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RESEARCH ARTICLE

Dynamical evolution and leading order gravitational wave emission of Riemann-S binaries

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Abstract An approximate strategy for studying the evolution of binary systems of extended objects is introduced. The stars are assumed to be polytropic ellipsoids. The surfaces of constant density maintain their ellipsoidal shape during the time evolution. The equations of hydrodynamics then reduce to a system of ordinary differential equations for the internal velocities, the principal axes of the stars and the orbital parameters. The equations of motion are given within Lagrangian and Hamiltonian formalism. The special case when both stars are axially symmetric fluid configurations is considered. Leading order gravitational radiation reaction is incorporated, where the quasi-static approximation is applied to the internal degrees of freedom of the stars. The influence of the stellar parameters, in particular the influence of the polytropic index n, on the leading order gravitational waveforms is studied.

Keywords Riemann-S binaries · Gravitational waves · Polytropic equation of state

1 Introduction

The search for gravitational waves is one of the most challenging projects of twenty first century physics. Inspiraling compact binaries are among the most promising sources for gravitational waves that could be detected by gravitational waves observatories, such as LIGO, VIRGO and GEO600. During most of the inspiral time the gravitational waves emitted by a compact binary system

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are much to weak, the frequencies are much to small to be detectable by existing gravitational wave observatories. However, at orbital distances corresponding to the last 10 or 15 min of the inspiral time, the orbital frequency increases from around 1 Hz up to 1,000 Hz, while in the same time the amplitude of the gravitational waves increases, thus improving the chance of actually detecting the weak signals.

In order to extract a possible signal from the detector noise the gravitational waveforms emitted by inspiraling binary systems must be known in great detail, in particular during the last stages of the inspiral process before the final plunge. The frequency gap covered by contemporary gravitational wave observatories ranges from 1 to 1,000 Hz. In this regime post-Newtonian (pN) effects become important and have to be included into the analysis. For non spinning point-particle binaries an analytic solution is available up to the 3rd pN approximation [1]. Incorporating the spin leads to enormeous complifications. Until now there exists a solution only for point-mass binaries with either two equal masses and arbitrary spins, or two arbitrary masses and only one spinning object [2].

However, while the components of black hole-black hole binaries can be treated as pointlike objects eventually up to the innermost stable circular orbit (ISCO), yet another effect has to be incorporated into the analysis for all other binary systems: the internal structure of the components. Lai and Wiseman [3] argue that during most of the final inspiral process even the neutron stars can be treated as pointlike objects. This is true if one considers only a few orbits. If, however, one is interested in the long-term evolution of the system, the tidal interaction of the neutron stars will lead to a phase shift in the gravitational waveforms, which is not neglegible anymore.

In this paper we shall focus on the problem of including tidal interaction into the equations of motion and we discuss the influence of the internal stellar dynamics on the binary's dynamics as well as on the gravitational waveforms in leading order approximation. PN effects will be included in a forthcoming paper.

A detailed description of the tidal interaction and the corresponding changes in the gravitational waveforms requires the application of three-dimensional numerical hydrodynamics. This is well beyond the scope of this paper. Instead we shall consider an approximative formulation of the problem. This strategy allows an analytic formulation of the equations of motion. Moreover, we are able to run long-term calculations of the system's dynamics with sufficient accuracy. There are basically two approaches for an approximate description of the tidal interaction. The first one is based on a linear adiabatic theory, describing small derivations from equilibrium. Assuming an adiabatic equation of state, the linearized equations of hydrodynamics have been derived by Ledoux and Walraven [4] and Dziembowski [5]. Press and Teukolsky [6] developed a technique for calculating normal modes of non-radial stellar oscillations in the presence of tidal forces. Using the formulation of Dziembowski [5], Kokkotas and Schäfer [7] studied the leading order dynamics and gravitational waveforms of an NS–NS binary system, in particular taking into account the influence of tidal interaction on the actual waveforms. Resonant tidal excitations of a binary neutron star with polytropic equations of state have been investigated by Ho and Lai [8] in leading order approximation.

