(\tau, \vec{\sigma}) = \sum_b \Big(\Big[g^{\mathrm{hom}}_{((a)(b))} + \cdots \Big]\, {}^3\bar{e}^r_{(b)} \Big)(\tau, \vec{\sigma})$ of $\bar{D}_{r(a)(b)}$. The associated residual

[10] Like the choice of the retarded, advanced or symmetric Green functions in the Lienard-Wiechert solution for an electro-magnetic field coupled to charged matter.

gauge freedom is connected to the group of 3-diffeomorphisms and not to a Lie group like in Yang-Mills theory (see the discussion in Sect. 5).

Due to the distributional nature of the Green function $\bar{\mathcal{G}}$ (whose flat limit is given in Eq. (E4) of Ref. [36]), required by the Shanmugadhasan canonical transformations (2.3) and (2.9), to avoid distributional problems in the expression of the super-Hamiltonian constraint and in the weak ADM energy we need a suitable choice of the arbitrary zero mode $g^{hom}_{((a)(b))}$, i.e. of $\tilde{g}_{(ce)}$, which will be done elsewhere when we will solve the theory in the weak field limit.

4 The final form of the super-Hamiltonian constraint and of the weak ADM energy in a completely fixed 3-orthogonal Schwinger time gauge

As already said, the gauge fixings $\varphi_{(a)}(\tau,\vec{\sigma}) \approx 0$, $\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0$, whose τ-constancy implies $\lambda_{\varphi(a)}(\tau,\vec{\sigma}) = 0$ and $\lambda_{\alpha(a)}(\tau,\vec{\sigma}) = 0$, define a *special Schwinger time gauge*. We assume to have eliminated these variables by going to Dirac brackets (we go on to denote them as Poisson brackets).

Let us now consider the *completely fixed 3-orthogonal Schwinger time gauge* defined by the gauge fixing

$$\theta^i(\tau,\vec{\sigma}) \approx 0, \quad \pi_{\tilde{\phi}}(\tau,\vec{\sigma}) \approx -\frac{c^3}{12\pi\, G}\, K(\tau,\vec{\sigma}), \tag{4.1}$$

i.e. with ${}^3\tilde{K}(\tau,\vec{\sigma}) \approx \epsilon\, K(\tau,\vec{\sigma})$. In this way the function $K(\tau,\vec{\sigma})$ *will show explicitly how the dynamics depends on the shape of* Σ_τ*, namely on the convention for the synchronization of clocks*.

4.1 The super-Hamiltonian constraint

In these completely fixed 3-orthogonal Schwinger time gauge, by using Eq. (3.8) (or Eq. (3.10) of Ref. [36]) for the solution of the super-momentum constraints, the super-Hamiltonian constraint (2.12), i.e. the Lichnerowicz equation for $\tilde{\phi}(\tau,\sigma) = \phi^6(\tau,\vec{\sigma})$, becomes[11]

$$\begin{aligned}
\mathcal{H}(\tau,\vec{\sigma}) \approx \epsilon \Bigg[& \frac{c^3}{16\pi\, G} \left(\tilde{\phi}^{1/6}\, [-8\, \hat{\triangle}[R_{\bar{a}}] + {}^3\hat{R}[R_{\bar{a}}]\,]\, \tilde{\phi}^{1/6}\right)(\tau,\vec{\sigma}) \\
& -\frac{1}{c}\, \mathcal{M}(\tau,\vec{\sigma}) + \frac{c^3}{24\,\pi\, G}\, \left(\tilde{\phi}\, K^2\right)(\tau,\vec{\sigma}) - \frac{4\pi\, G}{c^3} \left(\tilde{\phi}^{-1} \sum_{\bar{b}} \Pi^2_{\bar{b}}\right)(\tau,\vec{\sigma}) \\
& -\frac{4\pi\, G}{c^3}\, \tilde{\phi}^{-1}(\tau,\vec{\sigma})\, Z(\tau,\vec{\sigma}) \Bigg],
\end{aligned}$$

[11] The steps to get Eq. (4.2) from Eq. (2.12) are described in Eqs. (B6), (B7), (B8), (C5), (C6), (C7), (C9), (D10), (D11) of Appendices B, C and D of Ref. [36]. We do not give the final expression (B8) for $Z(\tau,\vec{\sigma})$, because, being rather complicated, its explicit dependence on $\tilde{\phi}$ (either algebraic or under integrals) is irrelevant for the general discussion.

$$Z(\tau, \vec{\sigma}) = Z_\theta(\tau, \vec{\sigma})|_{\theta=0} = \sum_{ab}^{b\neq a} \mathcal{S}^2_{((a)(b))}\, |_{\theta=0}\, (\tau, \vec{\sigma}), \tag{4.2}$$

where the $\tilde{\phi}$-dependent function Z takes into account the contribution of the Γ-Γ term $\mathcal{S}$, containing the inertial potentials present in the non-inertial frame. In particular Z contains terms linear in $K(\tau, \vec{\sigma})$.

Its unknown solution is a functional $\tilde{\phi}(\tau, \vec{\sigma}|R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K]$ of the gravitational tidal degrees of freedom $R_{\bar{a}}, \Pi_{\bar{a}}$, of both the mass density $\mathcal{M}$ and mass-current density $\mathcal{M}_r$ of the matter and of the gauge parameter K describing the shape of the hyper-surfaces Σ_τ (the convention for clock synchronization and for the Cauchy surface) having the given 3-orthogonal 3-coordinate system.

Even if Eq. (4.2) is a non-linear integro-differential equation for $\tilde{\phi}$, the presence of the Laplace-Beltrami operator on Σ_τ (with its associated theory of harmonic functions) suggests the plausibility that the assumed behavior $\tilde{\phi}(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} 1 + O(r^{-1})$ at spatial infinity will identify a unique solution.

4.2 The weak ADM energy

In these completely fixed 3-orthogonal Schwinger time gauges, by using Eq. (C9) of Ref. [36] for the term quadratic in the momenta and Eq. (C1) for the Γ-Γ term $\mathcal{S}$, the weak ADM energy (2.12) becomes [Z has been defined in Eq. (4.2)]

$$\begin{aligned} E_{\rm ADM} = \int d^3\sigma \Bigg[& \mathcal{M} - \frac{c^4}{24\pi G}\, \tilde{\phi}\, K^2 + \frac{4\pi G}{c^2}\, \tilde{\phi}^{-1} \Bigg[\sum_{\bar{b}} \Pi^2_{\bar{b}} + Z \Bigg] \\ & - \frac{c^4}{16\pi G}\, \tilde{\phi}^{1/3} \sum_a Q_a^{-2} \Bigg(20\, (\partial_a\, ln\, \tilde{\phi}^{1/6})^2 \\ & - 4 \sum_r (\partial_r\, ln\, \tilde{\phi}^{1/6})^2 + 8\, \partial_a\, ln\, \tilde{\phi}^{1/6} \sum_{\bar{b}} \gamma_{\bar{b}a}\, \partial_a\, R_{\bar{b}} \\ & - 2 \sum_r \partial_r\, ln\, \tilde{\phi}^{1/6} \sum_{\bar{b}} (\gamma_{\bar{b}a} + \gamma_{\bar{b}r})\, \partial_r\, R_{\bar{b}} + \Bigg(\sum_{\bar{b}} \gamma_{\bar{b}a}\, \partial_a\, R_{\bar{b}} \Bigg)^2 \\ & + \sum_{\bar{b}} (\partial_a\, R_{\bar{b}})^2 - \sum_r \Bigg(\sum_{\bar{b}} \gamma_{\bar{b}r}\, \partial_r\, R_{\bar{b}} \Bigg) \Bigg(\sum_{\bar{c}} \gamma_{\bar{c}a}\, \partial_r\, R_{\bar{c}} \Bigg) \Bigg) \Bigg] (\tau, \vec{\sigma}), \end{aligned} \tag{4.3}$$

While the terms coming from the Γ-Γ term $\mathcal{S}$ describe the relativistic version of the standard inertial potentials in this 3-coordinate system (expressed as functions of $\tilde{\phi}(\tau, \vec{\sigma})$ and of the tidal effects $R_{\bar{a}}(\tau, \vec{\sigma})$), the first line contains the dependence on the inertial potential $K(\tau, \vec{\sigma})$ (both explicitly, K^2, and inside Z) describing the choice of the instantaneous 3-space.

4.3 The rest-frame conditions and the spin of the 3-universe

By using Eqs. (A6), the rest-frame conditions for the 3-universe [3] in the York basis and in the completely fixed gauge (4.1) are $[\mathcal{S}_{((a)(b))}|_{\theta=0}$ is the contribution of the Γ-Γ term]

$$
\begin{aligned}
P^r_{\mathrm{ADM}} \approx P^r_{\mathrm{ADM}}\,|_{\theta=0} &= \int d^3\sigma\, \Big[\tilde{\phi}^{-2/3}\, Q_r^{-2}\, \mathcal{M}_r\Big](\tau, \vec{\sigma}) \\
&\quad - \int d^3\sigma \sum_{uv} \Bigg[\tilde{\phi}^{-2/3} \Bigg(\delta_{uv}\, Q_v^{-2} \Bigg(\sum_{\bar{b}} \gamma_{\bar{b}v}\, \Pi_{\bar{b}} - \frac{c^3}{12\pi\, G}\, \tilde{\phi}\, K\Bigg) \\
&\quad + Q_u\, Q_v\, \mathcal{S}_{((u)(v))}\,|_{\theta=0}\Bigg) \Bigg(\delta_{ru} \Bigg(\frac{1}{3}\, \partial_v\, ln\, \tilde{\phi} + \sum_{\bar{c}} \gamma_{\bar{c}r}\, \partial_v\, R_{\bar{c}}\Bigg) \\
&\quad + \delta_{rv} \Bigg(\frac{1}{3}\, \partial_u\, ln\, \tilde{\phi} + \sum_{\bar{c}} \gamma_{\bar{c}r}\, \partial_u\, R_{\bar{c}}\Bigg) \\
&\quad - \delta_{uv} \Bigg(\frac{1}{3}\, \partial_r\, ln\, \tilde{\phi} + \sum_{\bar{c}} \gamma_{\bar{c}r}\, \partial_r\, R_{\bar{c}}\Bigg) Q_u\, Q_v^{-1}\Bigg)\Bigg](\tau, \vec{\sigma}) \approx 0. \qquad (4.4)
\end{aligned}
$$

Like in special relativity [43], these 3 first class constraints imply that 3 variables $q^r_{\mathrm{ADM}}[R_{\bar{a}}, \Pi_{\bar{a}}, \ldots]$, describing the internal canonical 3-center of mass of the 3-universe, are *gauge variables* (they describe the arbitrariness in the choice of the observer used as origin of the 3-coordinates on Σ_τ). As shown in Ref. [43], the natural gauge fixings to eliminate them is to ask the vanishing of the boost generators in Eqs. (A6), i.e. $J^{\tau r}_{\mathrm{ADM}} \approx 0$. These conditions imply the vanishing of the internal Møller 3-center of energy so that it can be shown that then Eqs. (4.4) imply also $q^r_{\mathrm{ADM}} \approx 0$. In this way the observer may be identified with the decoupled 4-center of mass (more exactly with the covariant non-canonical Fokker-Pryce 4-center of inertia) of the universe.

See Eq. (4.5) of Ref. [36] for the expression of the spin (A6) of the 3-universe in the rest frame.

5 The shift and lapse functions

Let us now determine the lapse and shift functions of the completely fixed 3-orthogonal Schwinger time gauges of Sect. 4. In all the equations of this section $\tilde{\phi}(\tau, \vec{\sigma})$ should be replaced by the unknown solution $\tilde{\phi}(\tau, \vec{\sigma}|R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K]$ of the Lichnerowicz equation (4.2).

5.1 The 3-orthogonal gauges, the shift functions and a generalized Gribov ambiguity

If we use Eqs. (2.12) for the super-Hamiltonian constraint and the weak ADM energy in Eqs. (A7) and (A9), the Dirac Hamiltonian in the York basis can be written in the form

$$
\begin{aligned}
H_D = & \int d^3\sigma\ [(1+n)\,\mathcal{M}]\,(\tau,\vec{\sigma}) \\
& - \frac{c^4}{16\pi\,G} \int d^3\sigma\,\Big[\mathcal{S} + n\,\tilde{\phi}^{1/6}\,\Big(-8\,\hat{\triangle} + {}^3\hat{R}\Big)\,\tilde{\phi}^{1/6}\Big](\tau,\vec{\sigma}) \\
& + \frac{2\pi\,G}{c^2} \int d^3\sigma\,\Bigg[(1+n)\,\tilde{\phi}^{-1}\,\Bigg(-3\,(\tilde{\phi}\,\pi_{\tilde{\phi}})^2 + 2\sum_{\bar{b}} \Pi^2_{\bar{b}} \\
& + 2\sum_{abtwiuvj} \frac{\epsilon_{abt}\,\epsilon_{abu}\,V_{tw}(\theta^n)\,B_{iw}(\theta^n)\,V_{uv}(\theta^n)\,B_{jv}(\theta^n)\,\pi_i^{(\theta)}\,\pi_j^{(\theta)}}{\Big[Q_a\,Q_b^{-1} - Q_b\,Q_a^{-1}\Big]^2}\Bigg)\Bigg](\tau,\vec{\sigma}) \\
& + \int d^3\sigma\,\Bigg[\sum_a \bar{n}_{(a)}\,\bar{\tilde{\mathcal{H}}}_{(a)} + \lambda_n\,\pi_n + \sum_a \lambda_{\vec{n}(a)}\,\pi_{\vec{n}(a)}\Bigg](\tau,\vec{\sigma}),
\end{aligned}
\tag{5.1}
$$

where the super-momentum constraints $\bar{\tilde{\mathcal{H}}}_{(a)}(\tau,\vec{\sigma}) \approx 0$ are given by Eqs. (3.1), $\mathcal{S}$ in Eq. (B1) and ${}^3\hat{R}[\theta^n, R_{\bar{a}}]$ in Eq. (B2) of Appendix B of Ref. [36] [for $\theta^n(\tau,\vec{\sigma}) \approx 0$ see Eqs. (C1) and (C2)]. The Hamilton equations of motion must be evaluated with this Dirac Hamiltonian (see Sect. 6) and the solution (3.8) [or Eq. (3.10) of Ref. [36]] of the super-momentum constraints *can be used only after having evaluated the Poisson brackets*.

With the Hamiltonian (5.1) the time-constancy of the gauge fixings $\theta^n(\tau,\vec{\sigma}) \approx 0$, determining the shift functions, implies [here $\approx$ means by using these gauge fixings, the one of Eq. (4.1) and the solution (C3) of the super-momentum constraints]

$$
\begin{aligned}
\partial_\tau\,\theta^i(\tau,\vec{\sigma}) &= \{\theta^i(\tau,\vec{\sigma}), H_D\} \\
&= \sum_a \int d^3\sigma_1\,\bar{n}_{(a)}(\tau,\vec{\sigma}_1)\,\{\theta^i(\tau,\vec{\sigma}), \bar{\tilde{\mathcal{H}}}_{(a)}(\tau,\vec{\sigma}_1)\} \\
&\quad + \Big\{\theta^i(\tau,\vec{\sigma}), E_{\mathrm{ADM}} - \epsilon\,c \int d^3\sigma_1\,n(\tau,\vec{\sigma}_1)\,\mathcal{H}(\tau,\vec{\sigma}_1)\Big\} \\
&\approx \sum_a \int d^3\sigma_1\,\bar{n}_{(a)}(\tau,\vec{\sigma}_1)\,\tilde{Z}_{(a)i}(\tau,\vec{\sigma}_1)\,\delta^3(\vec{\sigma},\vec{\sigma}_1) - W_i(\tau,\vec{\sigma}) \\
&= \sum_a Z_{(a)i}(\tau,\vec{\sigma})\,\bar{n}_{(a)}(\tau,\vec{\sigma}) - W_i(\tau,\vec{\sigma}) \approx 0, \\
\tilde{Z}_{(a)i}(\tau,\vec{\sigma}) &\stackrel{\text{def}}{=} \sum_{rb} \Big[\Big(\delta_{ab}\,\partial_{1r} + \epsilon_{(a)(b)(c)}\,{}^3\bar{\omega}_{r(c)}\Big)\,G^{(o)r}_{(b)i}\Big](\tau,\vec{\sigma}) \\
&= \sum_{rb} \Big[\bar{D}_{r(a)(b)}\,G^{(o)r}_{(b)i}\Big](\tau,\vec{\sigma}),
\end{aligned}
$$

$$Z_{(a)i}(\tau,\vec{\sigma}) \stackrel{\text{def}}{=} -\sum_{arb}\Big[\,G^{(o)r}_{(b)i}\,\bar{D}_{r(b)(a)}\,\Big](\tau,\vec{\sigma}),$$

$$W_i(\tau,\vec{\sigma}) \stackrel{\text{def}}{=} -\left[\frac{\delta}{\delta\,\pi^{(\theta)}_i(\tau,\vec{\sigma})}\,\left(E_{\text{ADM}} + \int d^3\sigma_1\,(-\epsilon\,c\,n\,\mathcal{H})\,(\tau,\vec{\sigma}_1)\right)\right], \qquad (5.2)$$

where we used Eq. (3.1) and Eq. (2.13) but with the notation $G^{(o)r}_{(a)i} = -\tilde{\phi}^{-1/3}\,Q_r^{-1}\,\frac{\epsilon_{rai}}{Q_r\,Q_a^{-1} - Q_a\,Q_r^{-1}}$ of Eq. (B16).

