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RESEARCH ARTICLE

Thin shell wormholes in higher dimensional Einstein–Maxwell theory

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Abstract We construct thin shell Lorentzian wormholes in higher dimensional Einstein–Maxwell theory applying the 'Cut and Paste' technique proposed by Visser. The linearized stability is analyzed under radial perturbations around some assumed higher dimensional spherically symmetric static solution of the Einstein field equations in presence of Electromagnetic field. We determine the total amount of exotic matter, which is concentrated at the wormhole throat.

Keywords Thin shell wormholes · Electromagnetic field · Higher Dimension · Stability

1 Introduction

In a pioneer work, Morris and Thorne [1] have found traversable Lorentzian wormholes as the solutions of Einstein's field equations. These are hypothetical

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shortcuts between two regions (of the same Universe or may be of two separate Universes) connected by a throat. The throat of the wormholes is defined as a two dimensional hypersurface of minimal area and to hold such a wormhole open, violations of certain energy conditions are unavoidable i.e. the energy momentum tensor of the matter source of gravity violates the local and averaged null energy condition $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0, k_{\mu}k^{\nu} = 0$. Thus all traversable wormholes require exotic matter that violates the null energy condition. Recently, it has been shown that the requirements of exotic matter for the existence of a wormhole can be made infinitesimally small by a suitable choice of the geometry [2–3].

In recent past, Visser [4] has proposed another way, which is known as 'Cut and Paste' technique, of minimizing the usage of exotic matter to construct a wormhole in which the exotic matter is concentrated at the wormhole throat. In 'Cut and Paste' technique, the wormholes are theoretically constructed by cutting and pasting two manifolds to obtain geodesically complete new manifold with a throat placed in the joining shell [5]. Using Darmois–Israel [6] formalism, one can determine the surface stresses of the exotic matter (located in thin shell placed at the joining surface). Though we do not know about the equation of state of exotic matter, yet it is possible to investigate the stability of these thin wormholes. Following references [5–6], one can analyze the stability of these thin wormholes through linearized perturbations around static solutions of the Einstein field equations. Several authors have used surgical technique (Cut and Paste) to construct thin wormholes. Poisson and Visser [5] have analyzed the stability of a thin wormhole constructed by joining two Schwarzschild spacetimes. Eiroa and Romero [7] have extended the linearized stability analysis to Reissner–Nordström thin spacetimes. Eiroa and Simeone[8] have constructed the wormholes by cutting and pasting two metrics corresponding to a charged black hole which is a solution of low energy bosonic string theory, with vanishing antisymmetric field but including a Maxwell field. Also the same authors have analyzed cylindrically symmetric thin wormhole geometry associated to gauge cosmic strings [9]. Recently, Thibeault et al.[10] have studied the stability and energy conditions of five dimensional spherically symmetric thin shell wormholes in Einstein–Maxwell theory with addition of a Gauss Bonnet term. In this article, we study thin shell wormholes in higher dimensional Einstein–Maxwell theory i.e. wormholes constructed by cutting and pasting two metrics corresponding a higher dimensional Reissner–Nordström black hole. We are interested only to study the geometry of these objects. We do not explain about the mechanism that provide the exotic matter to them, but rather we focus on the total amount of exotic matter.

2 Reissner–Nordström black holes in higher dimension

The Reissner–Nordström black hole is a solution of the Einstein equation coupled to the Maxwell field. From the Einstein–Maxwell action in (D+2) dimension [11]

$$S = \int \mathrm{d}^{D+2}\sqrt{-g}\left[R - \frac{k}{8\pi}F_{ab}F^{ab}\right] \tag{1}$$

where

$$k = 8\pi G \tag{2}$$

$$F_{ab} = A_{a;b} - A_{b;a} \tag{3}$$

One can obtain the following Einstein–Maxwell equations:

$$R_{ab} - \frac{1}{2}g_{ab}R = \frac{k}{4\pi}\left[F_a^c F_{bc} - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right] \tag{4}$$

$$F^c_{a;c} = 0 \tag{5}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{6}$$

The only non-trivial components of F_{ab} are

$$F_{tr} = -F_{rt} = \frac{Q}{r^D} \tag{7}$$

where Q represents an isolated point charge.

