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RESEARCH ARTICLE

Exact solutions for the massless plane symmetric scalar field in general relativity, with cosmological constant

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Abstract The Klein–Gordon equations are solved for the case of a plane-symmetric static massless scalar field in general relativity with cosmological constant, generalizing the solutions found by Taub, Novotny and Horsky, and Singh. A separate class of solutions is obtained in which the metrics reduce to flat space in the limit that $\Lambda \to 0$. The static solutions can be used to generate time-dependent cosmological solutions, one of which exhibits rapid inflation followed by continued exponential expansion at all later times.

Keywords Exact solutions · General relativity · Massless scalar field · Dark energy · Domain walls · Cosmology

1 Introduction

A large number of exact solutions have been found of the Einstein Field Equations for various stress-energies [1]. Of particular interest are solutions that relate on some level to quantum theory, as such solutions may lead to a better understanding of how to combine quantum mechanics and general relativity into a single theory. In this paper, the solution of a static plane-symmetric minimally-coupled scalar field in general relativity is extended to include the possibility of a cosmological constant. These solutions can be interpreted as describing the space-time of domain walls.

Plane-symmetric space-times have been studied by numerous authors. Taub [2], in 1951, found the general vacuum solution. A generalization to a spacetime

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with cosmological constant was found by Novotny and Horsky [3] in 1974, and the solution for a plane-symmetric scalar field was found by Singh [4] in the same year. The generalization of Singh's solution presented here can be shown to contain all previous solutions as special cases. Further, a distinct family of solutions has been found that gives flat space in the limit as $\Lambda \to 0$.

Of perhaps greater interest than the static domain wall solutions are cosmological solutions generated from them. The method is exemplified in a paper by Vaidya and Som [5]. The solutions can be obtained via complex transformations, with the resulting spacetimes representing non-isotropic plane-symmetric cosmologies where the matter is given by the scalar field and cosmological constant term. One presented solution represents a cyclic universe, with alternating epoques of contraction and expansion, whereas the other exhibits rapid early inflation and subsequent accelerated expansion, suggesting the action of an inflaton field and dark energy.

2 Development of the equations

Conventions on curvature follow Carmeli [6]. The general metric for static plane symmetry can be taken to have the form

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}t^2 - \mathrm{e}^{\lambda}\mathrm{d}z^2 - Y^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{1}$$

where $\nu = \nu(z)$, $\lambda = \lambda(z)$, and $Y = Y(z)$. The Lagrangian density of the massless scalar field is

$$\pounds = \alpha g^{cd}\nabla_c\psi\nabla_d\psi\sqrt{-g} \tag{2}$$

where α is a constant. The stress-energy can be obtained by varying Eq. 2 with respect to the metric:

$$T_{ab} = \frac{\alpha_M}{\sqrt{-g}}\frac{\delta\pounds}{\delta g^{ab}} \tag{3}$$

The constant α_M determines how strongly the stress energy creates curvature. Here, it will be rolled into the constant α. The stress-energy is therefore:

$$T_{ab} = \alpha\nabla_a\psi\nabla_b\psi - \frac{1}{2}g_{ab}\left(\alpha\nabla_c\psi\nabla^c\psi\right) \tag{4}$$

Einstein's equation with cosmological constant is

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab} \tag{5}$$

whereas the equation of the massless scalar field is

$$\nabla^a\nabla_a\psi = 0 \tag{6}$$

Equations 2, 3, 4, 5 lead to the following three independent equations for the metric components:

$$-\mathrm{e}^{\nu-\lambda}\left(\frac{2Y''}{Y^2}-\frac{\lambda'Y'}{Y}+\frac{Y'^2}{Y^2}\right)+\Lambda\mathrm{e}^{\nu}=\frac{\kappa\alpha}{2}\mathrm{e}^{\nu-\lambda}\psi'^2 \tag{7}$$

$$\frac{Y'\nu'}{Y}+\frac{Y'^2}{Y^2}-\Lambda\mathrm{e}^{\lambda}=\frac{\kappa\alpha}{2}\psi'^2 \tag{8}$$

$$\frac{1}{2}Y^2\mathrm{e}^{-\lambda}\left(\frac{\nu'Y'}{Y}+\frac{2Y''}{Y}-\frac{\lambda'Y'}{Y}+\frac{1}{2}\nu'^2+\nu''-\frac{1}{2}\nu'\lambda'\right)-\Lambda Y^2=-\frac{\kappa\alpha}{2}Y^2\mathrm{e}^{-\lambda}\psi'^2 \tag{9}$$