Equations (3.8) and (C3) imply the following expression for $W_i(\tau,\vec{\sigma})$ [with the substitution $\tilde{\phi}\,\pi_{\tilde{\phi}} \mapsto -\frac{c^3}{12\pi\,G}\,\tilde{\phi}\,K$ into the functions $F_{(ab)}$ of Eq. (C3) in accord with Eq. (4.1): let us remark that these functions have a linear dependence on $\pi_{\tilde{\phi}}$, namely they *know the sign* of $K(\tau,\vec{\sigma})$]

$$\begin{aligned}
W_i(\tau,\vec{\sigma}) &= \left[-\frac{8\pi\,G}{c^2}\,\tilde{\phi}^{-1}\,(1+n)\sum_{abj}\frac{\epsilon_{abi}\,\epsilon_{abj}\,\pi^{(\theta)}_j}{[Q_a\,Q_b^{-1} - Q_b\,Q_a^{-1}]^2}\right](\tau,\vec{\sigma})\\
&\approx -\frac{16\pi\,G}{c^2}\,\tilde{\phi}^{-1}(\tau,\vec{\sigma})\,(1+n(\tau,\vec{\sigma}))\\
&\quad\times\sum_{ab}\frac{\epsilon_{iab}}{Q_a\,Q_b^{-1} - Q_b\,Q_a^{-1}}(\tau,\vec{\sigma})\,F_{(ab)}(\tau,\vec{\sigma}).
\end{aligned} \qquad (5.3)$$

Since Eqs. (B16) with $H^{(o)}_{(b)ri} = \frac{1}{2}\,\epsilon_{bri}\,\tilde{\phi}^{1/3}\,Q_r\,(Q_r\,Q_b^{-1} - Q_b\,Q_r^{-1})$, imply $\sum_{ar} H^{(o)}_{(a)rj}\,G^{(o)r}_{(a)i} = \delta_{ij}$ and $\sum_{ri} H^{(o)}_{(b)ri}\,G^{(o)r}_{(a)i} = \delta_{ab}$, we get

$$\begin{aligned}
\sum_a Z_{(a)i}\,\bar{n}_{(a)} &= \sum_{ar}\left[\sum_b G^{(o)r}_{(b)i}\left(\partial_r\,\delta_{ba} + \sum_c \epsilon_{(b)(a)(c)}\,{}^3\bar{\omega}_{r(c)}\right)\right]\bar{n}_{(a)}\\
&= \sum_s \partial_s\left(\sum_a G^{(o)s}_{(a)i}\,\bar{n}_{(a)}\right)\\
&\quad -\sum_{sd}\left(\sum_{bc}\epsilon_{(b)(c)(d)}\,G^{(o)s}_{(b)i}\,{}^3\bar{\omega}_{s(c)} + \partial_s\,G^{(o)s}_{(d)i}\right)\\
&\quad \times\sum_a\left(\sum_{rj} H^{(o)}_{(d)rj}\,G^{(o)r}_{(a)j}\right)\bar{n}_{(a)}\\
&\stackrel{\text{def}}{=} \sum_{jr}\tilde{D}_{rij}\left(\sum_a G^{(o)r}_{(a)j}\,\bar{n}_{(a)}\right) \approx W_i,
\end{aligned} \qquad (5.4)$$

where we have introduced the modified covariant derivative operator

$$
\begin{aligned}
\tilde{D}_{rij} &= \delta_{ij}\,\partial_r - T_{rij}, \quad Z_{(a)i} = -\sum_{arb} G^{(o)r}_{(b)i}\,\bar{D}_{r(b)(a)} = \sum_{jr} \tilde{D}_{rij}\,G^{(o)r}_{(a)j},\\
T_{rij} &= \sum_{sd}\left(\sum_{bc}\epsilon_{(b)(c)(d)}\,G^{(o)s}_{(b)i}\,{}^3\bar{\omega}_{s(c)} + \partial_s\,G^{(o)s}_{(d)i}\right) H^{(o)}_{(d)rj}\\
&= -\sum_{abcds}\epsilon_{abc}\,\epsilon_{asi}\,\epsilon_{crj}\,\epsilon_{sbd}\,Q_r\,Q_d^{-1}\,\partial_d\left(\frac{1}{3}\ln\tilde{\phi} + \Gamma^{(1)}_s\right)\frac{Q_r\,Q_c^{-1} - Q_c\,Q_r^{-1}}{Q_s\,Q_a^{-1} - Q_a\,Q_s^{-1}}\\
&\quad -\sum_{as}\epsilon_{asi}\,\epsilon_{arj}\,Q_r\,Q_s^{-1}\,\frac{Q_r\,Q_a^{-1} - Q_a\,Q_r^{-1}}{Q_s\,Q_a^{-1} - Q_a\,Q_s^{-1}}\left[\partial_s\,(\frac{1}{3}\ln\tilde{\phi} + \Gamma^{(1)}_s)\right.\\
&\quad \left. + \frac{Q_r\,Q_a^{-1} + Q_a\,Q_r^{-1}}{Q_s\,Q_a^{-1} - Q_a\,Q_s^{-1}}\,\partial_s\,(\Gamma^{(1)}_s - \Gamma^{(1)}_a)\right],\\
Q_a &= e^{\Gamma^{(1)}_a}, \quad \Gamma^{(1)}_a = \sum_{\bar{a}}\gamma_{\bar{a}a}\,R_{\bar{a}}, \quad \sum_a \Gamma^{(1)}_a = 0.
\end{aligned}
\tag{5.5}
$$

By using the Green function of the operator $\tilde{D}_{rij}$, defined in Eq. (D4), the shift functions can be determined and have the following expression as functions of the lapse function and of the dynamical variables (it is linear in n)

$$
\begin{aligned}
\bar{n}_{(a)}(\tau,\vec{\sigma}) &= \mathcal{N}_{(a)}(\tau,\vec{\sigma}\,|\tilde{\phi}, n, R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K]\\
&= f_{(a)}(\tau,\vec{\sigma}) + \sum_{ri} H^{(o)}_{(a)ri}(\tau,\vec{\sigma})\\
&\quad \times \sum_j \int d^3\sigma_1\,\tilde{\zeta}^r_{ij}(\vec{\sigma},\vec{\sigma}_1;\tau)\,W_j(\tau,\vec{\sigma}_1),\\
\sum_a \left[Z_{(a)i}\,f_{(a)}\right](\tau,\vec{\sigma}) &\overset{\text{def}}{=} -\sum_{abr}\left[G^{(o)r}_{(b)i}\,\bar{D}_{r(b)(a)}\,f_{(a)}\right](\tau,\vec{\sigma})\\
&= \sum_{rj}\left[\tilde{D}_{rij}\sum_a G^{(o)r}_{(a)j}\,f_{(a)}\right](\tau,\vec{\sigma})\\
&\overset{\text{def}}{=} \sum_{jr}\left[\tilde{D}_{rij}\,\tilde{f}^r_j\right](\tau,\vec{\sigma}) = 0,
\end{aligned}
\tag{5.6}
$$

with $f_{(a)}(\tau,\vec{\sigma}) = \sum_{rj}\left[H^{(o)}_{(a)rj}\,\tilde{f}^r_j\right](\tau,\vec{\sigma})$ *zero modes* of $Z_{(a)i}(\tau,\vec{\sigma})$, namely with $\tilde{f}^r_j(\tau,\vec{\sigma}) = \left[\sum_a G^{(o)r}_{(a)j}\,f_{(a)}\right](\tau,\vec{\sigma})$ *zero modes* of the operator $\tilde{D}_{rij}(\tau,\vec{\sigma})$.

Naturally the shift functions are defined modulo homogeneous solution of Eqs. (D4): $\tilde{\zeta}^r_{ij} \mapsto \tilde{\zeta}^r_{ij} + \tilde{\zeta}^{(\mathrm{hom})r}_{ij}$ with $\sum_{rj}\tilde{D}_{rij}(\tau,\vec{\sigma})\,\tilde{\zeta}^{(\mathrm{hom})r}_{jk}(\vec{\sigma},\vec{\sigma}_1;\tau) = 0$. Again this is a problem of choice of the initial conditions.

When the operator $Z_{(a)i}$ has zero modes, $\sum_a\left[Z_{(a)i}\,f_{(a)}\right](\tau,\vec{\sigma}) = 0$, also its adjoint operator $\tilde{Z}_{(a)i} = \sum_{rb}\bar{D}_{r(a)(b)}\,G^{(o)r}_{(b)i}$, appearing in Eq. (5.2), has zero modes $h_i(\tau,\vec{\sigma})$, i.e. $\sum_i \left[\tilde{Z}_{(a)i}\, h_i\right](\tau, \vec{\sigma}) = 0$. Then Eq. (5.2) imposes the following restriction on W_i

$$\int d^3\sigma \sum_i W_i(\tau, \vec{\sigma})\, h_i(\tau, \vec{\sigma}) = 0. \tag{5.7}$$

Therefore, in the 3-orthogonal gauges there is a *residual gauge freedom or generalized Gribov ambiguity* in the determination of the shift functions associated to the zero modes of the operator $Z_{(a)i}$ (or of $\tilde{D}_{rij}$). Since in general relativity the Gauss law constraints $\tilde{\bar{\mathcal{H}}}_{(a)}(\tau, \vec{\sigma}) \approx 0$ are a suitable reformulation of the constraints $\Theta_r(\tau, \vec{\sigma}) \approx 0$ generating 3-diffeomorphisms [see after Eq. (A4)], this extra ambiguity is connected to the group of 3-diffeomorphisms.

Given $\theta^i(\tau, \vec{\sigma})$ and its modification $\theta^i(\tau, \vec{\sigma}) + \sum_a Z'_{ai}(\tau, \vec{\sigma})\, \beta_a(\tau, \vec{\sigma})$ ($Z_{ai} = Z'_{ai}|_{\theta^r=0}$) induced by a (modified) 3-diffeomorphism [generated by $\int d^3\sigma \sum_a \beta_a(\tau, \vec{\sigma})\, \tilde{\bar{H}}_{(a)}(\tau, \vec{\sigma})$], we have that the vanishing of the first as a gauge fixing, $\theta^i(\tau, \vec{\sigma}) \approx 0$, implies the vanishing also of the modified one $\theta^i(\tau, \vec{\sigma}) + \sum_a Z_{ai}(\tau, \vec{\sigma})\, \beta_a(\tau, \vec{\sigma}) \approx 0$ when β_a coincides with one of the zero modes $f_{(a)}$. The same happens in Yang-Mills (YM) theory: for certain gauge potentials arising from special connections the gauge fixing $\vec{\partial} \cdot \vec{A}_a \approx 0$ (so that $\vec{A}_a = \vec{A}_{a\perp}$) implies that there are transformed gauge potentials $\vec{A}^U_a = \vec{A}_a + U^{-1}\, \vec{D}^{(\vec{A})}\, U$, $U = e^{i\alpha}$ also satisfying $\vec{\partial} \cdot \vec{A}^U_a \approx 0$ if $K(\vec{A}_\perp)\, \alpha = 0$ [$K(\vec{A}_\perp) = -\vec{\partial} \cdot \vec{D}^{(\vec{A}_\perp)}$]. In these cases the connection originating the gauge potential $\vec{A}_a$ has *gauge symmetries* (stability subgroup of gauge transformations) implying the existence of zero modes of the Faddeev-Popov operator $K(\vec{A}_\perp)$ and of the operator $\triangle(\vec{A}_\perp) = \vec{D}^{(\vec{A}_\perp)} \cdot \vec{D}^{(\vec{A}_\perp)}$. This leads to the Gribov ambiguity (see Ref. [19,20] for a review).[12]

Since the shift functions determine which points on different Σ_τ's have the same numerical value of the chosen 3-orthogonal 3-coordinates $\vec{\sigma}$ (and then the inertial gravito-magnetic potential), we see that, when $Z_{(a)i}$ has zero modes, there are as many independent 3-orthogonal gauges, and, therefore, non-inertial frames, as zero modes (each one with the gauge freedom in the choice of $\pi_{\tilde{\phi}}$, i.e. in the form of Σ_τ). It is an open problem whether this generalized Gribov ambiguity (gauge

[12] In the YM case the canonical variables are $A^o_a, \pi^o_a, \vec{A}_a, \vec{\pi}_a$ and the Gauss laws $\vec{\partial} \cdot \vec{\pi}_a \approx 0$ are secondary constraints implied by the primary ones $\pi^o_a \approx 0$. In ordinary Sobolev spaces the Gribov ambiguity creates problems in the analogue of Eq. (5.2), namely $\partial_\tau\, \vec{\partial} \cdot \vec{A}_a \approx 0$, needed for the determination of the gauge variables A^o_a's. The YM constraint manifold is a stratification of Gribov copies labeled by a winding number with the different sectors separated by Gribov horizons. In suitable weighted Sobolev spaces [18] the Faddeev-Popov does not have zero modes, and there is no Gribov ambiguity (the connections with gauge symmetries have a constant limit at spatial infinity not allowed in these spaces) and we have that the only solution of $\sum_b D_{rab}\, f_b = 0$ is $f_a = 0$. In these spaces is also absent the other aspect of the Gribov ambiguity, i.e. the existence of special field strengths stable ($F = F^U$) under a subgroup of gauge transformations (the problem of *gauge copies*). However also in YM, the absence of zero modes of the Faddeev-Popov operator does not fix the Green function appearing in the solution of the Gauss laws (the non-Abelian generalization of $\bar{\zeta}^r_{(a)(b)}$ of Eq. (D1) in flat space-time): there is the usual freedom (connected to the choice of the initial data) in the choice of homogeneous solutions.
See Refs. [18,44–49] for the known results on the zero modes of operators like $Z_{(a)i}$ and $\tilde{Z}_{(a)i}$ in the case of Yang-Mills and Einstein equations.

symmetries of the gauge variables $\theta^i(\tau, \vec{\sigma})$, i.e. existence of a stability subgroup of passive 3-diffeomorphisms, whose group-manifold has a mathematical structure not yet under control), whose possibility was noted in Ref. [13] (see p. 765), can be eliminated by a suitable restriction of the function space like it happens in the YM case (but here the gauge group is a Lie group with a well understood theory of associated principal bundles) with the restriction to suitable weighted Sobolev spaces [18]. Since our class of non-compact space-times does not admit [16,17] asymptotically vanishing Killing vectors, the only known result (see the first of Refs. [16,17], pp. 133–136) is that in weighted Sobolev spaces suitable elliptic operators acting on the simultaneity surfaces Σ_τ have no zero modes. If it is possible to apply these results to the covariant divergence $\bar{D}_{r(a)(b)}$ and to the operator $Z_{(a)i}$, given the assumed behavior $\bar{n}_{(a)}(\tau, \vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon}), \epsilon > 0$, at spatial infinity, also the zero modes (3.8) would be absent in these function spaces.