These equations admit a spherically symmetric static solution given by [11]

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega_D^2 \tag{8}$$

where $\mathrm{d}\Omega_D^2$ is the line element on the D unit sphere i.e.

$$\mathrm{d}\Omega_D^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\theta_2^2 + \cdots + \prod_{n=1}^{D-1}\sin^2\theta_n\mathrm{d}\theta_D^2 \tag{9}$$

The volume of the D unit sphere is given by

$$\Omega_D = 2\frac{\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})} \tag{10}$$

The expression of $f(r)$ is [11]

$$f(r) = 1 - \frac{\mu}{r^{D-1}} + \frac{q^2}{r^{2(D-1)}} \tag{11}$$

.

Here, μ is related to the mass M of the black hole as

$$\mu = \frac{16\pi GM}{D\Omega_D} \tag{12}$$

$$q^2 = \frac{8\pi GQ}{D(D-1)} \tag{13}$$

There are two roots of equation $f(r) = 0$ as

$$r_\pm = \left[\frac{\mu}{2} \pm \frac{\mu}{2}\left(1 - \frac{4q^2}{\mu^2}\right)\right]^{\frac{1}{D-1}} \tag{14}$$

When $\mu^2 < 4q^2$, we have two positive roots, one of which is an outer horizon r_+ while the other is inner horizon.

If $\mu^2 = 4q^2$, both horizons coincide at

$$r_+ = r_- = \left[\frac{8\pi GM}{D\Omega_D}\right]^{\frac{1}{D-1}} \tag{15}$$

3 The Darmois–Israel formalism and 'Cut and Paste' construction

From the higher dimensional Reissner–Nordström geometry, we can take two copies of the region with $r \geq a$:

$$M^\pm = (x \mid r \geq a)$$

and paste them at the hypersurface

$$\Sigma = \Sigma^\pm = (x \mid r = a)$$

We take $a > r_+$ to avoid horizon and this new construction produces a geodesically complete manifold $M = M^+ \bigcup M^-$ with a matter shell at the surface $r = a$, where the throat of the wormhole is located. Thus M is a manifold with two asymptotically flat regions connected by the throat. We shall use the Darmois–Israel formalism to determine the surface stress at the junction boundary. Now we choose the coordinates $\xi^i(\tau, \theta_1, \theta_2, \ldots, \theta_D)$ in Σ where the throat is located with τ is the proper time on the shell.

To analyze the dynamics of the wormhole, we let the radius of the throat be a function of the proper time $a = a(\tau)$.

The parametric equation for Σ is given by

$$\Sigma : F(r, \tau) = r - a(\tau) \tag{16}$$

The intrinsic surface stress energy tensor, S_{ij} is given by the Lanczos equation in the following form:

$$S^i_j = -\frac{1}{8\pi}\left(\kappa^i_j - \delta^i_j \kappa^k_k\right) \tag{17}$$

where $\kappa_{ij} = K^+_{ij} - K^-_{ij}$ i.e. the discontinuity in the second fundamental forms or extrinsic curvatures.

The extrinsic curvature associated with the two sides of the shell are

$$K^\pm_{ij} = -n^\pm_\nu \left[\frac{\partial^2 X_\nu}{\partial\xi^i \partial\xi^j} + \Gamma^\nu_{\alpha\beta}\frac{\partial X^\alpha}{\partial\xi^i}\frac{\partial X^\beta}{\partial\xi^j}\right]\bigg|_\Sigma \tag{18}$$

where $n^\pm_\nu$ are the unit normals to Σ,

$$n^\pm_\nu = \pm\left|g^{\alpha\beta}\frac{\partial F}{\partial X^\alpha}\frac{\partial F}{\partial X^\beta}\right|^{-\frac{1}{2}}\frac{\partial F}{\partial X^\nu} \tag{19}$$

with $n^\mu n_\mu = 1$.

The intrinsic metric on Σ is given by

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \mathrm{d}\Omega^2_D \tag{20}$$

From Lanczos equation, one obtain the surface stress energy tensor $S^i_j = \mathrm{diag}(-\sigma, p_{\theta_1}, p_{\theta_2}, \ldots, p_{\theta_D})$, where σ is the surface energy density and p is the surface pressure as

$$\sigma = -\frac{D}{4\pi a}\sqrt{f + \dot{a}^2} \tag{21}$$

$$p_{\theta_1} = p_{\theta_2} = \cdots = p_{\theta_D} = p = -\frac{D-1}{D}\sigma + \frac{1}{8\pi}\frac{2\ddot{a} + f'}{\sqrt{f + \dot{a}^2}} \tag{22}$$

where over dot and prime mean, respectively, the derivatives with respect to τ and r.