Expanding Eq. 6 gives the static plane-symmetric massless Klein-Gordon equation:

$$\psi''+\frac{2Y'}{Y}\psi'+\left(\frac{\nu'}{2}-\frac{\lambda'}{2}\right)\psi'=0 \tag{10}$$

Multiply Eq. 7 by $\mathrm{e}^{-\nu+\lambda}$ and add it to Eq. 8:

$$\frac{(\nu'+\lambda')Y'}{Y}-\frac{2Y''}{Y^2}=\kappa\alpha\psi'^2 \tag{11}$$

Multiply Eq. 7 by $-\mathrm{e}^{-\nu+\lambda}$ and add it to Eq. 8:

$$\frac{2Y''}{Y^2}+\frac{(\nu'-\lambda')Y'}{Y}+\frac{2Y'^2}{Y^2}-2\Lambda\mathrm{e}^{\lambda}=0 \tag{12}$$

Now specialize these equations by assuming that $Y(z)$ is used as the z-coordinate. Equations 9, 10, 11, 12 then read as follows:

$$\frac{1}{2}z^2\mathrm{e}^{-\lambda}\left(\frac{\nu'}{z}-\frac{\lambda'}{z}+\frac{1}{2}\nu'^2+\nu''-\frac{1}{2}\nu'\lambda'\right)+\Lambda z^2=\frac{\kappa\alpha}{2}z^2\mathrm{e}^{-\lambda}\psi'^2 \tag{13}$$

$$\psi''+\frac{2}{z}\psi'+\left(\frac{\nu'}{2}-\frac{\lambda'}{2}\right)\psi'=0 \tag{14}$$

$$\frac{(\nu'+\lambda')}{z}=\kappa\alpha\psi'^2 \tag{15}$$

$$\frac{(\nu'-\lambda')}{z}+\frac{2}{z^2}-2\Lambda\mathrm{e}^{\lambda}=0 \tag{16}$$

This system of equations (in different coordinates) was solved by Singh [4] for the $\Lambda=0$ case. The solution is

$$\mathrm{d}s^2=|z|^{\left(-1+\frac{1}{2}\kappa\alpha b^2\right)}\mathrm{d}t^2-|z|^{\left(1+\frac{1}{2}\kappa\alpha b^2\right)}\mathrm{d}z^2-z^2\left(\mathrm{d}x^2+\mathrm{d}y^2\right) \tag{17}$$

The parameter b is a constant of integration. Setting $\alpha = 0$ in Eq. 17 recovers the plane-symmetric solution found by Taub [2].

Generalizing the metric given in Eq. 17 to one having a cosmological constant proved intractable in these coordinates, so another way had to be found. First, divide Eq. 14 by ψ':

$$\frac{\psi''}{\psi'} = -\frac{2}{z} + \frac{\lambda'}{2} - \frac{\nu'}{2} \tag{18}$$

Equation 18 can be integrated, rearranged, and squared:

$$\mathrm{e}^{\lambda} = \frac{z^4 \mathrm{e}^{\nu} \psi'^2}{b^2} \tag{19}$$

where again, b is a constant of integration. Substitute this expression for e^{λ} into the metric, obtaining:

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}t^2 - \mathrm{e}^{\nu} z^4 \psi'^2 \mathrm{d}z^2 - z^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{20}$$

The constant b has been absorbed into the definition of coordinates by a trivial coordinate transformation. Set $u = \psi(z)$ so that $\mathrm{d}u = \psi'\mathrm{d}z$. With this transformation, the metric assumes a form

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\mathrm{d}t^2 - \mathrm{e}^{\nu} W^{-4}\mathrm{d}u^2 - W^{-2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{21}$$

where W is an unknown function of u. (A similar coordinate transformation was used by Wyman [7] in the spherical case, with $\Lambda = 0$.) Einstein's equations can again be generated with this form of the metric. Note that in these coordinates the gradient of the scalar field has components (0,1,0,0). (The various trivial coordinate changes carried out subsequently have only the effect of changing these components by an overall constant factor, which can be absorbed into the definition of α.) The equations are:

$$-W^3 W'\nu' + 2W^3 W'' - W^2 W'^2 + \Lambda \mathrm{e}^{\nu} = \frac{1}{2}\kappa\alpha W^4 \tag{22}$$

$$-\frac{W'\left(\nu' W - W'\right)}{W^2} - \Lambda \mathrm{e}^{\nu} W^{-4} = \frac{1}{2}\kappa\alpha \tag{23}$$

$$\frac{1}{2}\mathrm{e}^{-\nu} W\left(-2W'' + \nu'' W + 2W'\nu'\right) - \Lambda W^{-2} = -\frac{1}{2}\kappa\alpha W^2 \mathrm{e}^{-\nu} \tag{24}$$