However, there are scenarios when the linear adiabatic theory is not applicable. In particular the theory fails when the stellar oscillations cannot be considered as small derivations from equilibrium. Thus far only little investigation has been undertaken in this direction. For a rotating, oscillating disk of dust there exists an analytic solution up to first PN order [9]. A system of an rotating, osillating dusty disk and a point-mass object has been studied by the author in a recent paper [10]. A so-called 'affine stellar model' for polytropic, ellipsoidal configurations has been developed by Carter and Luminet [11]. Here the density contours in the star are assumed to form homologeous ellipsoids. The model allows for the discussion of arbitrary large amplitudes of the tidally generated oscillations, but fails to give accurate results for small oscillations. The excited mode roughly corresponds to the f-mode of the oscillations (see also [12]). Later on Lai and Shapiro [13] extended this model to allow for the investigation of Riemann-S binaries. In this paper we shall follow the approach of Lai and Shapiro [13], pointing out some discrepancies in the authors' formulation of the equations of motion as well as for the gravitational waveforms.

The paper is organized as follows: In Sect. 2 we briefly recapitulate the Lagrangian formulation of the dynamics of Riemann-S binaries derived by Lai and Shapiro in a series of papers [13–15]. In this model the stars are considered as rotating and oscillating triaxial ellipsoids with a polytropic equation of state. To keep the analysis as simple as possible, the tidal interaction potential is truncated, only the leading order (i.e. quadrupole) tidal interaction is taken into account. For later reasons we shall also present the Hamiltonian formulation of this problem. In particular we shall specialize to axially symmetric polytropic stars.[1] Section 3 is devoted to the derivation of the leading order gravitational reaction terms appearing in the dynamical equations. We shall point out some discrepancies between our approach and the approach taken by Lai and Shapiro. The leading order gravitational waveforms are derived in Sect. 4. In particular we shall apply the quasi-static approximation to the internal stellar dynamics. The corresponding equations are derived for general triaxial Riemann-S binaries, but for simplicity we restrict ourselves to axially symmetric configurations in the numerical calculations. The numerical results and some possible physical applications of our model are discussed in Sect. 5. In particular we study the influence of the polytropic index and hence of the equation of state on the dynamics as well as on the leading order gravitational waveforms. We shall show that the effect of the internal stellar structure on the long term evolution of the binary is quite significant. We argue that the exact knowledge of the polytropic equation of state is essential in order to calculate the leading order gravitational waveforms emitted by the binary system correctly.

[1] Just setting $a_1 = a_2$ in the equations of motions gives rise to divergent terms.

2 The Newtonian dynamics of Riemann-S binaries

2.1 Lagrangian formalism

On Newtonian order the Lagrangian equations of motion for compressible Riemann-S binaries have been derived by Lai and Shapiro in several papers (see [13–15]) and we shall therefore only shortly recapitulate their strategy. We then turn our attention to the Hamiltonian formulation of the problem which will be more suitable for numerical calculations. The case of axially symmetric, polytropic stars is of particular interest for us, but requires some special care in order not to overcount the degrees of freedom. Consider a binary consisting of two ellipsoidal fluid configurations M and M', assuming the stellar matter to obey a polytropic equation of state,

$$P = K\rho^{1+1/n}, \quad P' = K'\rho'^{1+1/n'}. \tag{1}$$

Here n and n' are the polytropic indices, while K and K' represent constants which are determined by the equilibrium radii of non-rotating, spherically symmetric polytropes of the same mass. A Riemann-S ellipsoid is characterized by the angular velocity $\Omega = \Omega\mathbf{e}_z$ of the ellipsoidal figure around the principal axes and the internal vorticity $\zeta = \zeta\mathbf{e}_z$. The model is based on two assumptions: First, we shall require the vorticity to be uniform, and second, the surfaces of constant density inside the stars are assumed to form self-similar ellipsoids. In other words, each star is described by only five degrees of freedom: the three semi-major axes a_1, a_2, a_3 and a'_1, a'_2, a'_3 respectively, and the two angles ψ and γ, which are introduced in Fig. 1.