The time-constancy of Eq. (5.6), i.e. the time-constancy of the induced gauge fixings, determining the shift functions, determines the Dirac multiplier $\lambda_{\vec{n}\,(a)}(\tau, \vec{\sigma})$, since we have

$$
\begin{aligned}
&\partial_\tau \Bigg[\bar{n}_{(a)}(\tau, \vec{\sigma}) - f_{(a)}(\tau, \vec{\sigma}) - \sum_{ri} H^{(o)}_{(a)ri}(\tau, \vec{\sigma}) \\
&\qquad \times \sum_j \int d^3\sigma_1\, \tilde{\zeta}^r_{ij}(\vec{\sigma}, \vec{\sigma}_1; \tau)\, W_j(\tau, \vec{\sigma}_1)\Bigg] \approx 0, \\
&\lambda_{\vec{n}\,(a)}(\tau, \vec{\sigma}) \approx \frac{\partial f_{(a)}(\tau, \vec{\sigma})}{\partial \tau} \\
&\qquad + \left\{ \sum_{ri} H^{(o)}_{(a)ri}(\tau, \vec{\sigma}) \sum_j \int d^3\sigma_1\, \tilde{\zeta}^r_{ij}(\vec{\sigma}, \vec{\sigma}_1; \tau)\, W_j(\tau, \vec{\sigma}_1),\, H_D \right\}.
\end{aligned}
\tag{5.8}
$$

Therefore $\lambda_{\vec{n}(a)}(\tau, \vec{\sigma})$ inherits the arbitrariness of $\bar{n}_{(a)}(\tau, \vec{\sigma})$.

5.2 The lapse function in the 3-orthogonal Schwinger time gauges

The time constancy of the other gauge fixing (4.1), evaluated with the Dirac Hamiltonian (5.1) determines the lapse function [the calculations can be found in Eqs. (D8) and (D9) of Appendix D of Ref. [36]]

$$
\begin{aligned}
\partial_\tau \left[\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) + \frac{c^3}{12\pi\, G}\, K(\tau, \vec{\sigma})\right] &= \frac{c^3}{12\pi\, G}\, \frac{\partial\, K(\tau, \vec{\sigma})}{\partial\, \tau} + \{\pi_{\tilde{\phi}}(\tau, \vec{\sigma}), H_D\} \\
&= \frac{c^3}{12\pi\, G}\, \frac{\partial\, K(\tau, \vec{\sigma})}{\partial\, \tau} - \frac{\delta}{\delta\, \phi(\tau, \vec{\sigma})}
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg(E_{\mathrm{ADM}} + \int d^3\sigma_1\, \Big[-\epsilon\, c\, n\, \mathcal{H} \\
&\quad + \sum_a \mathcal{N}_{(a)}[\tilde{\phi}, n, R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K]\, \bar{\tilde{\mathcal{H}}}_{(a)} \Big](\tau, \vec{\sigma}_1) \Bigg) \approx 0, \\
&\Downarrow \\
n(\tau, \vec{\sigma}) &\approx \mathcal{N}(\tau, \vec{\sigma} | \tilde{\phi}, R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K]. \qquad (5.9)
\end{aligned}
$$

The explicit expression for this linear integro-differential equation for the lapse function is given in Eq. (5.9) of Ref. [36].

It is impossible to judge whether Eq. (5.9), with the assumed behavior $n(\tau, \vec{\sigma}) \to_{r\to\infty} O(r^{-(2+\epsilon)}), \epsilon > 0$ at spatial infinity, admits a further residual gauge freedom (ambiguity in the determination of the proper time element $n(\tau, \vec{\sigma})\, d\tau$ in each point of Σ_τ, giving the packing of the Σ_τ's in the foliation), besides the generalized Gribov ambiguity for the shift functions.

The time-constancy of the induced gauge fixing (5.9) determines the Dirac multiplier $\lambda_n(\tau, \vec{\sigma})$

$$
\begin{aligned}
&\frac{\partial}{\partial \tau} [n(\tau, \vec{\sigma}) - \mathcal{N}(\tau, \vec{\sigma} | \tilde{\phi}, R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K]\,] \approx 0, \\
&\downarrow \\
&\lambda_n(\tau, \vec{\sigma}) \approx \{\mathcal{N}(\tau, \vec{\sigma} | \tilde{\phi}, R_{\bar{a}}, \Pi_{\bar{a}}, \mathcal{M}, \mathcal{M}_r, K], H_D\}. \qquad (5.10)
\end{aligned}
$$

6 Equations of motion for the tidal effects $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ in Schwinger time gauges

In this section we shall consider the Hamilton equations in the York basis both in arbitrary Schwinger time gauges and in a completely fixed 3-orthogonal Schwinger time gauge.

6.1 Equations of motion in the York basis

In the York canonical basis in an arbitrary Schwinger time gauge the effective Dirac Hamiltonian is given in Eq. (5.1). As a consequence the first half of Hamilton equations becomes [the equations for $\partial_\tau\, \theta^i$ are written using the results in Eq. (5.2)]

$$
\begin{aligned}
\partial_\tau\, n(\tau, \vec{\sigma}) &= \lambda_n(\tau, \vec{\sigma}), \\
\partial_\tau\, \bar{n}_{(a)}(\tau, \vec{\sigma}) &= \lambda_{\bar{n}(a)}(\tau, \vec{\sigma}), \\
\partial_\tau\, \tilde{\phi}^{1/6}(\tau, \vec{\sigma}) &= \Bigg[-\frac{2\pi\, G}{c^2}\, (1+n)\, \tilde{\phi}^{-1/6}\, \pi_{\tilde{\phi}} \\
&\quad - \frac{1}{6}\, \tilde{\phi}^{-1/6} \sum_{rb} V_{rb}(\theta^n)\, Q_b^{-1} \sum_a \bar{D}_{r(b)(a)}\, \bar{n}_{(a)} \Bigg](\tau, \vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
\partial_\tau\, \theta^i(\tau,\vec{\sigma}) &= \Bigg[\frac{8\pi\, G}{c^2}\,(1+n)\,\tilde{\phi}^{-1}\\
&\quad \times \sum_{abtwuvj} \frac{\epsilon_{abt}\,\epsilon_{abu}\, V_{tw}(\theta^n)\, B_{iw}(\theta^n)\, V_{uv}(\theta^n)\, B_{jv}(\theta^n)\, \pi_j^{(\theta)}}{[Q_a\, Q_b^{-1} - Q_b\, Q_a^{-1}]^2}\\
&\quad + \sum_a Z_{(a)i}\, \bar{n}_{(a)}\Bigg](\tau,\vec{\sigma}),\\
\partial_\tau\, R_{\bar{a}}(\tau,\vec{\sigma}) &= \Bigg[\frac{4\pi\, G}{c^2}\,(1+n)\,\tilde{\phi}^{-1}\,\Pi_{\bar{a}}\\
&\quad -\tilde{\phi}^{-1/3}\sum_{rb}\gamma_{\bar{a}b}\, V_{rb}(\theta^n)\, Q_b^{-1}\sum_a \bar{D}_{r(b)(a)}\,\bar{n}_{(a)}\Bigg](\tau,\vec{\sigma}),\\
&\Downarrow\\
\Pi_{\bar{a}}(\tau,\vec{\sigma}) &= \Bigg[\frac{c^2}{4\pi\, G}\,\frac{\tilde{\phi}^{2/3}}{1+n}\Bigg(\partial_\tau\, R_{\bar{a}} + \tilde{\phi}^{-1/3}\sum_{rb}\gamma_{\bar{a}b}\, V_{rb}(\theta^n)\, Q_b^{-1}\\
&\quad \times \sum_a \bar{D}_{r(b)(a)}\,\bar{n}_{(a)}\Bigg)\Bigg](\tau,\vec{\sigma}),\\
\pi_{\tilde{\phi}}(\tau,\vec{\sigma}) &= \Bigg[-\frac{c^2}{2\pi\, G}\,\frac{\tilde{\phi}^{-1/6}}{1+n}\Bigg(\partial_\tau\,\tilde{\phi}^{1/6} + \frac{1}{6}\,\tilde{\phi}^{-1/6}\sum_{rb} V_{rb}(\theta^n)\, Q_b^{-1}\\
&\quad \times \sum_a \bar{D}_{r(b)(a)}\,\bar{n}_{(a)}\Bigg)\Bigg](\tau,\vec{\sigma}).
\end{aligned}
\tag{6.1}
$$

In the last two lines we have given $\Pi_{\bar{a}}$ and $\pi_{\tilde{\phi}}$ in terms of the velocities and of the configuration variables. Also $\pi_i^{(\theta)}$ could be expressed in the same way, so that the solution (3.8) of the super-momentum constraints could be transformed on a statement about the velocities $\partial_\tau\, \theta^i$. As said in the previous section, the vanishing of these velocities become the equations for the shift functions of 3-orthogonal gauges.

The second half of Hamilton equations, to which the unsolved first class constraints have to be added, is

$$
\begin{aligned}
\partial_\tau\, \pi_{\tilde{\phi}}(\tau,\vec{\sigma}) &= -\frac{\delta\, H_D}{\delta\,\tilde{\phi}(\tau,\vec{\sigma})}, \quad \mathcal{H}(\tau,\vec{\sigma}) \approx 0,\\
\partial_\tau\, \pi_i^{(\theta)}(\tau,\vec{\sigma}) &= -\frac{\delta\, H_D}{\delta\,\theta^i(\tau,\vec{\sigma})}, \quad \bar{\tilde{\mathcal{H}}}_{(a)}(\tau,\vec{\sigma}) \approx 0,\\
\partial_\tau\, \Pi_{\bar{a}}(\tau,\vec{\sigma}) &= -\frac{\delta\, H_D}{\delta\, R_{\bar{a}}(\tau,\vec{\sigma})},
\end{aligned}
\tag{6.2}
$$

where the super-Hamiltonian and super-momentum constraints have the forms given in Eqs. (2.12) and (3.1), respectively. The equation for $\partial_\tau\, \pi_{\tilde{\phi}}$ may be obtained by using Eq. (5.9).

The content of the equations for $\partial_\tau \tilde{\phi}$ in Eqs. (6.1) and for $\partial_\tau \pi_i^{(\theta)}$ in Eqs. (6.2) is the preservation in time of the super-Hamiltonian and super-momentum constraints, respectively.

By using the expression of $\Pi_{\bar{a}}(\tau, \vec{\sigma})$ given in Eqs. (6.1) and Eq. (D12) of Appendix D of Ref. [36] for $\delta\, H_D/\delta\, R_{\bar{a}}(\tau, \vec{\sigma})$, the equations $\partial_\tau\, \Pi_{\bar{a}}(\tau, \vec{\sigma}) = -\delta\, H_D/\delta\, R_{\bar{a}}(\tau, \vec{\sigma})$ assume the following form $[Q_a = e^{\sum_{\bar{a}} \gamma_{\bar{a}a} R_{\bar{a}}},\ Q_1\, Q_2\, Q_3 = 1]$

$$
\begin{aligned}
&\left[\partial_\tau^2\, R_{\bar{a}} + \sum_{rs\bar{b}} A_{rs\bar{a}\bar{b}}\, \partial_r\, \partial_s\, R_{\bar{b}}\right](\tau, \vec{\sigma})\\
&\quad = \left[\sum_{\bar{b}} B_{\bar{a}\bar{b}}\, \partial_\tau\, R_{\bar{b}} + \sum_{r\bar{b}\bar{c}} B_{r\bar{a}\bar{b}\bar{c}}\, \partial_\tau\, R_{\bar{b}}\, \partial_r\, R_{\bar{c}}\right.\\
&\qquad \left. + \sum_{rs\bar{b}\bar{c}} C_{rs\bar{a}\bar{b}\bar{c}}\, \partial_r\, R_{\bar{b}}\, \partial_s\, R_{\bar{c}} + \sum_{r\bar{b}} C_{r\bar{a}\bar{b}}\, \partial_r\, R_{\bar{b}} + F_{\bar{a}}\right](\tau, \vec{\sigma}),\\
&A_{rs\bar{a}\bar{b}} \quad \text{functions of} \quad Q_a,\, \tilde{\phi},\, n,\, \theta^i,\, \partial_u\, \tilde{\phi},\, \partial_u\, n,\, \partial_u\, \theta^i,\, \partial_u\, \partial_v\, \tilde{\phi},\, \partial_u\, \partial_v\, n,\, \partial_u\, \partial_v\, \theta^i,\\
&B_{\bar{a}\bar{b}},\, B_{r\bar{a}\bar{b}\bar{c}},\, C_{r\bar{a}\bar{b}},\, C_{rs\bar{a}\bar{b}\bar{c}} \quad \text{functions of the same variables and of}\ \pi_{\tilde{\phi}}, \pi_i^{(\theta)},\\
&\quad \bar{n}_{(a)},\, \partial_u\, \bar{n}_{(a)},\\
&F_{\bar{a}} \quad \text{functions of the previous variables and of} \quad \mathcal{M},\, \mathcal{M}_v.
\end{aligned}
\tag{6.3}
$$

The hyperbolic equations (6.3) show explicitly that the equations of motion for the two tidal degrees of freedom $R_{\bar{a}}(\tau, \vec{\sigma})$ of the gravitational field *depend upon the arbitrary gauge variables (the inertial effects)* n, $\bar{n}_{(a)}$, θ^i, $\pi_{\tilde{\phi}}$, and on the unknowns $\tilde{\phi}$ and $\pi_i^{(\theta)}$ in the super-Hamiltonian and super-momentum constraints. In particular the term in $\delta\, H_D/\delta\, R_{\bar{a}}(\tau, \vec{\sigma})$ coming from the super-momentum constraints (3.1) (see Eq. (D12) of Ref. [36] for its expression in the 3-orthogonal gauges) *depends linearly on* $\pi_{\tilde{\phi}}$: since its *sign* (i.e. the sign of the trace of the extrinsic curvature of the simultaneity surface) is not fixed, $\pi_{\tilde{\phi}}$ *describes a relativistic inertial force which may vary from attractive to repulsive* from a region of Σ_τ to another one with an opposite sign of ${}^3K(\tau, \vec{\sigma})$.

Therefore, *to get a deterministic evolution we must go to a completely fixed gauge*. The same holds for Einstein's equations, but only at the Hamiltonian level it can be made explicit.

A naive background-independent linearization of Eqs. (6.3) along the lines of Ref. [25] could be attempted by requiring $|R_{\bar{a}}(\tau, \vec{\sigma})| \ll 1$ [so that $Q_a \approx 1 + \sum_{\bar{a}} \gamma_{\bar{a}a}\, R_{\bar{a}}$][13] producing equations of the type $\Big[\partial_\tau^2\, R_{\bar{a}} + A^{(o)}_{rs}\, \partial_r\, \partial_s\, R_{\bar{a}} + \cdots + M^{(o)}\, R_{\bar{a}} + F^{(o)}\Big](\tau, \vec{\sigma}) = 0$ (the quantities $A^{(o)}_{rs}, \ldots,$ are evaluated for $Q_a \to 1$) with a *pseudo-squared-mass term* $M^{(o)}$ depending upon $\mathcal{M}^{(o)}$ (the metric-independent part

[13] The presence of the denominators $(Q_a\, Q_b^{-1} - Q_b\, Q_a^{-1})^{-k}$, $k = 1, 2, 3$, in Eqs. (6.1) and Eq. (D12) of Appendix D of Ref. [36] suggests the necessity of a point canonical transformation from the tidal variables $R_{\bar{a}}$ to new variables more suitable for the linearization. This problem will be studied elsewhere.

of $\mathcal{M}$), $\mathcal{M}_r$ and the gauge variables, i.e. upon the inertial effects.[14] This type of term will appear also in completely fixed gauges. Let us remark that in the standard linearization on a background one ignores the dependence of the matter energy-momentum tensor $T^{\mu\nu}$ on the 4-metric: it too would generate a similar pseudo-squared-mass term.

In conclusion Eqs. (6.3) show that the refusal of particle physicists[15] to accept the geometrical view of the gravitational field with its reduction to a spin 2 (massless graviton) theory in an inertial frame of the background Minkowski (or DeSitter) space-time, is not acceptable as already noticed long time ago in Ref. [51]. Inertial effects and the coupling to matter give a *non-inertial-frame-dependent* description of the tidal degrees of freedom even in the limit of the relativistic linearized theory, which has to be defined and understood before going to the post-Newtonian limit, the only one required till now by the solar system tests of general relativity.

6.2 Equations of motion in the 3-orthogonal Schwinger time gauges

Let us now look at the Hamilton equations in the completely fixed 3-orthogonal time gauge.