With a little algebra, these equations can be recast as follows:

$$-\nu'\frac{W'}{W} + \frac{2W''}{W} - \frac{W'^2}{W^2} = -\frac{\Lambda \mathrm{e}^{\nu}}{W^4} + \frac{1}{2}\kappa\alpha \tag{25}$$

$$-\nu'\frac{W'}{W}+\frac{W'^2}{W^2}=\frac{\Lambda e^{\nu}}{W^4}+\frac{1}{2}\kappa\alpha \tag{26}$$

$$-\frac{W''}{W}+\frac{1}{2}\nu''+\nu'\frac{W'}{W}=\frac{\Lambda e^{\nu}}{W^4}-\frac{1}{2}\kappa\alpha \tag{27}$$

Add Eqs. 25 and 26 and divide by 2:

$$-\nu'\frac{W'}{W}+\frac{W''}{W}=\frac{\kappa\alpha}{2} \tag{28}$$

Now add Eqs. 27 and 28, obtaining the following important result:

$$\frac{1}{2}\nu''=\frac{\Lambda e^{\nu}}{W^4} \tag{29}$$

The expression in Eq. 29 can be used to eliminate all the Λ−terms, so the three Einstein Equations become:

$$-\nu'\frac{W'}{W}+\frac{2W''}{W}-\frac{W'^2}{W^2}=-\frac{1}{2}\nu''+\frac{\kappa\alpha}{2} \tag{30}$$

$$-\nu'\frac{W'}{W}+\frac{W'^2}{W^2}=\frac{1}{2}\nu''+\frac{\kappa\alpha}{2} \tag{31}$$

$$-\frac{W''}{W}+\nu'\frac{W'}{W}=-\frac{\kappa\alpha}{2} \tag{32}$$

It can be readily shown, now, that any one of Eqs. 30, 31, 32 can be obtained from the other two, so Eqs. 31, 32 will be carried forward. Adding them, arrive at

$$\frac{1}{2}\nu''=\frac{W'^2}{W^2}-\frac{W''}{W}=-\left(\frac{W'}{W}\right)' \tag{33}$$

Equation 33 can be readily integrated, yielding the following relationship between ν and W:

$$\nu=\ln W^{-2}+c_1u+c_2 \tag{34}$$

or

$$W=e^{-\nu/2+\frac{1}{2}c_1u+\frac{1}{2}c_2} \tag{35}$$

The constant c_2 can be eliminated by trivial coordinate transformations, hence will be dropped. The expression for W in Eq. 35 and its first and second derivatives can be used to replace W in Einstein's equations in favor of ν and its derivatives. All three of Eqs. 30, 31, 32 then assume the same form, which is:

$$\nu''-\frac{3}{2}\nu'^2+2c_1\nu'-\frac{c_1^2}{2}+\kappa\alpha=0 \tag{36}$$

3 Solution of the field equations

There are a number of tricks that can be tried in solving Eq. 36, but it's important to choose the correct one, because some tries, such as choosing $c_1^2/2 = \kappa\alpha$, lead to contradictions, typically that $\Lambda = 0$. This failing also occurs with the obvious guess $\nu = Au + B$, which in fact is a solution of the $\Lambda = 0$ equations. Completing the square (see Sect. 3.2) yields solutions that do not generalize previous work.

3.1 First solution: ansatz on the form of ν

The difficulty in integrating Eq. 36 lies in the accompanying constant terms. As the try $c_1^2/2 = \kappa\alpha$ fails, it's natural to introduce a new parameter, β, and an associated function f:

$$\nu = f + \beta u \tag{37}$$

Substituting this expression into Eq. 36, obtain:

$$f'' - \frac{3}{2}f'^2 + (2c_1 - 3\beta)f' + \left(2c_1\beta - \frac{3}{2}\beta^2 + \kappa\alpha - \frac{1}{2}c_1^2\right) = 0 \tag{38}$$

To proceed, choose c_1 and β so that the constant expression is identically zero:

$$2c_1\beta - \frac{3}{2}\beta^2 + \kappa\alpha - \frac{1}{2}c_1^2 = 0 \tag{39}$$