From Eqs. (21) and (22), one can verify the energy conservation equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\sigma a^D) + p\frac{\mathrm{d}}{\mathrm{d}\tau}(a^D) = 0 \tag{23}$$

or

$$\dot{\sigma} + D\frac{\dot{a}}{a}(p + \sigma) = 0 \tag{24}$$

The first term represents the variation of the internal energy of the throat and the second term is the work done by the throat's internal forces. Negative energy density in Eq. (21) implies the existence of exotic matter at the shell.

4 Linearized stability analysis

Rearranging Eq. (21), we obtain the thin shell's equation of motion

$$\dot{a}^2 + V(a) = 0 \tag{25}$$

Here the potential is defined as

$$V(a) = f(a) - \frac{16\pi^2 a^2 \sigma^2(a)}{D^2} \tag{26}$$

Linearizing around a static solution situated at a_0, one can expand V(a) around a_0 to yield

$$V = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + 0[(a - a_0)^3] \tag{27}$$

where prime denotes derivative with respect to a.

Since we are linearizing around a static solution at $a = a_0$, we have $V(a_0) = 0$ and $V'(a_0) = 0$. The stable equilibrium configurations correspond to the condition $V''(a_0) > 0$. Now we define a parameter β, which is interpreted as the speed of sound, by the relation

$$\beta^2(\sigma) = \left.\frac{\partial p}{\partial \sigma}\right|_{\sigma} \tag{28}$$

Using conservation Eq. (24), we have

$$\begin{aligned} V''(a) = f'' - \frac{32\pi^2 a^2 \sigma^2}{D^2} - \frac{128\pi^2 a \sigma \sigma'}{D^2} - \frac{32\pi^2 a^2 (\sigma')^2}{D^2} \\ - \frac{32\pi^2 a^2 \sigma}{D^2}\left[\frac{D}{a^2}(p + \sigma) - \frac{D}{a}\sigma'\left(1 + \beta^2\right)\right] \end{aligned} \tag{29}$$

The wormhole solution is stable if $V''(a_0) > 0$ i.e. if

$$\beta_0^2 < \frac{1}{D(a_0 f_0' - 2f_0)}\left[a_0 f_0' - 2f_0 - a_0^2 f_0'' + \frac{a_0^2 (f_0')^2}{2f_0}\right] - 1 \tag{30}$$

or

$$\beta_0^2 < \frac{1}{D} - 1 + \frac{\left[\frac{2\mu(D-1)(D-2)}{a_0^{D-1}} + \frac{\mu^2(D-1)(9D-15)}{a_0^{2D-2}} - \frac{2\mu q^2(D-1)(4D-7)}{a_0^{3D-3}} + \frac{4q^4(D-1)(D-2)}{a_0^{4D-4}}\right]}{2\left(1 - \frac{\mu}{a_0^{D-1}} + \frac{q^2}{a_0^{2(D-1)}}\right)\left(2 - \frac{\mu(D+1)}{a_0^{D-1}} + \frac{2q^2 D}{a_0^{2(D-1)}}\right)} \tag{31}$$

Thus if one treats a_0, D and the parameters related to the Reissner–Nordström black hole are specified quantities, then the stability of the configuration requires the above restriction on the parameter β_0.

5 Energy condition and exotic matter

Weak energy condition (WEC) implies that for all time like vectors x^μ, $T_{\mu\nu}x^\mu x^\nu \geq 0$. In an orthonormal basis WEC reads $\rho \geq 0, \rho + p_i \geq 0 \, \forall i$, where ρ is the energy density and p_i , the principal pressures. Null energy condition (NEC) states that $T_{\mu\nu}k^\mu k^\nu \geq 0$ for all null vectors k^μ. In an orthonormal frame $T_{\mu\nu}k^\mu k^\nu \geq 0$ takes the form $\rho + p_i \geq 0 \forall i$. (The WEC implies by continuity the NEC). In the case of thin wormhole constructed above, we have [from Eqs. (21) and (22)] $\sigma < 0$ and $\sigma + p < 0$ i.e. matter occupies in the shell violates WEC and NEC, in other words, shell contains exotic matter. The only contributor in the stress tensor out side the shell is electromagnetic field. Now from the field equations $R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab}$, we can write $T_{ab} = T_{ab}^{EM} = \frac{1}{4\pi}[F_a^c F_{bc} - \frac{1}{4}g_{ab}F_{cd}F^{cd}]$.