Let us remark that it is not convenient to use the Dirac brackets eliminating the super-Hamiltonian and super-momentum constraints and their respective gauge fixings (4.1), because otherwise the tidal variables $R_{\bar{a}}$ and $\Pi_{\bar{a}}$ would not be any more canonical and the search of the final canonical Dirac observables $\tilde{R}_{\bar{a}}$, $\tilde{\Pi}_{\bar{a}}$, would be extremely difficult. Therefore we can use these constraints and the final gauge fixing for $\pi_{\tilde{\phi}}(\tau, \vec{\sigma})$ and $\theta^i(\tau, \vec{\sigma})$ only after the evaluation of the Poisson brackets.

Therefore the Hamilton equations of motion with the Dirac Hamiltonian (5.1) are

$$
\begin{aligned}
\partial_\tau\, R_{\bar{a}}(\tau, \vec{\sigma}) &= \{R_{\bar{a}}(\tau, \vec{\sigma}), H_D\} = \frac{\delta\, H_D}{\delta\, \Pi_{\bar{a}}(\tau, \vec{\sigma})} \\
&= \Bigg[\frac{4\pi\, G}{c^2}\, (1+n)\, \tilde{\phi}^{-1}\, \Pi_{\bar{a}} - \tilde{\phi}^{-1/3} \sum_{rb} \gamma_{\bar{a}b}\, V_{rb}(\theta^n)\, Q_b^{-1} \\
&\qquad \times \sum_a \bar{D}_{r(b)(a)}\, \bar{n}_{(a)}\Bigg](\tau, \vec{\sigma}), \\
\partial_\tau\, \Pi_{\bar{a}}(\tau, \vec{\sigma}) &= \{\Pi_{\bar{a}}(\tau, \vec{\sigma}), H_D\} = -\frac{\delta\, H_D}{\delta\, R_{\bar{a}}(\tau, \vec{\sigma})}.
\end{aligned}
\tag{6.4}
$$

where now the functional derivatives, given by Eqs. (D12) of Appendix D of Ref. [36], are evaluated by using the gauge fixings (4.1) and the solution (5.6) for the shift functions $\bar{n}_{(a)}$ (after a choice for the residual gauge freedom). To these equations we must add:

---

[14] Since the sign of the non-inertial-frame-dependent term $M^{(o)}(\tau, \vec{\sigma})$ is unknown and may vary from a region of Σ_τ to another one, we have not used a notation like in the Klein-Gordon equation $(\Box + m^2)\, \phi = 0$.

[15] See Feynman's statement [50] that *the geometrical interpretation is not really necessary or essential to physics*.

(i) the coupled Hamilton equations for the matter;
(ii) the Lichnerowicz equation (4.2) for $\tilde{\phi}$;
(iii) the equation (5.9) for the lapse function n.

All these equations depend on the solution (3.8) (or Eq. (3.10) of Ref. [36]) of the super-momentum constraints, on the choice of the zero modes (3.8), (5.6) and on the choice of the three Green functions (D1), (D2) and (D4).

Having completely fixed the gauge, we have chosen a well defined non-inertial frame and a well defined pattern of inertial potentials in the density of the weak ADM energy (the Γ-Γ term $\mathcal{S}$), in terms of the generalized tidal effects $R_{\bar{a}}(\tau,\vec{\sigma})$, $\Pi_{\bar{a}}(\tau,\vec{\sigma})$. As a consequence in Eqs. (6.4) there are *relativistic inertial forces* associated to the chosen gauge and a well defined deterministic evolution.

Modulo the ambiguities in the shift functions and in the solution of the equations (ii) and (iii), the resulting Hamilton equations (6.4) and (i) are a hyperbolic system of partial differential equations ensuring a deterministic evolution for $\tau \geq \tau_o$ of the tidal effects $R_{\bar{a}}(\tau,\vec{\sigma})$, $\Pi_{\bar{a}}(\tau,\vec{\sigma})$ and of the matter from a set of Cauchy data for $R_{\bar{a}}(\tau_o,\vec{\sigma})$, $\Pi_{\bar{a}}(\tau_o,\vec{\sigma})$ and the matter on a Cauchy surface Σ_{τ_o}.

The solution of all these equations is equivalent to a solution ${}^4g_{\mu\nu}$ of Einstein's equations written in the radar 4-coordinate system associated to the chosen 3-orthogonal non-inertial frame. This leads to an Einstein space-time, whose *chrono-geometrical structure* $ds^2 = {}^4g_{\mu\nu}(x)\,dx^\mu\,dx^\nu$ *is dynamically determined by the solution*. In particular, there is a *dynamical emergence of 3-space* [4–7]: the leaves of the $3+1$ splitting determined by the solution in the adapted radar 4-coordinates (i.e. the dynamically selected non-inertial frame centered on some time-like observer) are the instantaneous 3-spaces (the 3-universe) corresponding to a dynamical convention for the synchronization of distant clocks. One of the leaves is the Cauchy surface of the solution.

Since Eqs. (4.2), (5.6) and (5.9) imply that both n and $\bar{n}_{(a)}$ depend upon the momenta $\Pi_{\bar{a}}$, it becomes non trivial to re-express them in terms of the velocities $\partial_\tau R_{\bar{a}}$, of $R_{\bar{a}}$ and of the matter, like it was possible in Eqs. (6.1) before fixing the gauge. This is the price to be paid to have deterministic evolution. As a consequence the analogue of Eq. (6.3) becomes extremely complicated and much more non-linear. However the background-independent linearization of Eqs. (6.4) will lead to a linearized equation with the same type of behavior as the linearization of Eq. (6.3).

7 Conclusions

As shown in Ref. [4–7], ADM canonical gravity is sufficiently developed on both the theoretical and interpretational levels so that it is now possible to see which are the implications of a coherent and systematic use of constraint theory. We can finally give the Hamiltonian re-interpretation of all the procedures developed till now in the covariant Lagrangian approach, even if some of them are understood only at the theoretical level without suitable approximation schemes for practical calculations (for instance a weak field background-independent Post-Minkowskian approximation with relativistic matter motion is now under investigation). In this paper we give an alternative formulation of the York-Lichnerowicz conformal approach clarifying all its aspects like the elusive York map and which is the natural scheme for gauge fixing. Regarding this last point, so relevant for numerical gravity, we show that the determination of the lapse and shift functions is implied by the gauge fixing constraints for the super-Hamiltonian and super-momentum secondary constraints and should not be given independently as it happens in most of the treatments of numerical gravity.[16] Also the harmonic gauges, so relevant for the covariant approach and its Post-Newtonian approximations, have been shown to belong to a peculiar family of Hamiltonian gauge fixings without analog in finite-dimensional constrained systems.

As a consequence, we now have a good understanding of the Hamiltonian framework and we can try to face concrete problems ranging from $\frac{1}{c^3}$ relativistic effects near the geoid [52] (inertial effects, clock synchronization) to the notion of simultaneity to be used in astrophysics and cosmology (with the associated problem of which is the 1-way propagation velocity of electromagnetic and gravitational signals) and to the weak field approximation but with relativistic motion (fast binaries and relativistic quadrupole emission formula).

The rest-frame instant form of tetrad gravity developed in Refs. [2,3,25] for the canonical treatment of vacuum Einstein's equations in Christodoulou-Klainermann space-times and emphasizing the role of the non-inertial frames (the only one allowed by the equivalence principle), has been modified in this paper so to allow the inclusion of matter. A new parametrization of the 3-metric has made possible the explicit construction of a York map as a partial Shanmugadhasan canonical transformation. This map, end point of the Lichnerowicz–York conformal approach [30,31,34], had been shown to correspond to a canonical transformation [35], but no-one had been able to build it.

In the York canonical basis we have the identification of three groups of variables (all of them have a well defined expression in terms of the original variables, differently from the canonical basis of Refs. [3,25] for which only the inverse canonical transformation was explicitly known):

(i) The conformal factor $\phi(\tau,\vec{\sigma})$ of the 3-metric or, better, the volume element $\tilde{\phi} = \phi^6$ on Σ_τ (the unknown in the super-Hamiltonian constraint, namely the Lichnerowicz equation), and three momenta $\pi_i^{(\theta)}(\tau,\vec{\sigma})$ (the unknowns in the super-hamiltonian constraints).

(ii) The 14 gauge variables describing *generalized inertial effects* in the non-inertial frames identified by the admissible 3 + 1 splittings of space-time. They are 13 configurational variables plus the momentum $\pi_{\tilde{\phi}}(\tau,\vec{\sigma})$ proportional to the York time ${}^3K(\tau,\vec{\sigma})$, whose fixation amounts to a convention for the synchronization of distant clocks and to the identification of the instantaneous 3-space. The meaning of the other 13 gauge variables $\alpha_{(a)}(\tau,\vec{\sigma})$, $\varphi_{(a)}(\tau,\vec{\sigma})$, $\theta^i(\tau,\vec{\sigma})$, $N(\tau,\vec{\sigma}) = 1 + n(\tau,\vec{\sigma})$, $n_{(a)}(\tau,\vec{\sigma})$ has been clarified in Subsection C of Sect. 2.

(iii) two pairs of canonical (in general non-covariant) variables describing the genuine physical degrees of freedom of the gravitational field (*generalized tidal effects*). The two configurational ones are determined by the eigen-values

[16] A priori any set of gauge fixing constraints, satisfying an orbit condition, is possible. However, as it happens using coordinates not adapted to the existing structures, in this way there the risk that *coordinate singularities* will develop in the time evolution, as often happens in numerical gravity.

of the 3-metric. Since the Shanmugadhasan canonical transformation is adapted only to 10 of the 14 first-class constraints, they are not the final Dirac observables. However, if we fix completely the gauge freedom and we go to Dirac brackets, they become 4 functions of the Dirac observables of that gauge, whose identification amounts to find a Darboux basis for the Dirac brackets.

In the rest-frame instant form of tetrad gravity [1,3] the Dirac Hamiltonian contains also the weak ADM energy $E_{\mathrm{ADM}} = \int d^3\sigma\, \mathcal{E}_{\mathrm{ADM}}(\tau, \vec{\sigma})$, besides all the first class constraints. The ADM energy density $\mathcal{E}_{\mathrm{ADM}}(\tau, \vec{\sigma})$ depends on all the gauge variables. Since a completely fixed Hamiltonian gauge corresponds to the choice of a global non-inertial frame, in which the observers have fixed metrological conventions, it is natural that the ADM energy density is a gauge-dependent quantity (the problem of energy in general relativity): it contains the inertial potentials generating the inertial effects (for instance the Γ-Γ term $\mathcal{S}$ give rise to the coordinate-dependent pattern of relativistic Coriolis, centrifugal ... forces).

We have given the general solution of the super-momentum constraints in the York canonical basis and the explicit form of the super-Hamiltonian constraint, of the weak ADM energy and of the Hamilton equations for the tidal degrees of freedom of the gravitational field in a *family of completely fixed 3-orthogonal Schwinger time gauges* (*the 3-metric is diagonal;* $\theta^i(\tau, \vec{\sigma}) \approx 0$) *parametrized by the gauge function* ${}^3K(\tau, \vec{\sigma})$ (so that the convention for clock synchronization varies smoothly from one gauge to another one). Unfortunately till now we do not yet know how to make calculations in the family of Hamiltonian harmonic gauges, so that it is not possible to compare the results with those in harmonic coordinates.

The study of the equations for the shift functions, emerging from the preservation in time of the gauge fixings $\theta^i(\tau, \vec{\sigma}) \approx 0$ of the 3-orthogonal gauges, shows the appearance of a generalized Gribov ambiguity connected to the gauge freedom in the choice of the 3-coordinates on the simultaneity surfaces Σ_τ (the 3-diffeomorphism subgroup of the gauge transformations), like the ordinary Gribov ambiguity of Yang-Mills theory is connected to the freedom of non-abelian gauge transformations. It is connected to the existence of *zero modes* of the covariant divergence, which imply the non-uniqueness of the momenta $\pi_i^{(\theta)}(\tau, \vec{\sigma})$ given by the solution (3.7) of the super-momentum constraints [see the $g^{\mathrm{hom}}_{((a)(b))}(\tau, \vec{\sigma})$'s in Eq. (3.8)].

The possibility of such an ambiguity in general relativity is pointed out in Ref. [13] (see p. 765). As a consequence, the 3-orthogonal 3-coordinate system is identified on the Cauchy surface Σ_{τ_o} not only by the gauge fixings $\theta^i(\tau, \vec{\sigma}) \approx 0$, but also by gauge fixings modified by the addition of zero mode terms as shown in Subsection A of Sect. 5. Since to each such gauge fixing are associated different shift functions[17] (5.6), whose difference is connected with the the zero modes $f_{(a)}(\tau, \vec{\sigma})$ of the operator $Z_{(a)i}$ of Eqs. (5.2), (5.4), and since this ambiguity is inherited by the lapse function, it turns out that there are inequivalent $3+1$ splittings (i.e. non-inertial frames) of M^4 (*Gribov copies*) with the same 3-orthogonal 3-coordinates. In the copies with $f_{(a)}(\tau, \vec{\sigma}) \neq 0$ there are the restrictions (5.7) on the dynamical variables.

[17] This is a byproduct of the natural scheme for the gauge fixings implied by constraint theory.

Till now we were able to identify the generalized Gribov ambiguity only in the 3-orthogonal gauges. It would be important to check whether it arises also in other gauges, to exclude the possibility that the 3-orthogonal gauges are globally ill-defined due to some unknown pathology.

While in Yang-Mills theory the choice [18] of suitable weighted Sobolev spaces eliminates the ordinary Gribov ambiguity , it is not clear if the assumed direction-independent behavior of the various fields at spatial infinity (required by the absence of super-translations) is enough to eliminate the generalized Gribov ambiguity of canonical gravity. In canonical gravity, where there are no asymptotically vanishing Killing vectors [16,17] in our class of space-times, the use of weighted Sobolev spaces (see the first paper in Ref. [16,17], pp. 133–136) implies the absence of zero modes for suitable elliptic operators acting on the simultaneity surfaces Σ_τ. It is an open problem whether there are weighted Sobolev spaces compatible with the assumed direction-independent behavior at spatial infinity for the cotriads and the lapse and shift functions and such that the zero modes $g^r_{(a)}(\tau, \vec{\sigma})$ of the covariant divergence and the asymptotically vanishing ones $f_{(a)}(\tau, \vec{\sigma})$ of the operator $Z_{(a)i}$ are expelled from the function space.

To find the suitable function space for gravity plus the standard model of elementary particles, in which there are neither ordinary or generalized Gribov ambiguities nor Killing vectors, could be a difficult task, since the group manifold in large of the diffeomorphisms is not under mathematical control and we do not have the well understood topological properties of the principal fiber bundles of Yang-Mills theory.

Three independent Green functions,[18] each one defined modulo solutions of the corresponding homogeneous equation, appear inside the Hamilton equations, we will have to specify not only the initial data for the dynamical variables but also which type of conditions we have to assume on the gravitational fields at $\tau \rightarrow -\infty$ (in the linearized theory in harmonic coordinates one usually uses *retarded* conditions on the incoming radiation at minus null infinity).

Then we have written the Hamilton equations for the tidal variables in the York canonical basis in arbitrary Schwinger time gauges for tetrad gravity and explicitly shown that to get a deterministic evolution we must completely fix the gauge, i.e. we must choose a well defined non-inertial frame with its pattern of inertial forces. Given Cauchy data for the tidal variables (and matter, if present) on an instantaneous 3-space Σ_{τ_o}, one Einstein space-time is identified by solving these equations.

There are strong indications that in a generic gauge the background-independent linearization along the lines of Ref. [25] will lead to the appearance of a gauge-dependent pseudo-square-mass term. This result, joined with Ref. [51], makes the refusal of the geometrical view of the gravitational field, with its replacement with a linear spin 2 theory in an *inertial frame* of a flat background space-time, unacceptable. This refusal is induced by the fact that till now we are able to define the creation and annihilation operators for quantum fields only in such inertial frames, where there is a well posed notion of Fourier transform. The first step towards a better approximation, even if still with a background, would be the definition of quantum fields in non-inertial

[18] Two are needed for the solution of the super-momentum constraint and one for the determination of the shift functions in the 3-orthogonal gauges.

frames in Minkowski space-time. But this is still an unsolved problem due to the Torre–Varadarajan no-go theorem [53,54], showing that in general there is no unitary evolution in the Tomonaga–Schwinger formalism. The open problem, discussed in Ref. [22, 23] treating the quantization of particles in non-inertial frames, is to identify the family of non-inertial frames admitting a unitary evolution. But then the effective non-inertial Hamiltonian density will be frame-dependent also in flat space-time, due to the inertial potentials like it happens in general relativity, with the same interpretational problems.