It turns out later that a relationship between the constants c_1, β, and $\kappa\alpha$ must be derived in order to give the proper connections to less general metrics. Next, divide Eq. 38 through by f':

$$\frac{f''}{f'} - \frac{3}{2}f' + \gamma = 0 \tag{40}$$

where $\gamma = 2c_1 - 3\beta$. The resulting equation can be integrated, yielding:

$$\nu = \ln\left(A\mathrm{e}^{-\gamma u} + B\right)^{-2/3} + \beta u \tag{41}$$

Substituting this expression back into the field equation,it is found that a solution results, but only when the $\gamma = 2c_1 - 3\beta$ and $\kappa\alpha = \frac{3}{2}\beta^2 + \frac{1}{2}c_1^2 - 2c_1\beta$ hold identically, as they should. Simple coordinate transformations can eliminate one of the two constants A and B, so the constant A will be dropped. The metric can then be written:

$$\mathrm{d}s^2 = \frac{\mathrm{e}^{\beta u}}{\left(\mathrm{e}^{-\gamma u} + B\right)^{2/3}}\mathrm{d}t^2 - \frac{\mathrm{e}^{-\gamma u}}{\left(\mathrm{e}^{-\gamma u} + B\right)^2}\mathrm{d}u^2 - \frac{\mathrm{e}^{(-\frac{1}{2}\beta - \frac{1}{2}\gamma)u}}{\left(\mathrm{e}^{-\gamma u} + B\right)^{2/3}}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{42}$$

where from Eq. 39 the condition

$$\beta = -\frac{\gamma}{3} \pm \left(\frac{4\gamma^2}{9} - \frac{8}{3}\kappa\alpha\right)^{\frac{1}{2}} \tag{43}$$

must be satisfied. Hence the metric in Eq. 42 is a one-parameter family of solutions for a minimally-coupled plane-symmetric massless scalar field in general relativity, with cosmological constant. Equation 29 yields a condition on γ:

$$-\frac{1}{3}\gamma^2 B = \Lambda \tag{44}$$

For concreteness, the cosmological constant may be assumed to be carried by the constant B. It is of value to check that previously-found metrics can be obtained from this one.

3.1.1 Case 1: Reduction to Taub's metric, $\Lambda = 0, \alpha = 0$

If $\Lambda = 0$, then $B = 0$, and further, $\beta = -\frac{1}{3}\gamma \pm \frac{2}{3}\gamma$. Hence there are two possibilities, $\beta = \frac{1}{3}\gamma$ and $\beta = -\gamma$. It turns out that the minus sign, giving $\beta = -\gamma$, leads to the Taub metric. Substitution leads to

$$\mathrm{d}s^2 = \mathrm{e}^{-\frac{\gamma}{3}u}\mathrm{d}t^2 - \mathrm{e}^{\gamma u}\mathrm{d}u^2 - \mathrm{e}^{\frac{2}{3}\gamma u}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{45}$$

The substitution $z = \mathrm{e}^{\frac{\gamma}{3}u}$ then gives Eq. 17 with $\alpha = 0$, which is the Taub metric.

3.1.2 Case 2: Reduction to Novotny and Horsky's metric: $\Lambda \neq 0, \alpha = 0$

This time $B \neq 0$, but again $\beta = \frac{1}{3}\gamma$ or $\beta = -\gamma$. As before in Case 1, $\beta = -\gamma$, leads to the less general metric. Making the substitutions results in:

$$\mathrm{d}s^2 = \frac{\mathrm{e}^{-\gamma u}}{\left(\mathrm{e}^{-\gamma u} + B\right)^{2/3}}\mathrm{d}t^2 - \frac{\mathrm{e}^{\gamma u}}{\left(\mathrm{e}^{-\gamma u} + B\right)^2}\mathrm{d}u^2 - \left(\mathrm{e}^{-\gamma u} + B\right)^{-2/3}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{46}$$

Setting $\mathrm{e}^{-\gamma u} + B = B\csc^2(az)$ then leads to Novotny and Horsky's metric.