Consider now an isolated star M. If the assumptions given above are to be fullfilled, it is immediately clear that the fluid velocity in the body fixed system, which rotates with angular velocity $\Omega = \Omega\mathbf{e}_3$ ($\Omega = \dot{\gamma}$) with respect to an inertial system centered at M, obeys the following ansatz:

$$\mathbf{u}_c = Q_1 x_2 \mathbf{e}_1 + Q_2 x_1 \mathbf{e}_2. \tag{2}$$

Here Q_1, Q_2 are constants and x_1, x_2 denote coordinates in the body fixed frame. The second requirement yields

$$Q_1 = -\frac{a_1^2}{a_1^2 + a_2^2}\,\zeta = \frac{a_1}{a_2}\,\Lambda, \quad Q_2 = \frac{a_2^2}{a_1^2 + a_2^2}\,\zeta = -\frac{a_2}{a_1}\,\Lambda,$$

where ζ is the vorticity in the body fixed frame and $\Lambda = \dot{\psi}$ is the angular velocity of the internal fluidal motion. The angle ψ describes the rotation of the $x_{\bar{1}}$-axis of the figure of the body with respect to a matter-fixed system. On the other hand, observing the stellar matter in an inertial system centered at M it's velocity is given by

$$\mathbf{u}_{\mathrm{IS}} = \mathbf{u}_c + \Omega \wedge \mathbf{r}. \tag{3}$$

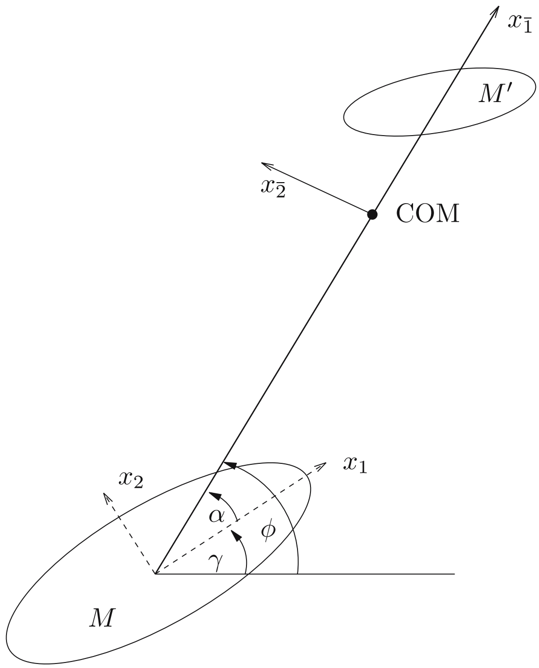

Fig. 1 The figure shows the coordinate systems used throughout this paper. The $x_{\bar{1}}$-axes of the comoving system $\bar{\Sigma}$ is chosen along the axes between the centers of M and M' and includes with the X-axes of the inertial system the angle ϕ. *Dotted lines* represent the axes of the body-fixed system. COM denotes the center of mass

The rotational energy of the star reads

$$T_{\mathrm{rot}} = \frac{1}{2}\int \rho\, \mathbf{u}_{\mathrm{IS}} \cdot \mathbf{u}_{\mathrm{IS}} \mathrm{d}V = \frac{\kappa_n M}{10}\left(a_1^2 + a_2^2\right)\left(\Lambda^2 + \Omega^2\right) - \frac{2}{5}\kappa_n M a_1 a_2 \Lambda\Omega, \tag{4}$$

where $\kappa_n \le 1$ is a constant, which can be obtained from the Lane–Emden equation (see Appendix). It can be easily seen that the inertial tensor $I_{ij} = \int \rho x_i x_j \mathrm{d}V$ in the body fixed system takes a very simple form,

$$I_{ij} = \int \rho x_i x_j \mathrm{d}V = \frac{\kappa_n M}{5} a_i^2 \delta_{ij}, \quad \kappa_n := \frac{5}{3}\frac{\int_0^{\xi_1} \theta^n \xi^4 \mathrm{d}\xi}{\xi_1^4 |\theta_1'|}. \tag{5}$$

In particular, for $a_1 \equiv a_2$ the rotational energy reduces to

$$T_{\mathrm{rot}} = \frac{\kappa_n M}{5}(\Omega - \Lambda)^2 a_1^2. \tag{6}$$

Note that if the angular velocity Ω of the ellipsoidal figure and the angular velocity Λ of the internal fluidal motion are equal, the rotational energy of an axially symmetric star vanishes. The total kinetic energy of star M is given by

$$T_s = \frac{\kappa_n M}{10}\left(a_1^2 + a_2^2\right)\left(\Omega^2 + \Lambda^2\right) - \frac{2}{5}\kappa_n M a_1 a_2 \Lambda\Omega + \frac{\kappa_n M}{10}\left(\dot{a}_1^2 + \dot{a}_2^2 + \dot{a}_3^2\right), \quad (7)$$