As shown in Refs. [4–7], each independent solution of Einstein equations corresponds to an equivalence class of gauge equivalent Cauchy data on simultaneity surfaces leaves of the $3+1$ splittings connected by the gauge transformations admitted by the solution. Therefore each solution admits *preferred dynamical non-inertial frames corresponding to the dynamical chrono-geometrical structure of the solution* (including dynamically determined conventions for the synchronization of clocks implying a *dynamical emergence of a notion of instantaneous 3-space*, absent in special relativity).

Let us add a final remark on the dependence of the Hamilton equations (6.4) on the gauge function $K(\tau, \vec{\sigma})$, both explicitly as a consequence of Eq. (3.1) and implicitly through the shift and lapse functions, in the family of 3-coordinate systems where ${}^3K(\tau, \vec{\sigma}) = \epsilon\, K(\tau, \vec{\sigma})$ [see Eqs. (4.1)]. Each choice of ${}^3K(\tau, \vec{\sigma})$ corresponds to the presence of *inertial forces, attractive or repulsive according to the sign of* ${}^3K(\tau, \vec{\sigma})$*, dictated by the convention chosen for the synchronization of distant clocks, i.e. for the identification of the instantaneous 3-space*. This *non-local* effect has no non-relativistic counterpart, because Newtonian physics in Galilei space-time makes use of an absolute notion of simultaneity and there is an *absolute Euclidean 3-space*. Since the post-Newtonian calculations in harmonic coordinates [41,42] agree with the ADM ones [38,39] using ${}^3K(\tau, \vec{\sigma}) = 0$ at the 3PN order and since there are no calculations at fixed 3-coordinates but with varying ${}^3K(\tau, \vec{\sigma})$, we do not know the influence of this inertial effect on the gravitational dynamics.

It is important to find a relativistic solution of the Hamilton equations in these gauges, for instance in the weak field approximation but with relativistic motion, so to be able to understand this effect. Such a solution would allow to study the motion of a test particle along a time-like geodesic spiralling around a compact mass distribution visualized in an instantaneous 3-space Σ_τ in a family of completely fixed gauges like the ones of Eqs. (4.1) depending in a continuous way on the function $K(\tau, \vec{\sigma})$. As a consequence, we would find how the velocity of the test particle depends on the instantaneous distance inside Σ_τ (along a space-like 3-geodesic) of the test particle from the center of mass of the matter distribution and how this dependence changes as a function of $K(\tau, \vec{\sigma})$, i.e. of the definition of the instantaneous 3-space (the clock synchronization convention). Therefore, for the first time we could explicitly check which are the (weak field) general relativistic deviations from the Kepler virial theorem, which is used in the interpretation of the observational data about the rotation curves of galaxies [55]. Furthermore, in this calculation one should replace the instantaneous distance in Σ_τ, the general relativistic alternative to the absolute Euclidean distance of Newton theory, with the *luminosity distance* (a property of a congruence of light rays), the only one definable from the observed electro-magnetic signals. Do the general relativistic deviations go in the direction of reducing the quantity named *dark matter*? Or does it (or part of it) correspond to an inertial appearance, as it happens for the gravito-magnetic frame dragging? Which is the dependence of these deviations on the clock synchronization convention, i.e. on the definition of the instantaneous 3-spaces and of the associated pattern of inertial forces? Are we sure that the till now undetected WIMPs are the explaination of dark matter or, in other words, are we sure that the prevailing interpretation of the observational data is the correct one? If the quoted deviations will turn out to be negligible, this would reduce the strength of points of view like the relativistic version [56–58] of the non-relativistic MOND model[19] [59,60] or like the gravito-magnetic relativistic inertial effect of Ref. [61,62]. Otherwise there should be some coordinate-independent signature of dark matter (for instance an effective mass higher of the rest mass for ordinary matter), like it happens with the Lense-Thirring effect, a consequence of the gravito-magnetic gauge variables in presence of matter.

Moreover, the gauge dependence [including a dependence upon ${}^3K(\tau, \vec{\sigma})$] of the ADM energy density $\mathcal{E}_{\mathrm{ADM}}(\tau, \vec{\sigma})$, namely its dependence on the chosen non-inertial frame,[20] should play some role in the understanding of what is the *dark energy*, which in some way has to take into account the gravitational energy. We have to understand whether our results may help to clarify the kinematical back-reaction effects appearing in the scenario of Ref. [63], where cosmology is seen as an effective description emerging from a coarse graining starting from the gravitational field at small scales and going to larger and larger scales.

A general open problem in the astrophysical and cosmological contexts is what has to be understood with the word "*observable*": usually it is said that it must be *4-coordinate independent* (see the description of quantities connected with obseved light rays). In the context of general relativity this means independent from the 4-diffeomorphisms at the Lagrangian level, i.e. independent from the Hamiltonian gauge transformations (namely independent from the inertial effects) at the canonical level. But, apart from Einstein's *point-coincidence quantities* (what do they mean in cosmology?), we do not yet have control on this subject: also the coordinate-independent Weyl scalars of the Newman–Penrose approach [64] (used in the framework of gravitational waves) are gauge-dependent on the chice of the null tetrads. We are just beginning to understand the non-covariant coordinate-dependent Dirac observables, invariant under Hamiltonian gauge transformations, but we are still far away from identifying the coordinate- and Hamiltonian-gauge-transformation independent Bergmann observables (see Refs. [4–7,10,11] about what is known on the four eigenvalues of the Weyl tensor). As a conclusion, it is not clear to us how many interpretational problems are hidden behind the empirical notions of dark matter and dark energy.

[19] It is based on a modification of Newton's law in an inertial frame of the absolute Euclidean 3-space of Newton physics. While in the MOND model one modifies the acceleration side of the equations of motion, in general relativity it is the force side to be modified by the inertial effects.

[20] Let us remark that this already happens for the Hamiltonian describing the evolution of both relativistic and non-relativistic particles in non-inertial frames [22,23].

Appendix A: Notations for tetrad gravity

A.1 Tetrads, cotetrads and the 4- and 3- metric

We shall use the signature $\epsilon\,(+---)$ for the 4-metric, with $\epsilon = \pm$, according to particle physics and general relativity conventions respectively.

After an admissible $3+1$ splitting of space-time with space-like hyper-surfaces Σ_τ, we introduce adapted coordinates, namely the *radar 4-coordinates* $\sigma^A = (\tau; \sigma^r)$[21] adapted to the $3+1$ splitting and centered on an arbitrary time-like observer (they define a *non-inertial frame* centered on the observer, so that they are *observer and frame- dependent*).

Namely, instead of local coordinates x^μ for M^4, we use local coordinates σ^A on $R \times \Sigma \approx M^4$ with $\Sigma \approx R^3$ $[x^\mu = z^\mu(\sigma)$ with inverse $\sigma^A = \sigma^A(x)]$, i.e. a Σ_τ-*adapted holonomic coordinate basis* for vector fields $\partial_A = \frac{\partial}{\partial \sigma^A} \in T(R\times\Sigma) \mapsto b^\mu_A(\sigma)\partial_\mu = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A}\partial_\mu \in TM^4$, and for differential one-forms $dx^\mu \in T^*M^4 \mapsto d\sigma^A = b^A_\mu(\sigma)dx^\mu = \frac{\partial \sigma^A(z)}{\partial z^\mu}dx^\mu \in T^*(R\times\Sigma)$.

As shown in Ref. [2], the general cotetrads ${}^4E^{(\alpha)}_\mu$ (dual to the tetrads ${}^4E^\mu_{(\alpha)}$), appearing in the 4-metric of the ADM action principle, are connected to the cotetrads ${}^4\overset{\circ}{E}{}^{(\alpha)}_A$ (and tetrads) adapted to the $3+1$ splitting (the time-like tetrad is normal to Σ_τ) by a point-dependent standard Lorentz boost for time-like orbits acting on the flat indices (it sends the unit future-pointing time-like vector $\overset{o}{V}{}^{(\alpha)} = (1; 0)$ into the unit time-like vector $V^{(\alpha)} = l^A\, {}^4E^{(\alpha)}_A = \left(\sqrt{1+\sum_a \varphi^2_{(a)}}; \varphi^{(a)} = -\epsilon\, \varphi_{(a)}\right)$, where l^A is the unit future-pointing normal to Σ_τ)

$$
\begin{aligned}
&{}^4E^{(\alpha)}_A = L^{(\alpha)}{}_{(\beta)}(\varphi_{(a)})\, {}^4\overset{\circ}{E}{}^{(\beta)}_A, \quad {}^4E^A_{(\alpha)} = {}^4\overset{\circ}{E}{}^A_{(\beta)}\, L^{(\beta)}{}_{(\alpha)}(\varphi_{(a)}),\\
&g_{AB} = {}^4g_{AB} = {}^4E^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4E^{(\beta)}_B = {}^4\overset{\circ}{E}{}^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4\overset{\circ}{E}{}^{(\beta)}_B
\end{aligned}
\tag{A1}
$$

The adapted tetrads and cotetrads (corresponding to the *Schwinger time gauge* of tetrad gravity) are expressed at the Hamiltonian level in terms the lapse $N = 1+n > 0$ (so that $N\,d\tau$ is positive from Σ_τ to $\Sigma_{\tau+d\tau}$) and shift $n_{(a)} = {}^3e_{(a)r}\, N^r = {}^3e^r_{(a)}\, N_r$ functions and of cotriads ${}^3e_{(a)r}$ (dual to the triads ${}^3e^r_{(a)}$) on Σ_τ

$$
\begin{aligned}
&{}^4\overset{\circ}{E}{}^A_{(o)} = \frac{1}{N}\,(1; -n_{(a)}\, {}^3e^r_{(a)}) = l^A, \quad {}^4\overset{\circ}{E}{}^A_{(a)} = (0; {}^3e^r_{(a)}),\\
&{}^4\overset{\circ}{E}{}^{(o)}_A = N\,(1;0) = \epsilon\, l_A, \quad {}^4\overset{\circ}{E}{}^{(a)}_A = (n_{(a)}; {}^3e_{(a)r}),
\end{aligned}
\tag{A2}
$$

[21] For the sake of simplicity we shall use the notation $\vec{\sigma}$ for $\{\sigma^r\}$. (α) and (a) are flat 4- and 3-indices, respectively; μ is a world 4-index; A is a Σ_τ-adapted world 4-index.

As a consequence, our configuration variables for tetrad gravity are n, $n_{(a)}$, ${}^3e_{(a)r}$ and the boost parameters $\varphi_{(a)}$. The future-oriented unit normal to Σ_τ is $l_A = \epsilon\, N\, (1; 0)$ $(g^{AB}\, l_A\, l_B = \epsilon)$, $l^A = \epsilon\, N\, g^{A\tau} = \frac{1}{N}\, (1; -n^r) = \frac{1}{N}\, (1; -n_{(a)}\, {}^3e^r_{(a)})$.

As shown in Ref. [2], the induced 4-metric ${}^4g_{AB}$ and the inverse 4-metric ${}^4g^{AB}$ become in the adapted basis[22]

$$\begin{aligned}
g_{\tau\tau} &= {}^4g_{\tau\tau} = \epsilon\, [N^2 - {}^3g^{rs}\, n_r\, n_s] = \epsilon\, [N^2 - n_{(a)}\, n_{(a)}],\\
g_{\tau r} &= {}^4g_{\tau r} = -\epsilon\, n_r = -\epsilon\, n_{(a)}\, {}^3e_{(a)r},\\
g_{rs} &= {}^4g_{rs} = -\epsilon\, {}^3g_{rs} = -\epsilon\, {}^3e_{(a)r}\, {}^3e_{(a)s},\\
g^{\tau\tau} &= {}^4g^{\tau\tau} = \frac{\epsilon}{N^2}, \quad g^{\tau r} = {}^4g^{\tau r} = -\epsilon\, \frac{n^r}{N^2} = -\epsilon\, \frac{{}^3e^r_{(a)}\, n_{(a)}}{N^2},\\
g^{rs} &= {}^4g^{rs} = -\epsilon\, \left({}^3g^{rs} - \frac{n^r\, n^s}{N^2}\right) = -\epsilon\, {}^3e^r_{(a)}\, {}^3e^s_{(b)}\, \left(\delta_{(a)(b)} - \frac{n_{(a)}\, n_{(b)}}{N^2}\right).
\end{aligned} \tag{A3}$$

The 3-metric ${}^3g_{rs}$ has signature $(+++)$, so that we will put all the flat 3-indices *down*.

A.2 The constraints and the Dirac Hamiltonian

As shown in Refs. [2,3,25–28], in presence of matter with Hamiltonian mass-energy density $\mathcal{M}(\tau, \sigma)$ and 3-momentum density $\mathcal{M}_r(\tau, \vec{\sigma})$ ($\mathcal{M}$ depends on the 4-metric but not on its gradients) the primary and secondary constraints are[23]

$$\begin{aligned}
&\pi_n(\tau, \vec{\sigma}) \approx 0, \quad \pi_{\vec{n}(a)}(\tau, \vec{\sigma}) \approx 0, \quad \pi_{\vec{\varphi}(a)}(\tau, \vec{\sigma}) \approx 0,\\
&M_{(a)}(\tau, \vec{\sigma}) = \sum_{bcr} \epsilon_{(a)(b)(c)}\, [{}^3e_{(b)r}\, {}^3\tilde{\pi}^r_{(c)}](\tau, \vec{\sigma}) \approx 0,\\
&\mathcal{H}(\tau, \vec{\sigma}) = \epsilon\, \Bigg[\frac{c^3}{16\pi\, G}\, {}^3e\, {}^3R - \frac{1}{c}\, \mathcal{M}\\
&\qquad - \frac{2\pi\, G}{c^3\, {}^3e}\, {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(b)}\, {}^3e_{(c)s}\, {}^3\tilde{\pi}^s_{(d)}\Bigg](\tau, \vec{\sigma}) \approx 0,\\
&\mathcal{H}_{(a)}(\tau, \vec{\sigma}) = \left[\sum_{rb} D_{r(a)(b)}\, {}^3\tilde{\pi}^r_{(b)} - {}^3e^v_{(a)}\, \mathcal{M}_v\right](\tau, \vec{\sigma}) \approx 0,
\end{aligned} \tag{A4}$$

In Ref. [2] it is shown that the super-momentum constraints $\mathcal{H}_{(a)}(\tau, \vec{\sigma}) \approx 0$ are not the Hamiltonian generators of passive 3-diffeomorphims: the actual generators

[22] If ${}^4\gamma^{rs}$ is the inverse of the spatial part of the 4-metric $({}^4\gamma^{ru}\, {}^4g_{us} = \delta^r_s)$, the inverse of the 3-metric is ${}^3g^{rs} = -\epsilon\, {}^4\gamma^{rs}$ $({}^3g^{ru}\, {}^3g_{us} = \delta^r_s)$.

[23] ${}^3G_{o(a)(b)(c)(d)} = \delta_{(a)(c)}\, \delta_{(b)(d)} + \delta_{(a)(d)}\, \delta_{(b)(c)} - \delta_{(a)(b)}\, \delta_{(c)(d)}$ is the flat Wheeler-DeWitt supermetric. The covariant derivative is $D_{r(a)(b)} = \delta_{ab}\, \partial_r + \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(c)}$, where ${}^3\omega_{r(a)}$ is the (cotriad dependent) 3-spin connection.

are $\Theta_r(\tau,\vec{\sigma}) = \sum_{as} \Big[{}^3\tilde{\pi}^s_{(a)}\, \partial_r\, {}^3e_{(a)s} - \partial_s\, ({}^3e_{(a)r}\, {}^3\tilde{\pi}^s_{(a)})\Big](\tau,\vec{\sigma}) \approx 0$ and that we have $\mathcal{H}_{(a)}(\tau,\vec{\sigma}) = -\Big[{}^3e^r_{(a)}\, \big(\Theta_r + \sum_b {}^3\omega_{r(b)}\, M_{(b)}\big)\Big](\tau,\vec{\sigma})$.