Now, the energy density $\rho = T_{tt}$, the radial pressure $p_R = T_{rr}$ and the tangential pressure $p_t = p_\theta = p_\phi$ are given by $\rho^{\mathrm{EM}} = p_t^{\mathrm{EM}} = -p_r^{\mathrm{EM}} = \frac{q^2}{r^{2D}}$. Thus $\rho^{\mathrm{EM}} > 0$, $\rho^{\mathrm{EM}} + p_t^{\mathrm{EM}} > 0$ and $\rho^{\mathrm{EM}} + p_r^{\mathrm{EM}} = 0$ i.e. the NEC and WEC are satisfied out side the shell. Hence the exotic matter is confined within the shell. The total amount of exotic matter can be quantified by the integrals [3] $\int \rho\sqrt{-g}\mathrm{d}^{D+1}x$, $\int[\rho + p_i]\sqrt{-g}d^{D+1}x$, where g is the determinant of the metric tensor. To quantify the amount of exotic matter, we use the following integral (NEC violating matter is related only on p_r and not the transverse components):

$$\Omega = \int [\rho + p_r]\sqrt{-g}\mathrm{d}^{D+1}x \tag{32}$$

Following Eiroa and Simone [8], we introduce new radial coordinate $R = \pm(r - a)$ in M ($\pm$ for $M^\pm$, respectively) as

$$\Omega = \int_0^{2\pi}\int_0^{\pi} \cdots \int_0^{\pi}\int_{-\infty}^{\infty} [\rho + p_r]\sqrt{-g}dR\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_D \tag{33}$$

Since the shell does not exert radial pressure and the energy density is located on a thin shell surface, so that $\rho = \delta(R)\sigma_0$, then we have

$$\begin{aligned}\Omega &= \int_0^{2\pi}\int_0^{\pi} \cdots\cdot \int_0^{\pi} [\rho\sqrt{-g}]|_{r=a_0}\mathrm{d}\theta_1\mathrm{d}\theta_2\cdots\mathrm{d}\theta_D \\ &= a_0^D\sigma_0 \times \text{area of the unit D-sphere} \\ &= 2a_0^D\sigma_0\frac{\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}\end{aligned}$$

Thus one gets

$$\Omega = -Da_0^{D-1}\sqrt{f_0}\frac{\pi^{\frac{D+1}{2}}}{2\Gamma(\frac{D+1}{2})} \tag{34}$$

Using Eq. (10), we have

$$\Omega = -Da_0^{D-1}\sqrt{1 - \frac{\mu}{a_0^{D-1}} + \frac{q^2}{a_0^{2(D-1)}}}\frac{\pi^{\frac{D+1}{2}}}{2\Gamma(\frac{D+1}{2})} \tag{35}$$

Since the total amount of exotic matter Ω is proportional to $\sqrt{f_0}$, then Ω approaches to zero when wormhole radius tends to the event horizon (i.e. when $a_0 \to r_+$). So one can get vanishing amount of exotic matter by taking a_0 near r_+.

6 Concluding remarks

Recently, several theoretical physicists are interested to obtain wormholes by surgically grafting two identical copies of various well known spacetimes. In this report, we have constructed thin wormhole in higher dimensional Einstein–Maxwell theory. We analyze the dynamical stability of the thin shell, considering linearized radial perturbations around stable solutions. To analyze this, we define a parameter $\beta^2 = \frac{p'}{\sigma'}$ as a parametrization of the stability of equilibrium. We have obtained a restriction on β^2 to get stable equilibrium of the thin wormhole [see eq.(31)]. We have shown that matter within the shell violates the WEC and NEC but matter out side the shell obeys the NEC and WEC. Thus the exotic matter is confined only within the shell. Since the viability of traversable wormholes are linked to the total amount of exotic matter for their construction, we have calculated an integral measuring of the total amount of exotic matter. Finally, we have shown that total amount of exotic matter needed to support traversable wormhole can be made infinitesimal small by taking wormhole radius near the throat.

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