and the Lagrangian L_s of the star reads

$$L_s = T_s - U - W. \quad (8)$$

The gravitational self-energy W, and U, the internal energy of the star, have been computed by [13,19]. They are given by

$$U = \int n \frac{P}{\rho}\,\mathrm{d}m = k_1 K M \rho_c^{1/n}, \quad (9)$$

where $k_1 = n(n+1)\xi_1|\theta_1'|/(5-n)$ is constant, and

$$W = -\frac{3}{5-n}\frac{GM^2}{2R^3}\mathcal{J}. \quad (10)$$

Here $R = (a_1 a_2 a_3)^{1/3}$ is the mean radius of the ellipsoid and $\mathcal{J}$ is given by $\mathcal{J} = a_1^2 A_1 + a_2^2 A_2 + a_3^2 A_3$. The coefficients A_i, defined by Chandrasekhar [16], are given in the Appendix.

Now let us come back to the binary system. The tidal interaction potential is clearly dominated by the quadrupole interaction, which will be the only terms to be included into our calculations. Thus the orbital Lagrangian reads

$$L_{\mathrm{orb}} = \frac{\mu}{2}\dot{r}^2 + \frac{\mu r^2}{2}\dot{\phi}^2 - W_{\mathrm{int}}, \quad (11)$$

where $\mu = MM'/(M+M')$ is the reduced mass, and the interaction potential W_{int} is

$$W_{\mathrm{int}} = -\frac{GMM'}{r} - \frac{GMM'\kappa_n}{10r^3}\left[a_1^2\left(3\cos^2\alpha - 1\right) + a_2^2\left(3\sin^2\alpha - 1\right) - a_3^2\right] - \frac{GMM'\kappa_n'}{10r^3}\left[a_1'^2\left(3\cos^2\alpha' - 1\right) + a_2'^2(3\sin^2\alpha' - 1) - a_3'^2\right]. \quad (12)$$

The meaning of the angle $\alpha = \phi - \gamma$ can be read off from Fig. 1. It is now easy to write down the Lagrangian of a general Riemann-S binary according to

$$
\begin{aligned}
L = & \frac{\mu}{2}\dot{r}^2 + \frac{\mu r^2}{2}\dot{\phi}^2 + \frac{G\mu\mathcal{M}}{r} + \frac{G\mu\mathcal{M}\kappa_n}{10r^3}\Big[a_1^2\left(3\,\cos^2\,\alpha - 1\right) \\
& + a_2^2\left(3\,\sin^2\,\alpha - 1\right) - a_3^2)\Big] + \frac{G\mu\mathcal{M}\kappa_n'}{10r^3}\Big[a_1'^2\left(3\,\cos^2\,\alpha' - 1\right) \\
& + a_2'^2\left(3\sin^2\alpha' - 1\right) - a_3'^2\Big] + \frac{\kappa_n M}{10}\left(a_1^2 + a_2^2\right)\left(\Lambda^2 + \Omega^2\right) - \frac{2}{5}\kappa_n M a_1 a_2 \Lambda\Omega \\
& + \frac{\kappa_n M}{10}\left(\dot{a}_1^2 + \dot{a}_2^2 + \dot{a}_3^2\right) - k_1 K M \rho_c^{1/n} + \frac{3}{5-n}\frac{GM^2}{2R^3}\mathcal{J} \\
& + \frac{\kappa_n' M'}{10}\left(a_1'^2 + a_2'^2\right)\left(\Omega'^2 + \Lambda'^2\right) - \frac{2}{5}\kappa_n' M' a_1' a_2' \Lambda'\Omega' - k_1' K' M' \rho_c'^{1/n'} \\
& + \frac{\kappa_n' M'}{10}\left(\dot{a}_1'^2 + \dot{a}_2'^2 + \dot{a}_3'^2\right) + \frac{3}{5-n'}\frac{GM'^2}{2R'^3}\mathcal{J}', \qquad (13)
\end{aligned}
$$

where $\mathcal{M} = M + M'$. It is straightforward to derive the equations of motion governing the dynamics of the binary system. Let us first focus on the most general case when all three semi-major axes a_i are different. With the central density ρ_c being proportional to $1/(a_1 a_2 a_3)$ and using

$$
\frac{\partial \mathcal{J}}{\partial a_i} = \frac{1}{a_i}\left(\mathcal{J} - a_i^2 A_i\right)
$$