The constraints $M_{(a)}(\tau,\vec{\sigma}) \approx 0$ generate O(3)- rotations, which vary the angles $\alpha_{(a)}(\tau,\vec{\sigma})$ hidden inside the cotriads. The boost parameters $\varphi_{(a)}(\tau,\vec{\sigma})$ and the angles $\alpha_{(a)}(\tau,\vec{\sigma})$ describe the O(3,1) gauge freedom of the tetrads in their flat indices (α) in each point of Σ_τ. The constraints $M_{(a)}(\tau,\vec{\sigma}) \approx 0$ and $\pi_{\vec{\varphi}\,(a)}(\tau,\vec{\sigma}) \approx 0$ replace the standard generators of the O(3,1) (proper Lorentz group) gauge transformations ($\varphi_{(a)}$ and $\alpha_{(a)}$ are our parametrization of the six gauge variables also appearing in the Newman-Penrose formalism, where they label the arbitrariness in the choice of the null tetrads).

Therefore, with our parametrization the independent configuration variables and the conjugate momenta of our canonical basis are

$\varphi_{(a)}$	n	$n_{(a)}$	${}^3e_{(a)r}$
≈ 0	≈ 0	≈ 0	${}^3\tilde{\pi}^r_{(a)}$

(A5)

This is a Shanmugadhasan canonical basis already naturally adapted to seven of the primary constraints. See Refs. [1,3] for the assumed (direction independent) behavior at spatial infinity of these variables: the basic information is ${}^3e_{(a)r}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} (1+\frac{M}{2r})\delta_{(a)r} + O(r^{-3/2})$, $N(\tau,\vec{\sigma}) = 1 + n(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} 1 + O(r^{-(2+\epsilon)})$ $(\epsilon > 0)$, $n_r(\tau,\vec{\sigma}) = n_{(a)}(\tau,\vec{\sigma})\, {}^3e_{(a)r}(\tau,\vec{\sigma}) \rightarrow_{r\rightarrow\infty} O(r^{-\epsilon})$ $(r = |\vec{\sigma}|)$.

From Eqs. (25) of Ref. [3] the weak or volume form of the ADM Poincaré charges of metric gravity is $[\gamma = |\det {}^3g_{rs}| = ({}^3e)^2 = \phi^{12},\ {}^3e = \det {}^3e_{(a)r}]$

$$\begin{aligned}
E_{\mathrm{ADM}} &= -\epsilon\, c\, P^\tau_{\mathrm{ADM}} = \int d^3\sigma \Bigg[\mathcal{M} - \frac{c^4}{16\pi\, G}\sqrt{\gamma}\sum_{rsuv} {}^3g^{rs}({}^3\Gamma^u_{rv}\, {}^3\Gamma^v_{su} - {}^3\Gamma^u_{rs}\, {}^3\Gamma^v_{vu}) \\
&\qquad + \frac{8\pi\, G}{c^2\sqrt{\gamma}}\sum_{rsuv} {}^3G_{rsuv}\, {}^3\tilde{\Pi}^{rs}\, {}^3\tilde{\Pi}^{uv}\Bigg](\tau,\vec{\sigma}),\\
P^r_{\mathrm{ADM}} &= -2\int d^3\sigma \left[\sum_{su} {}^3\Gamma^r_{su}(\tau,\vec{\sigma})\, {}^3\tilde{\Pi}^{su} - \frac{1}{2}\sum_s {}^3g^{rs}\mathcal{M}_s\right](\tau,\vec{\sigma}),\\
J^{\tau r}_{\mathrm{ADM}} &= -J^{r\tau}_{\mathrm{ADM}} = \int d^3\sigma \Bigg(\sigma^r \left[\frac{c^3}{16\pi\, G}\sqrt{\gamma}\sum_{nsuv} {}^3g^{ns}({}^3\Gamma^u_{nv}\, {}^3\Gamma^v_{su} - {}^3\Gamma^u_{ns}\, {}^3\Gamma^v_{vu})\right.\\
&\qquad \left. - \frac{8\pi\, G}{c^3\sqrt{\gamma}}\sum_{nsuv} {}^3G_{nsuv}\, {}^3\tilde{\Pi}^{ns}\, {}^3\tilde{\Pi}^{uv} - \frac{1}{c}\mathcal{M}\right]\\
&\qquad + \frac{c^3}{16\pi\, G}\sum_{nsuv} \delta^r_u({}^3g_{vs} - \delta_{vs})\partial_n\left[\sqrt{\gamma}({}^3g^{ns}\, {}^3g^{uv} - {}^3g^{nu}\, {}^3g^{sv})\right]\Bigg)(\tau,\vec{\sigma}),
\end{aligned}$$

$$J^{rs}_{\text{ADM}} = \int d^3\sigma \Bigg[\sum_{uv} (\sigma^r\, {}^3\Gamma^s_{uv} - \sigma^s\, {}^3\Gamma^r_{uv})\, {}^3\tilde{\Pi}^{uv} - \frac{1}{2} \sum_u (\sigma^r\, {}^3g^{su} - \sigma^s\, {}^3g^{ru})\, \mathcal{M}_u \Bigg](\tau, \vec{\sigma}). \tag{A6}$$

These weak Poincaré charges are expressed in terms of cotriads ${}^3e_{(a)r}$ and their conjugate momenta ${}^3\tilde{\pi}^r_{(a)}$, by using ${}^3g_{rs} = \sum_a {}^3e_{(a)r}\, {}^3e_{(a)s}$, ${}^3\tilde{\Pi}^{rs} = \frac{1}{4} \sum_a [{}^3e^r_{(a)}\, {}^3\tilde{\pi}^s_{(a)} + {}^3e^s_{(a)}\, {}^3\tilde{\pi}^r_{(a)}]$ (see Eq. (12) of Ref. [3]).

The Dirac Hamiltonian is (the λ's are arbitrary Dirac multipliers[24])

$$H_D = E_{\text{ADM}} + \int d^3\sigma\, \big[-\epsilon\, c\, n\, \mathcal{H} + n_{(a)}\, \mathcal{H}_{(a)}\big](\tau, \vec{\sigma}) + \int d^3\sigma\, \big[\lambda_n\, \pi_n + \lambda_{\vec{n}(a)}\, \pi_{\vec{n}(a)} + \lambda_{\vec{\varphi}(a)}\, \pi_{\vec{\varphi}(a)} + \lambda_{(a)}\, M_{(a)}\big](\tau, \vec{\sigma}), \tag{A7}$$

where the explicit form of the weak ADM energy in tetrad gravity is

$$\begin{aligned} E_{\text{ADM}} &= \int d^3\sigma \Bigg[\mathcal{M} - \frac{c^4}{16\pi\, G}\, \mathcal{S} + \frac{2\pi\, G}{c^2\, {}^3e} \times \sum_{abcd} {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(b)}\, {}^3e_{(c)s}\, {}^3\tilde{\pi}^r_{(d)} \Bigg](\tau, \vec{\sigma}),\\ \mathcal{S}(\tau, \vec{\sigma}) &= \Bigg[{}^3e \sum_{rsuv} {}^3e^r_{(a)}\, {}^3e^s_{(a)} \left({}^3\Gamma^u_{rv}\, {}^3\Gamma^v_{su} - {}^3\Gamma^u_{rs}\, {}^3\Gamma^v_{uv}\right)\Bigg](\tau, \vec{\sigma}). \end{aligned} \tag{A8}$$

It is the sum of the matter mass density, of the Γ-Γ potential term $-\frac{c^4}{16\pi\, G} \int d^3\sigma\, \mathcal{S}(\tau, \vec{\sigma})$ and of the kinetic term quadratic in the momenta.

As a consequence we have

$$\begin{aligned} &E_{\text{ADM}} + \int d^3\sigma\, [-\epsilon\, c\, n\, \mathcal{H}](\tau, \vec{\sigma})\\ &= \int d^3\sigma\, [(1+n)\, \mathcal{M}](\tau, \vec{\sigma})\\ &\quad - \frac{c^4}{16\pi\, G} \int d^3\sigma\, \left(\mathcal{S} + {}^3e\, n\, {}^3R\right)(\tau, \vec{\sigma}) + \frac{2\pi\, G}{c^2} \int d^3\sigma \Bigg[\frac{1}{{}^3e}(1+n)\\ &\quad \times \sum_{abcd} {}^3G_{o(a)(b)(c)(d)}\, {}^3e_{(a)r}\, {}^3\tilde{\pi}^r_{(b)}\, {}^3e_{(c)s}\, {}^3\tilde{\pi}^r_{(d)}\Bigg](\tau, \vec{\sigma}). \end{aligned} \tag{A9}$$

[24] In canonical metric gravity they are only 4 (not 8), namely the Hamiltonian gauge group has 8 generators (both the primary and secondary constraints) but the same number of parameters (i.e. arbitrary functions) like the 4-diffeomorphism group of the covariant Lagrangian approach. The configurational lapse and shift variables in front of the secondary constraints are the effective parameters, because the kinematical part of the Hamilton equations implies that the Dirac multipliers are their τ-derivatives.

The extrinsic curvature of the hyper-surface Σ_τ and the first half of Hamilton equations are [2] (see Sect. 2 for the definition of the $\alpha_{(a)}$ -independent "barred" variables)

$$\begin{aligned}
{}^3K_{rs} &= \epsilon\,\frac{4\pi\,G}{c^3\,{}^3\bar{e}}\sum_{abu}\Big[\Big({}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(b)s} + {}^3\bar{e}_{(a)s}\,{}^3\bar{e}_{(b)r}\Big)\\
&\qquad \times {}^3\bar{e}_{(a)u}\,\bar{\pi}^u_{(b)} - {}^3\bar{e}_{(a)r}\,{}^3\bar{e}_{(a)s}\,{}^3\bar{e}_{(b)u}\,\bar{\pi}^u_{(b)}\Big],\\
{}^3K &= -\epsilon\,\frac{4\pi\,G}{c^3}\,\frac{\sum_{ar}{}^3\bar{e}_{(a)r}\,\bar{\pi}^r_{(a)}}{{}^3\bar{e}},\\
\partial_\tau\,n(\tau,\vec{\sigma}) &= \{n(\tau,\vec{\sigma}), H_D\} = \lambda_n(\tau,\vec{\sigma}),\\
\partial_\tau\,n_{(a)}(\tau,\vec{\sigma}) &= \{n_{(a)}(\tau,\vec{\sigma}), H_D\} = \lambda_{\vec{n}(a)}(\tau,\vec{\sigma}),\\
\partial_\tau\,\varphi_{(a)}(\tau,\vec{\sigma}) &= \{\varphi_{(a)}(\tau,\vec{\sigma}), H_D\} = \lambda_{\vec{\varphi}(a)}(\tau,\vec{\sigma}),\\
\partial_\tau\,{}^3e_{(a)r}(\tau,\vec{\sigma}) &= \{{}^3e_{(a)r}(\tau,\vec{\sigma}), H_D\} = \Bigg[\epsilon_{(a)(b)(c)}\,\lambda_{(b)}\,{}^3e_{(c)r}\\
&\qquad -N\sum_s{}^3K_{rs}\,{}^3e^s_{(a)} + \partial_r\,n_{(a)}\\
&\qquad +\sum_{bs}n_{(b)}\,{}^3e^s_{(b)}\,(\partial_s\,{}^3e_{(a)r} - \partial_r\,{}^3e_{(a)s})\Bigg](\tau,\vec{\sigma}),\\
\Rightarrow\quad \partial_\tau\,{}^3g_{rs}(\tau,\vec{\sigma}) &= \Big[n_{r|s} + n_{s|r} - 2\,N\,{}^3K_{rs}\Big](\tau,\vec{\sigma}). \qquad \text{(A10)}
\end{aligned}$$

The gauge-fixing procedure illustrated in the Introduction implies that at the end the Dirac multipliers are consistently determined by the preservation in time of the gauge-fixing constraints [3].

Appendix B: The canonical transformation (2.3)

B.1 Its determination

By putting Eq. (2.4) into Eqs. (2.6) we get the following three sets of equations for the kernels K, G, F

$$\begin{aligned}
&\sum_{sha}\epsilon_{(k)(h)(a)}\,{}^3e_{(h)s}(\tau,\vec{\sigma})\,K^s_{(a)b}(\tau,\vec{\sigma}) = 0,\\
&\sum_{sha}\epsilon_{(k)(h)(a)}\,{}^3e_{(h)s}(\tau,\vec{\sigma})\,G^s_{(a)b}(\tau,\vec{\sigma}) = 0,\\
&\sum_{khsa}\Big[A_{(k)(c)}(\alpha_{(e)})\,\epsilon_{(k)(h)(a)}\,{}^3e_{(h)s}\,F^s_{(a)(b)}\Big](\tau,\vec{\sigma}) = -\delta_{(b)(c)},\\
&\sum_{ar}K^r_{(a)c}(\tau,\vec{\sigma})\,R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\,V_{rb}(\theta^n(\tau,\vec{\sigma})) = \delta_{cb},
\end{aligned}$$

$$
\begin{aligned}
&\sum_{ar} G^r_{(a)i}(\tau,\vec{\sigma})\, R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, V_{rb}(\theta^n(\tau,\vec{\sigma})) = 0,\\
&\sum_{ar} F^r_{(a)(b)}(\tau,\vec{\sigma})\, R_{(a)(b)}(\alpha_{(e)}(\tau,\vec{\sigma}))\, V_{rb}(\theta^n(\tau,\vec{\sigma})) = 0,\\
&\sum_{rla} \epsilon_{mlr}\, {}^3e_{(a)l}(\tau,\vec{\sigma})\, K^r_{(a)c}(\tau,\vec{\sigma}) = 0,\\
&\sum_{rla} \epsilon_{mlr}\, {}^3e_{(a)l}(\tau,\vec{\sigma})\, G^r_{(a)i}(\tau,\vec{\sigma}) = -B_{mi}(\theta^n(\tau,\vec{\sigma})),\\
&\sum_{rla} \epsilon_{mlr}\, {}^3e_{(a)l}(\tau,\vec{\sigma})\, F^r_{(a)(c)}(\tau,\vec{\sigma}) = 0.
\end{aligned}
\tag{B1}
$$

The solutions of the first three equations (B1) are

$$
\begin{aligned}
K^s_{(a)b} &= \sum_h {}^3e^s_{(h)}\, \tilde{K}_{(h)(a)b}, \quad \tilde{K}_{(h)(a)b} = \tilde{K}_{(a)(h)b} = \tilde{K}_{((h)(a))b},\\
G^s_{(a)i} &= \sum_h {}^3e^s_{(h)}\, \tilde{G}_{(h)(a)i}, \quad \tilde{G}_{(h)(a)i} = \tilde{G}_{(a)(h)i} = \tilde{G}_{((h)(a))i},\\
F^s_{(a)(b)} &= -\frac{1}{2}\sum_{kh} \epsilon_{(h)(a)(k)}\, B_{(b)(k)}(\alpha_{(e)})\, {}^3e^s_{(h)} + \sum_h \tilde{\Lambda}_{(h)(a)(b)}\, {}^3e^s_{(h)}\\
&\stackrel{\text{def}}{=} \sum_h {}^3e^s_{(h)} \left[Z_{(h)(a)(b)} + \tilde{\Lambda}_{(h)(a)(b)}\right],\\
\tilde{\Lambda}_{(h)(a)(b)} &= \tilde{\Lambda}_{(a)(h)(b)} = \tilde{\Lambda}_{((h)(a))(b)}, \quad Z_{(h)(a)(b)} = -Z_{(a)(h)(b)} = Z_{[(h)(a)](b)}.
\end{aligned}
\tag{B2}
$$