3.1.3 Case 3: Reduction to Singh's metric: $\Lambda = 0$ *and* $\alpha \neq 0$

This time taking the negative root in Eq. 43 leads to $\gamma = 0$ and an imaginary β. The positive root, however, leads to the Singh metric. Representing the root term by C for convenience, the metric is:

$$\mathrm{d}s^2 = \mathrm{e}^{\left(\frac{1}{3}\gamma + C\right)u}\mathrm{d}t^2 - \mathrm{e}^{\gamma u}\mathrm{d}u^2 - \mathrm{e}^{\left(\frac{\gamma}{3} - \frac{C}{2}\right)u}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{47}$$

Choose $z = \mathrm{e}^{\left(\frac{\gamma}{6} - \frac{C}{4}\right)u}$. The metric becomes

$$\mathrm{d}s^2 = z^{\frac{\gamma/3+C}{\gamma/6-C/4}}\mathrm{d}t^2 - z^{\frac{(2/3)\gamma+C/2}{\gamma/6-C/4}}\mathrm{d}z^2 - z^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{48}$$

Set

$$\frac{\frac{1}{3}\gamma + C}{\gamma/6 - C/4} = -1 + D \tag{49}$$

and

$$\frac{\frac{2}{3}\gamma + \frac{1}{2}C}{\gamma/6 - C/4} = 1 + D \tag{50}$$

The latter two substitutions are made in order to see if it is possible to get the metric in the form of Eq. 17. A little algebra shows that Eqs. 49 and 50 are in fact identical, so that the two equations can be solved consistently, obtaining the same answer for D in terms of the other constants. Setting $D = \kappa\alpha b^2$, it can be shown that $\gamma^2 = \left(\kappa\alpha b^2 + 3\right)^2/2b^2$ yields Singh's metric in the coordinates of Eq. 17.

3.2 Second solution: completing the square

A second solution for the metric can be found by completing the square. This method of solution would appear to be the most natural choice, but leads to a solution unrelated to the less general exact solutions. Start again from Eq. 36 and complete the square:

$$\nu'' = \frac{3}{2}\left(\nu' - \frac{2}{3}c_1\right)^2 - \frac{1}{6}c_1^2 - \kappa\alpha \tag{51}$$

Now define $p' = \nu' - \frac{2}{3}c_1$. The above equation becomes:

$$p'' = \frac{3}{2}p'^2 - |\gamma| \tag{52}$$

where $\gamma = \frac{c_1^2}{6} + \kappa\alpha$. Note here that γ is positive definite, because one term is squared and α must be positive so as to yield a positive energy for the scalar field stress energy. (Relaxing this condition, which may serve some function in studies of exotic matter or fields, would lead to two other solutions. See also Sect. 3.3, where negative energy gives rise to inflationary cosmologies.) This differential equation is mathematically similar to the problem of an object falling under constant acceleration in the presence of air friction. To solve Eq. 52, substitute

$$p' = \sqrt{\frac{2|\gamma|}{3}}\coth\theta \tag{53}$$

Obtain

$$\frac{d\theta}{du} = -\sqrt{\frac{3|\gamma|}{2}}$$

and

$$\theta = -\sqrt{\frac{3|\gamma|}{2}}u + b$$

and finally

$$\nu = \frac{2}{3}c_1 u - \frac{2}{3}\ln\left|\sinh\left(-\sqrt{\frac{3|\gamma|}{2}}u + b\right)\right| + c_0 \tag{54}$$

The constant of integration c_0 can be eliminated by trivial coordinate transformations. The metric is given by

$$\begin{aligned} ds^2 = &\frac{e^{\frac{2}{3}c_1 u}}{|\sinh(-\delta u + b)|^{2/3}}dt^2 - \frac{du^2}{|\sinh(-\delta u + b)|^2} \\ &- \frac{e^{-\frac{1}{3}c_1 u}}{|\sinh(-\delta u + b)|^{2/3}}\left(dx^2 + dy^2\right) \end{aligned} \tag{55}$$

where $\delta = \sqrt{3|\gamma|/2}$. Metric and Riemann tensor components diverge at $u = b/\delta$. Equation 29 yields the following condition:

$$\frac{c_1^2}{6} + \kappa\alpha = 2\Lambda \tag{56}$$

From Eq. 56 it is evident only $\Lambda > 0$ is permitted, and that as $\Lambda \to 0$, c_1, α, $\gamma \to 0$, yielding flat space. In view of this last relationship it appears this solution is special and distinct from the previous class. This solution does not generalize those of Taub, Singh, and Novotny and Horsky, which led to the investigation of the first class.

A second, distinct solution can be obtained by making the substitution

$$p' = \sqrt{\frac{2|\gamma|}{3}}\tanh\theta \tag{57}$$

Just as in the problem of an object falling under constant acceleration through a gravity field, this assumption leads to a different solution that is complementary to that obtained previously.