The second set of three equations (B1) may be rewritten in the form

$$
\begin{aligned}
&\sum_{ha} \mathcal{M}_{(h)(a)b}\, \tilde{K}_{((h)(a))c} = \delta_{bc},\\
&\sum_{ha} \mathcal{M}_{(h)(a)b}\, \tilde{G}_{((h)(a))i} = 0,\\
&\sum_{ha} \mathcal{M}_{(h)(a)b} \left[Z_{[(h)(a)](c)} + \tilde{\Lambda}_{((h)(a))(c)}\right] = 0,\\
&\text{with}\\
&\mathcal{M}_{(h)(a)b} = \sum_r {}^3e^r_{(h)}\, R_{(a)(b)}(\alpha_{(e)})\, V_{rb}(\theta^n)\\
&\qquad = \frac{R_{(a)(b)}(\alpha_{(e)})\, R_{(h)(b)}(\alpha_{(e)})}{\Lambda^b} = \mathcal{M}_{((h)(a))b},
\end{aligned}
\tag{B3}
$$

and has the following solutions

$$
\tilde{K}_{((h)(a))c} = \sum_{lm} R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, K'_{((l)(m))c}
$$

$$
\begin{aligned}
&= \sum_{lm}^{l\neq m} R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, K'_{((l)(m))c} + R_{(h)(c)}(\alpha_{(e)})\, R_{(a)(c)}(\alpha_{(e)})\, \Lambda^c,\\
&\quad K'_{(l)(l)c} = \delta_{lc}\, \Lambda^c,\\
\tilde{G}_{((h)(a))i} &= \sum_{lm}^{l\neq m} R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, G'_{((l)(m))i}, \quad G'_{(l)(l)i} = 0,\\
\tilde{\Lambda}_{((h)(a))c} &= \sum_{lm}^{l\neq m} R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, \Lambda'_{((l)(m))c}, \quad \Lambda'_{(l)(l)c} = 0,\\
Z_{[(h)(a)](c)} &= -\frac{1}{2}\, \epsilon_{(h)(a)(k)}\, B_{(c)(k)}(\alpha_{(e)})\\
&= \sum_{lm}^{l\neq m} R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, Z'_{[(l)(m)](c)}.
\end{aligned} \tag{B4}
$$

By defining

$$
\mathcal{N}^m_{(h)(a)} = \sum_{rl} \epsilon_{mlr}\, {}^3e_{(a)l}\, {}^3e^r_{(h)} = \sum_{bk} R_{(a)(b)}(\alpha_{(e)})\, R_{(h)(k)}(\alpha_{(e)})\, \mathcal{N}'^{\,m}_{(b)(k)},
$$

with

$$
\mathcal{N}'^{\,m}_{(b)(k)} = \left[\sum_{rl} \epsilon_{mlr}\, V_{lb}(\theta^n)\, V_{rk}(\theta^n)\right] \frac{\Lambda_b}{\Lambda_k} \overset{\text{def}}{=} Q^m_{bk}\, \frac{\Lambda_b}{\Lambda_k}, \quad Q^m_{bk} = -Q^m_{kb} = Q^m_{[bk]}, \tag{B5}
$$

the third set of three equations (B1) may be written in the form

$$
\begin{aligned}
&\sum_{bk}^{b\neq k} \mathcal{N}'^{\,m}_{(b)(k)}\, K'_{((k)(b))c} = 0,\\
&\sum_{bk}^{b\neq k} \mathcal{N}'^{\,m}_{(b)(k)}\, G'_{((k)(b))i} = -B_{mi}(\theta^n),\\
&\sum_{bk}^{b\neq k} \mathcal{N}'^{\,m}_{(b)(k)}\, \left[Z'_{[(k)(b)](c)} + \Lambda'_{((k)(b))(c)}\right] = 0,
\end{aligned} \tag{B6}
$$

which does not contain the already known components $K'_{(b)(b)c} = \delta_{bc}\, \Lambda^b$, $G'_{(b)(b)i} = 0$, $\Lambda'_{(b)(b)(c)} = 0$.

The solution of Eqs. (B6) are

$$
\overset{k\neq b}{Z'_{[(k)(b)](c)}} = -\frac{1}{2} \sum_m B_{(c)(m)}(\alpha_{(e)}) \sum_{ha} \epsilon_{(h)(a)(m)}\, R_{(h)(k)}(\alpha_{(e)})\, R_{(a)(b)}(\alpha_{(e)}),
$$

$$W'_{(k)(b)i} = -\frac{1}{2} \sum_{tuw} \epsilon_{tuw}\, B_{iw}(\theta^n)\, \frac{\Lambda_k}{\Lambda_b}\, V_{uk}(\theta^n)\, V_{tb}(\theta^n), \quad W'_{(b)(b)i} = 0,$$

$$K'_{((k)(b))c} = 0, \quad \Lambda'_{((k)(b))(c)} = \frac{\frac{\Lambda_k}{\Lambda_b} + \frac{\Lambda_b}{\Lambda_k}}{\frac{\Lambda_k}{\Lambda_b} - \frac{\Lambda_b}{\Lambda_k}}\, Z'_{[(k)(b)](c)},$$

$$G'_{((k)(b))i} = \frac{2\,\frac{\Lambda_b}{\Lambda_k}}{\frac{\Lambda_k}{\Lambda_b} - \frac{\Lambda_b}{\Lambda_k}}\, W'_{(k)(b)i} = \sum_{tw} \frac{\epsilon_{bkt}\, V_{tw}(\theta^n)\, B_{iw}(\theta^n)}{\frac{\Lambda_k}{\Lambda_b} - \frac{\Lambda_b}{\Lambda_k}}. \tag{B7}$$

Therefore the kernels in Eq. (2.4) are

$$K^r_{(a)b} = \sum_h {}^3e^r_{(h)}\, R_{(h)(b)}(\alpha_{(e)})\, R_{(a)(b)}(\alpha_{(e)})\, \Lambda_b = R_{(a)(b)}(\alpha_{(e)})\, V_{rb}(\theta^n),$$

$$\begin{aligned} G^r_{(a)i} &= \sum_{ml}^{m\neq l} \sum_{htuw} {}^3e^r_{(h)}\, R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, \frac{\epsilon_{tuw}\, B_{iw}(\theta^n)\, V_{ul}(\theta^n)\, V_{tm}(\theta^n)}{\frac{\Lambda_l}{\Lambda_m} - \frac{\Lambda_m}{\Lambda_l}} \\ &= \sum_{lb}^{l\neq b} \sum_{tw} R_{(a)(b)}(\alpha_{(e)})\, \frac{V_{rl}(\theta^n)\, \epsilon_{blt}\, V_{tw}(\theta^n)\, B_{iw}(\theta^n)}{\Lambda_l \left(\frac{\Lambda_l}{\Lambda_b} - \frac{\Lambda_b}{\Lambda_l}\right)}, \end{aligned}$$

$$\begin{aligned} F^r_{(a)(c)} &= -\sum_{lm}^{l\neq m} \sum_{huvk} {}^3e^r_{(h)}\, \frac{\frac{\Lambda_l}{\Lambda_m}}{\frac{\Lambda_l}{\Lambda_m} - \frac{\Lambda_m}{\Lambda_l}} \\ &\quad \times R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, R_{(u)(l)}(\alpha_{(e)})\, R_{(v)(m)}(\alpha_{(e)})\, \epsilon_{(u)(v)(k)}\, B_{(c)(k)}(\alpha_{(e)}) \\ &= -\sum_{lb}^{l\neq b} R_{(a)(b)}(\alpha_{(e)}) \sum_t \frac{V_{rl}(\theta^n)\, \epsilon_{(l)(b)(t)}\, R_{(t)(k)}(\alpha_{(e)})\, B_{(c)(k)}(\alpha_{(e)})}{\Lambda_l \left(\frac{\Lambda_l}{\Lambda_b} - \frac{\Lambda_b}{\Lambda_l}\right)}. \end{aligned} \tag{B8}$$

B.2 Inversion of $G^r_{(a)i}$

Let us look for a kernel $H_{(a)rj}$, which is an inverse of $G^r_{(a)i}$ in the following sense (Eqs. (B2) and (B4) are used)

$$\begin{aligned} \delta_{ij} &= \sum_{ar} H_{(a)rj}\, G^r_{(a)i} \\ &= \sum_{ar} H_{(a)rj} \sum_h {}^3e^r_{(h)} \sum_{lm}^{l\neq m} R_{(h)(l)}(\alpha_{(e)})\, R_{(a)(m)}(\alpha_{(e)})\, G'_{((l)(m))i} \\ &= \sum_{lm}^{l\neq m} H'_{(l)(m)j}\, G'_{((l)(m))i}, \end{aligned} \tag{B9}$$

where we have introduced the kernel $H^{'}_{(l)(m)j}$

$$H^{'}_{(l)(l)j} = 0, \quad H^{'}_{(l)(m)j} = \sum_{arh} H_{(a)rj}\, {}^3\bar{e}^r_{(l)}\, R_{(a)(m)}(\alpha_{(e)}), \quad for\ l \neq m. \tag{B10}$$

Since $G^{'}_{(l)(l)i} = 0$, we can define the following 3×3 matrix

$$\mathcal{G}_{li} = G^{'}_{((b)(k))i}, \quad l \neq b, l \neq k, b \neq k. \tag{B11}$$

Since the first two lines of Eqs. (B6) suggest the following ansatz

$$H^{'}_{(l)(m)j} = H^{'}_{((l)(m))i} = -\frac{1}{2} \sum_t A_{jt}(\theta^n) \left[\mathcal{N}^{'\,t}_{(l)(m)} + \mathcal{N}^{'\,t}_{(m)(l)}\right], \tag{B12}$$

we can also define the 3×3 matrix

$$\mathcal{H}_{jl} = H^{'}_{((b)(k))j}, \quad l \neq b, l \neq k, b \neq k. \tag{B13}$$

As a consequence Eqs. (B9) are satisfied because the second of Eqs. (B6) implies

$$\sum_l \mathcal{H}_{jl}\, \mathcal{G}_{li} = -\sum_{tbk} A_{jt}(\theta^n)\, \mathcal{N}^{'\,t}_{(b)(k)}\, G^{'}_{((b)(k))i} = \sum_t A_{jt}(\theta^n)\, B_{ti}(\theta^n) = \delta_{ij}. \tag{B14}$$

Then the first of Eqs. (B6) implies

$$\sum_j v_j\, \mathcal{H}_{jl} = 0 \Rightarrow v_j = 0 \Rightarrow \det(\mathcal{H}_{jl}) \neq 0, \tag{B15}$$

so that we have also $\det(\mathcal{G}_{li}) \neq 0$, i.e. $\mathcal{G}_{li}\, v_i = 0$ implies $v_i = 0$.

Therefore we get (also the expressions in the 3-orthogonal gauges $\theta^i(\tau, \vec{\sigma}) \approx 0$ are given)

$$\begin{aligned}
H_{(a)rj} &= \sum_{lm}^{l\neq m} R_{(a)(m)}(\alpha_{(e)})\, {}^3\bar{e}_{(l)r}\, H^{'}_{((l)(m))j} \\
&= -\frac{1}{2} \sum_{lm}^{l\neq m} R_{(a)(m)}(\alpha_{(e)})\, {}^3\bar{e}_{(l)r} \times \sum_t A_{jt}(\theta^n) \left[\mathcal{N}^{'\,t}_{(l)(m)} + \mathcal{N}^{'\,t}_{(m)(l)}\right] \\
&= -\frac{1}{2} \sum_{lm}^{l\neq m} R_{(a)(m)}(\alpha_{(e)})\, V_{rl}(\theta^n)\, \Lambda_l \sum_t A_{jt}(\theta^n) \\
&\quad \times \sum_{uv} \epsilon_{tvu} \left[\frac{\Lambda_l}{\Lambda_m} V_{um}(\theta^n)\, V_{vl}(\theta^n) + \frac{\Lambda_m}{\Lambda_l} V_{ul}(\theta^n)\, V_{vm}(\theta^n)\right] \\
\rightarrow_{\alpha_{(e)}, \theta^n \rightarrow 0}\ H^{(o)}_{(a)rj} &= \frac{1}{2} \epsilon_{arj}\, \Lambda_r \left(\frac{\Lambda_r}{\Lambda_a} - \frac{\Lambda_a}{\Lambda_r}\right) = \Lambda_r\, H^{'}_{((r)(a))j}|_{\alpha_{(e)} = \theta^n = 0},
\end{aligned}$$

$$G^r_{(a)i} \to_{\alpha_{(e)},\theta^n\to 0} G^{(o)r}_{(a)i} = \frac{\epsilon_{ari}}{\Lambda_r\left(\frac{\Lambda_r}{\Lambda_a} - \frac{\Lambda_a}{\Lambda_r}\right)},$$
$$\sum_{ar} H_{(a)rj}|_{\theta^n=0}\ \ G^r_{(a)i}|_{\theta^n=0} = \delta_{ij},\quad \sum_{ri} H_{(b)ri}|_{\theta^n=0}\ G^r_{(a)i}|_{\theta^n=0}\ = \delta_{ab}. \tag{B16}$$

B.3 The spin connection in the York basis

When $\alpha(a)(\tau,\vec{\sigma}) = 0$, the spin connection on Σ_τ is given by (also its expression in the 3-orthogonal gauges is given)

$$\begin{aligned}
{}^3\bar{\omega}_{r(a)} &= \frac{1}{2}\sum_{bc} \epsilon_{(a)(b)(c)}\, {}^3\bar{\omega}_{r(b)(c)} = \frac{1}{2}\epsilon_{(a)(b)(c)} \sum_u \Bigg[{}^3\bar{e}^u_{(b)} (\partial_r\, {}^3\bar{e}_{(c)u} - \partial_u\, {}^3\bar{e}_{(c)r}) \\
&\quad + \frac{1}{2}\sum_v {}^3\bar{e}^u_{(b)}\, {}^3\bar{e}^v_{(c)}\, {}^3\bar{e}_{(d)r} (\partial_v\, {}^3\bar{e}_{(d)u} - \partial_u\, {}^3\bar{e}_{(d)v}) \Bigg] \\
&= \frac{1}{2}\sum_{bcu} \epsilon_{(a)(b)(c)}\, V_{ub}(\theta^n) \Bigg[Q_c\, Q_b^{-1} \\
&\quad \times \Bigg(\frac{1}{3} [V_{uc}(\theta^n)\, \partial_r\, ln\, \tilde{\phi} - V_{rc}(\theta^n)\, \partial_u\, ln\, \tilde{\phi}] \\
&\quad + \sum_{\bar{b}} \gamma_{\bar{b}c} [V_{uc}(\theta^n)\, \partial_r\, R_{\bar{b}} - V_{rc}(\theta^n)\, \partial_u\, R_{\bar{b}}] \\
&\quad + \partial_r\, V_{uc}(\theta^n) - \partial_u\, V_{rc}(\theta^n) \Bigg) + \frac{1}{2}\sum_{vd} Q_d^2\, Q_b^{-1}\, Q_c^{-1}\, V_{vc}(\theta^n)\, V_{rd}(\theta^n) \\
&\quad \times \Bigg(\frac{1}{3} [V_{ud}(\theta^n)\, \partial_v\, ln\, \tilde{\phi} - V_{vd}(\theta^n)\, \partial_u\, ln\, \tilde{\phi}] \\
&\quad + \sum_{\bar{b}} \gamma_{\bar{b}d} [V_{ud}(\theta^n)\, \partial_v\, R_{\bar{b}} - V_{vd}(\theta^n)\, \partial_u\, R_{\bar{b}}] \\
&\quad + \partial_v\, V_{ud}(\theta^n) - \partial_u\, V_{vd}(\theta^n) \Bigg) \Bigg] \\
&\quad \to_{\theta^n\to 0} -\sum_b \epsilon_{rab}\, Q_r\, Q_b^{-1}\, \partial_b \left(\frac{1}{3}\, ln\, \tilde{\phi} + \sum_{\bar{b}} \gamma_{\bar{b}r}\, R_{\bar{b}} \right).
\end{aligned} \tag{B17}$$

Appendix C: The 3-geometry in the 3-orthogonal gauges

As shown in Appendices B, C and D of Ref. [36], in the 3-orthogonal gauges we have the following expression for the 3-Christoffel symbols and the Γ-Γ potential (in this appendix we use $\phi = \tilde{\phi}^{1/6}$)

$$
\begin{aligned}
{}^3\Gamma^r_{uv} &\rightarrow_{\theta^n\rightarrow 0} \delta_{ru}\left(2\,\partial_v\, ln\,\phi + \sum_{\bar a}\gamma_{\bar ar}\,\partial_v\, R_{\bar a}\right) + \delta_{rv}\left(2\,\partial_u\, ln\,\phi + \sum_{\bar a}\gamma_{\bar ar}\,\partial_u\, R_{\bar a}\right)\\
&\quad -\delta_{uv}\left(2\,\partial_r\, ln\,\phi + \sum_{\bar a}\gamma_{\bar ar}\,\partial_r\, R_{\bar a}\right) Q_u\, Q_r^{-1},\\
\sum_v {}^3\Gamma^v_{uv} &\rightarrow_{\theta^n\rightarrow 0} 6\,\partial_u\, ln\,\phi,\\
\mathcal{S} &\rightarrow_{\theta^n\rightarrow 0}\\
&\quad \phi^2 \sum_a Q_a^{-2}\left(20\,(\partial_a\, ln\,\phi)^2 - 4\sum_r (\partial_r\, ln\,\phi)^2 + 8\,\partial_a\, ln\,\phi \sum_{\bar b}\gamma_{\bar ba}\,\partial_a\, R_{\bar b}\right.\\
&\quad -2\sum_r \partial_r\, ln\,\phi \sum_{\bar b}(\gamma_{\bar ba}+\gamma_{\bar br})\,\partial_r\, R_{\bar b} + \left(\sum_{\bar b}\gamma_{\bar ba}\,\partial_a\, R_{\bar b}\right)^2\\
&\quad \left. + \sum_{\bar b}(\partial_a\, R_{\bar b})^2 - \sum_r \left(\sum_{\bar b}\gamma_{\bar br}\,\partial_r\, R_{\bar b}\right)\left(\sum_{\bar c}\gamma_{\bar ca}\,\partial_r\, R_{\bar c}\right)\right).
\end{aligned}
\tag{C1}
$$

From Eqs. (223) of Appendix A of Ref. [3] for $\theta^n = 0$ we get[25]

$$
\begin{aligned}
&{}^3\hat g_{rs} = Q_r^2\,\delta_{rs}, \quad \det {}^3\hat g_{rs} = 1, \quad \Rightarrow \sum_r {}^3\hat\Gamma^r_{rs} = 0,\\
&{}^3R[\phi, R_{\bar a}] = {}^3R[\theta^n = 0, \phi, R_{\bar a}] = \phi^{-5}\,[-8\,\hat\triangle[R_{\bar a}]\,\phi + {}^3\hat R[R_{\bar a}]\phi]\\
&\quad = -\sum_{uv}\left(\left(2\,\partial_v\, ln\,\phi + \sum_{\bar a}\gamma_{\bar au}\,\partial_v\, R_{\bar a}\right)\left(4\,\partial_v\, ln\,\phi - \sum_{\bar b}\gamma_{\bar bu}\,\partial_v\, R_{\bar b}\right)\right.\\
&\qquad + \phi^{-4}\, Q_v^2\left[2\,\partial_v^2\, ln\,\phi + \sum_{\bar a}\gamma_{\bar au}\,\partial_v^2\, R_{\bar a}\right.\\
&\qquad + 2\left(2\,\partial_v\, ln\,\phi + \sum_{\bar a}\gamma_{\bar au}\,\partial_v\, R_{\bar a}\right)\sum_{\bar b}(\gamma_{\bar bu}-\gamma_{\bar bv})\,\partial_v\, R_{\bar b}\\
&\qquad \left.\left. -\left(2\,\partial_v\, ln\,\phi + \sum_{\bar a}\gamma_{\bar av}\,\partial_v\, R_{\bar a}\right)\left(2\,\partial_v\, ln\,\phi + \sum_{\bar b}\gamma_{\bar bu}\,\partial_v\, R_{\bar b}\right)\right]\right)\\
&\qquad + \phi^{-4}\sum_u Q_u^2\left[-2\,\partial_u^2\, ln\,\phi + 2\sum_{\bar a}\gamma_{\bar au}\,\partial_u^2\, R_{\bar a}\right.\\
&\qquad \left. + \left(2\,\partial_u\, ln\,\phi + \sum_{\bar a}\gamma_{\bar au}\,\partial_u\, R_{\bar a}\right)\left(2\,\partial_u\, ln\,\phi - 2\sum_{\bar b}\gamma_{\bar bu}\,\partial_u\, R_{\bar b}\right)\right],
\end{aligned}
$$

[25] The conformal decomposition ${}^3g_{rs} = \phi^4\,{}^3\hat g_{rs}$ implies (see Eqs. (189)–(190) of Ref. [3]) ${}^3\Gamma^u_{rs} = {}^3\hat\Gamma^u_{rs} + 2\left(\delta_{ur}\,\partial_s\, ln\,\phi + \delta_{us}\,\partial_r\, ln\,\phi - \sum_{abv} V_{ra}(\theta^n)\, V_{sa}(\theta^n)\, V_{ub}(\theta^n)\, V_{vb}(\theta^n)\, Q_b^2\, Q_a^{-2}\,\partial_v\, ln\,\phi\right)$ and ${}^3R[\theta^n, \phi, R_{\bar a}] = \phi^{-5}\left[-8\,\hat\triangle\,\phi + {}^3\hat R\,\phi\right]$, where ${}^3\hat R = {}^3\hat R[\theta^n, R_{\bar a}]$ and $\hat\triangle = \partial_r\,({}^3\hat g^{rs}\,\partial_s)$ are the scalar curvature and the Laplace-Beltrami operator associated with the 3-metric ${}^3\hat g_{rs}$, respectively. $\hat\triangle - \frac{1}{8}\,{}^3\hat R$ is a conformally invariant operator.

$$
\begin{aligned}
{}^3\hat{R}[R_{\bar{a}}] &= \lim_{\phi\to 1}\ {}^3R[\phi, R_{\bar{a}}]\\
&= \sum_u \left(1 - 2\, Q_u^{-2} \sum_{\bar{b}} (\partial_u\, R_{\bar{b}})^2\right)\\
&\quad + 2 \sum_u Q_u^{-2} \sum_{\bar{a}} \gamma_{\bar{a}u} \Big[\partial_u^2\, R_{\bar{a}} + \sum_{\bar{b}} \gamma_{\bar{b}u}\, \partial_u\, R_{\bar{a}}\, \partial_u\, R_{\bar{b}}\Big],\\
\hat{\triangle}[R_{\bar{a}}] &= \partial_r\, [{}^3\hat{g}^{rs}\, \partial_s] = {}^3\hat{g}^{rs}\, {}^3\hat{\nabla}_r\, {}^3\hat{\nabla}_s = \sum_r Q_r^{-2} \Big[\partial_r^2 - 2 \sum_{\bar{b}} \gamma_{\bar{b}r}\, \partial_r\, R_{\bar{b}}\, \partial_r\Big].
\end{aligned} \tag{C2}
$$

Let us remark that we have ${}^3R[\phi, R_{\bar{a}} = 0] = -24 \sum_u (\partial_u\, ln\, \phi)^2 - 8\phi^{-4} \sum_u [\partial_u^2\, ln\, \phi - 2\, (\partial_u\, ln\, \phi)^2] \to_{\phi\to 1} {}^3R[1, 0] = 0$.

The solution (3.8) of the super-momentum constraints becomes

$$
\begin{aligned}
&\pi_i^{(\theta)}(\tau, \vec{\sigma}) \to_{\theta^n \to 0} \sum_{ab} \Big[\epsilon_{iab}\, Q_a\, Q_b^{-1}\Big] (\tau, \vec{\sigma})\\
&\times \Bigg[\sum_d \int d^3\sigma_1\, \bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau)\, \Big[\tilde{\phi}^{-1/3}\, Q_d^{-1}\, \mathcal{M}_d\\
&-\sum_e \bar{D}_{e(d)(e)}\, \tilde{\phi}^{-1/3}\, Q_e^{-1} \left(\tilde{\phi}\, \pi_{\tilde{\phi}} + \sum_{\bar{b}} \gamma_{\bar{b}e}\, \Pi_{\bar{b}}\right)\Big] (\tau, \vec{\sigma}_1)\\
&+\sum_{ec}^{c\neq e} \int d^3\sigma_1 \Bigg(\delta_{c(a}\, \delta_{b)e}\, \delta^3(\vec{\sigma}, \vec{\sigma}_1)\\
&+\sum_d \bar{\mathcal{G}}_{((a)(b))(d)}(\vec{\sigma}, \vec{\sigma}_1; \tau)\, \frac{1}{2}\Big[\bar{D}_{c(d)(e)}\, \tilde{\phi}^{-1/3}\, Q_c^{-1} + \bar{D}_{e(d)(c)}\, \tilde{\phi}^{-1/3}\, Q_e^{-1}\Big] (\tau, \vec{\sigma}_1)\Bigg)\\
&\times \Bigg(-\tilde{g}_{ce}(\tau, \vec{\sigma}_1) + \sum_f \int d^3\sigma_2\, \frac{1}{2}\, \Big[\left(\tilde{\phi}^{1/3}\, Q_e\right) (\tau, \vec{\sigma}_1)\, \bar{\zeta}^e_{(c)(f)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)\\
&+\left(\tilde{\phi}^{1/3}\, Q_c\right) (\tau, \vec{\sigma}_1)\, \bar{\zeta}^c_{(e)(f)}(\vec{\sigma}_1, \vec{\sigma}_2; \tau)\Big] \left(\tilde{\phi}^{-1/3}\, Q_f^{-1}\, \mathcal{M}_f\right) (\tau, \vec{\sigma}_2)\Bigg)\ \Bigg]\\
&\stackrel{\text{def}}{=} \sum_{ab} \Big[\epsilon_{iab}\, Q_a\, Q_b^{-1}\, F_{(ab)}\Big] (\tau, \vec{\sigma}),
\end{aligned} \tag{C3}
$$

where the last line defines the function $F_{(a)(b)}$.

The expression in the York basis of the extrinsic curvature (2.12) (after having used the solution of the super-momentum constraints) is given in Eq. (C8) of Appendix C of Ref. [36].

Appendix D: The Green functions

The Green function $\bar{\zeta}^r_{(a)(b)}$ of Ref. [3] in the 3-orthogonal gauges is

$$
\begin{aligned}
\bar{\zeta}^r_{(a)(b)}(\vec{\sigma}, \vec{\sigma}_1; \tau) &= \bar{\zeta}^r_{(a)(b)}(\vec{\sigma}, \vec{\sigma}_1; \tau | \theta^n, \phi, R_{\bar{a}}] \\
&= d^r_{\gamma_{PP_1}}(\vec{\sigma}, \vec{\sigma}_1) \left(P_{\gamma_{PP_1}}\, e^{\int_{\vec{\sigma}_1}^{\vec{\sigma}} d\sigma_2^w\, {}^3\bar{\omega}_{w(c)}(\tau, \vec{\sigma}_2) \hat{R}^{(c)}} \right)_{(a)(b)} \\
&= d^r_{\gamma_{PP_1}}(\vec{\sigma}, \vec{\sigma}_1) \sum_{n=0}^{\infty} \int_0^1 ds_n \cdots \int_0^1 ds_1\, \frac{d\sigma_2^{i_n}(s_n)}{ds_n}\, {}^3\bar{\omega}_{i_n(c_n)}(\tau, \vec{\sigma}_2(s_n)) \cdots \\
&\quad \times \frac{d\sigma_2^{i_1}(s_1)}{ds_1}\, {}^3\bar{\omega}_{i_1(c_1)}(\tau, \vec{\sigma}_2(s_1)) \left(\hat{R}^{(c_n)} \cdots \hat{R}^{(c_1)} \right)_{(a)(b)}, \\
\left(R^{(c)} \right)_{(a)(b)} &= \epsilon_{(a)(b)(c)}, \\
&\quad \vec{\sigma}_2(s)\ geodesics\ \gamma_{PP_1},\ \vec{\sigma}_2(s=0) = \vec{\sigma}_1,\ \vec{\sigma}_2(s=1) = \vec{\sigma},
\end{aligned}
$$

$$
\begin{aligned}
\theta^n = R_{\bar{a}} = 0 \ \rightarrow \ & d^r_{\gamma_{PP_1}}(\vec{\sigma}, \vec{\sigma}_1)|_{R_{\bar{a}}=0} \\
& \left(P_{\gamma_{PP_1}}\, e^{2 \int_{\vec{\sigma}_1}^{\vec{\sigma}} d\sigma_2^w\, \epsilon_{(c)(m)(n)} \delta_{(m)w} \sum_u \delta_{(n)u} \partial_u ln\, \phi(\tau, \vec{\sigma}_2) \hat{R}^{(c)}} \right)_{(a)(b)},
\end{aligned}
$$

$$
\begin{aligned}
&\theta^n = R_{\bar{a}} = 0, \phi = 1 \rightarrow \ \zeta^{(o)r}_{(a)(b)}(\vec{\sigma}, \vec{\sigma}_1) = -\delta_{(a)(b)}\, c^r(\vec{\sigma} - \vec{\sigma}_1), \\
&\sum_r \partial_r\, c^r(\vec{\sigma}) = -\delta^3(\vec{\sigma}), \quad c^r(\vec{\sigma}) = -\frac{\sigma^r}{4\pi\, |\vec{\sigma}|^3}, \\
&\sum_{rb} \bar{D}_{r(a)(b)}(\tau, \vec{\sigma})\, \bar{\zeta}^r_{(b)(c)}(\vec{\sigma}, \vec{\sigma}_1; \tau) = -\delta_{(a)(c)}\, \delta^3(\vec{\sigma}, \vec{\sigma}_1),
\end{aligned}
\tag{D1}
$$

where d^r is the Synge bitensor tangent to the geodesics γ_{PP_1}, joining the point $\vec{\sigma}$ to a generic point $\vec{\sigma}_1$ on the same Σ_τ. The Green function is defined modulo solutions of the homogeneous equation $\sum_{rb} \bar{D}_{r(a)(b)}(\tau, \vec{\sigma})\, \bar{\zeta}^{(\mathrm{hom})r}_{(b)(c)}(\vec{\sigma}, \vec{\sigma}_1; \tau) = 0$.

The Green function appearing in Eqs. (3.5) satisfies the following equation

$$\sum_{rbc}^{b\neq c}\Big[\bar{D}_{r(a)((b)}\ {}^3\bar{e}^r_{(c))}\Big](\tau,\vec{\sigma})\,\bar{\mathcal{G}}_{((b)(c))(d)}(\vec{\sigma},\vec{\sigma}_1;\tau)$$

$$= \tilde{\phi}^{-1/3}(\tau,\vec{\sigma})\sum_{rbc}^{b\neq c}\Bigg[\hat{D}_{r(a)((b)} - 2\Bigg(\delta_{(a)((b)}\,\partial_r\, ln\,\phi + \sum_u ({}^3\hat{e}_{(a)r}\ {}^3\hat{e}^u_{((b)}$$

$$- {}^3\hat{e}_{((b)r}\ {}^3\hat{e}^u_{(a)})\,\partial_u\, ln\,\phi\Bigg)\Bigg](\tau,\vec{\sigma})\,{}^3\hat{e}^r_{(c))}(\tau,\vec{\sigma})$$

$$\bar{\mathcal{G}}_{((b)(c))(d)}(\vec{\sigma},\vec{\sigma}_1;\tau) = -\delta_{ad}\,\delta^3(\vec{\sigma},\vec{\sigma}_1). \tag{D2}$$

Its explicit form is not yet known. However we know an inhomogeneous solution $d_{((b)(c))(d)}(\vec{\sigma},\vec{\sigma}_1)$ (see Eq. (E4) of Ref. [36]) in the flat Minkowski limit, where Eq. (D2) becomes

$$\sum_{bc}^{b\neq c}(\delta_{ab}\,\partial_c + \delta_{ac}\,\partial_b)\,d_{((b)(c))(d)}(\vec{\sigma},\vec{\sigma}_1) = -2\,\delta_{ad}\,\delta^3(\vec{\sigma},\vec{\sigma}_1). \tag{D3}$$

The Green function of the modified covariant derivative operator $\tilde{D}_{rij}$ of Eq. (5.5) is

$$\sum_{rk}\tilde{D}_{rik}(\tau,\vec{\sigma})\,\tilde{\zeta}^r_{kj}(\vec{\sigma},\vec{\sigma}_1;\tau) = \delta_{ij}\,\delta^3(\vec{\sigma},\vec{\sigma}_1),$$

$$\tilde{\zeta}^r_{ij}(\vec{\sigma},\vec{\sigma}_1;\tau) = d^r_{\gamma_{PP_1}}(\vec{\sigma},\vec{\sigma}_1)\sum_{n=0}^{\infty}\int_0^1 ds_n\int_0^1 ds_{n-1}\cdots\int_0^1 ds_1\,\frac{d\sigma_2^{r_n}(s_n)}{ds_n}T_{r_n i j_n}(\tau,\vec{\sigma}_2(s_n))$$

$$\times\frac{d\sigma_2^{r_{n-1}}(s_{n-1})}{ds_{n-1}}T_{r_{n-1}j_n j_{n-1}}(\tau,\vec{\sigma}_2(s_n))\cdots\frac{d\sigma_2^{r_1}(s_1)}{ds_1}\,T_{r_1 j_1 j}(\tau,\vec{\sigma}_2(s_1)), \tag{D4}$$

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