The solution is

$$
\begin{aligned}
\mathrm{d}s^2 = &\frac{\mathrm{e}^{\frac{2}{3}c_1 u}}{|\cosh\left(-\delta u + b\right)|^{2/3}}\mathrm{d}t^2 - \frac{du^2}{|\cosh\left(-\delta u + b\right)|^2} \\
&- \frac{\mathrm{e}^{-\frac{1}{3}c_1 u}}{|\cosh\left(-\delta u + b\right)|^{2/3}}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)
\end{aligned}
\tag{58}
$$

Unlike the sinh solution, this solution does not result in divergences in the metric components when the argument goes to zero.

3.3 Cosmological solutions

It is conceivable that some of these solutions are related to cosmological solutions with a time-dependent scalar field and cosmological constant. In certain cases, such as the plane-symmetric perfect fluid solution of Taub and Tabensky [8], a fluid equation of state can admit a formulation as a scalar potential. The solutions derived above can be converted to time-dependent metrics via the complex transformation $u \to iw$ and $t \to iv$, as in Vaidya and Som [5]. Further, to avoid imaginary terms in the metric, one or more constants would also have to be chosen to be imaginary. The resulting spacetimes are non-isotropic plane-symmetric cosmologies.

The cosmological solutions merit a more complete study, but two interesting examples will be presented here. Start with the cosh solution given by Eq. 58. First, make the analogous Vaidya–Som coordinate transformations $t \to iv$, $u \to iw$, and $\delta \to i\delta$, setting $c_1 = 0$ to make the solution isotropic. The metric can then be converted to Robertson–Walker form with

$$
\frac{\mathrm{d}w}{\cosh(\delta w + b)} = \mathrm{d}\tau
$$

After some algebra, the metric is given by

$$
\mathrm{d}s^2 = \mathrm{d}\tau^2 - \cos^{2/3}(\delta(\tau - c_0))\left(\mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2\right) \tag{59}
$$

where c_0 is an arbitary constant. Arriving at this form requires the implicit assumption that the scalar field contributes negative stress energy, as follows from $c_1 = 0$ and $\delta \to i\delta$. Further, by Eq. 56, the cosmological constant would have to be negative. Equation 59 is a cyclic universe similar to the Robertson–Walker dust cosmology.

The sinh solution yields a very different cosmology. After making the same substitutions as before, convert to Robertson–Walker form with

$$
\frac{-\mathrm{d}w}{\sinh(\delta w + b)} = \mathrm{d}\tau \tag{60}
$$

This results in a metric given by

$$ds^2 = d\tau^2 - \left(e^{2\delta\tau} - 1\right)^{1/3} \left(dv^2 + dx^2 + dy^2\right) \tag{61}$$

This solution of Eq. 61 exhibits rapid inflation in the early universe, followed by continued exponential expansion at all later times. (The sign in Eq. 60 can be chosen otherwise, and results in the time-reverse of this solution.) In this way, the massless scalar field acts as both a source for an inflaton field and dark energy. The more general solution based on an ansatz may allow a tuning of these results so as to fit observations. Further study of the cosmological implications of these solutions will be explored in subsequent work.

4 Concluding remarks

Two distinct classes of solutions to the problem of a plane-symmetric static field in general relativity with cosmological constant have been found. The first of these solutions involves two parameters, β and γ, which are related to a non-trivial integration constant, designated c_1. Various choices of β as a function of γ (and implicitly, of c_1) demonstrated that the presented solution was a generalization of the metrics previously derived by Taub, Novotny and Horsky, and Singh. The solutions are static domain wall spacetimes.

By analytic continuation, corresponding time-dependent solutions can be found. One of these spacetimes was shown to exhibit very rapid inflation in the early universe followed by continued exponential inflation. In these solutions the cosmological constant and scalar field are mutually dependent, each vanishing if the other does. Further study of these spacetimes would be of interest, as they appear to have some of the features required by cosmological observations.

Analogous exact solutions for spherical and hyperbolic symmetry would be desirable. The current work, for example, bears similarity to work done on spherically-symmetric scalar fields by Wyman [7] and others [9]–[12] who independently found solutions for spherically-symmetric scalar fields with $\Lambda = 0$. These equations have been examined and are at a higher level of complexity. Nonetheless, there is hope for such solutions, and an effort is underway.

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