(see Appendix) the equations of motion derived from the Lagrangian (13) read

$$
\begin{aligned}
\ddot{a}_1 = & \frac{GM'}{r^3}\,a_1\left(3\,\cos^2\,\alpha - 1\right) + a_1\left(\Omega^2 + \Lambda^2\right) - 2a_2\Lambda\Omega + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_1} \\
& - \frac{3GM}{\kappa_n\left(1 - \frac{n}{5}\right)}\frac{a_1 A_1}{2R^3}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{a}_2 = & \frac{GM'}{r^3}\,a_2\left(3\,\sin^2\alpha - 1\right) + a_2(\Lambda^2 + \Omega^2) - 2a_1\Lambda\Omega + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_2} \\
& - \frac{3GM}{\kappa_n\left(1 - \frac{n}{5}\right)}\frac{a_2 A_2}{2R^3}, \qquad (15)
\end{aligned}
$$

$$
\ddot{a}_3 = -\frac{GM'}{r^3}\,a_3 + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_3} - \frac{3GM}{\kappa_n\left(1 - \frac{n}{5}\right)}\frac{a_3 A_3}{2R^3}, \qquad (16)
$$

$$
\ddot{\psi} = \frac{1}{a_1^2 - a_2^2}\left[\frac{3GM'}{r^3}\,a_1 a_2\,\sin 2\alpha - 2(a_1\dot{a}_1 - a_2\dot{a}_2)\Lambda - 2(\dot{a}_1 a_2 - a_1\dot{a}_2)\Omega\right], \qquad (17)
$$

$$
\ddot{\gamma} = \frac{1}{a_1^2 - a_2^2}\left[\frac{3GM'}{2r^3}\left(a_1^2 + a_2^2\right)\sin\,2\alpha - 2\left(a_1\dot{a}_1 - a_2\dot{a}_2\right)\Omega + 2(a_1\dot{a}_2 - \dot{a}_1 a_2)\Lambda\right], \qquad (18)
$$

$$
\begin{aligned}
\ddot{r} = & r\dot{\phi}^2 - \frac{G\mathcal{M}}{r^2} - \frac{3}{10}\frac{G\mathcal{M}}{r^4}\Big\{\kappa_n\left[a_1^2\left(3\cos^2\alpha - 1\right) + a_2^2\left(3\,\sin^2\,\alpha - 1\right) - a_3^2\right] \\
& + \kappa_n'\left[a_1'^2\left(3\cos^2\alpha' - 1\right) + a_2'^2\left(3\sin^2\alpha' - 1\right) - a_3'^2\right]\Big\}, \qquad (19)
\end{aligned}
$$

$$
\ddot{\phi} = -2\,\frac{\dot{r}\dot{\phi}}{r} - \frac{3}{10}\frac{G\mathcal{M}}{r^5}\left\{\left(a_1^2 - a_2^2\right)\kappa_n\,\sin\,2\alpha + \left(a_1'^2 - a_2'^2\right)\kappa_n'\,\sin\,2\alpha'\right\}, \qquad (20)
$$

with $\Lambda = \dot{\psi}$ and $\Omega = \dot{\gamma}$. The corresponding equations for a_i', ψ' and γ' are obtained by replacing unprimed variables by primed ones. Note that these equations apply whenever $a_1 \neq a_2$.

A particularly simple situation occurs, when the semi-major axes a_1 and a_2 are equal. In this case the corresponding star degenerates to an axially symmetric ellipsoid and the meaning of the angle α is obsolete. Although our analysis can be easily extended to the triaxial case, we shall restrict ourselves to the axially symmetric one in Sect. 4 and 5, since this is the simplest model which exhibits, nevertheless, the most important features of general Riemann-S binaries. The assumption $a_1 = a_2$ is of course only an approximation, since tidal interaction would inavoidably disturb an originally axially symmetric star. If, however, the angular velocity of the stellar rotation is considerably larger than the orbital angular velocity, the tidally induced deformation of the star can be neglected. Considering the equations of motion (17)–(20) it is easy to see that one cannot just set $a_1 = a_2$ here. This is due to the fact that the original variables a_1, a_2 and γ respectively ψ are not suitable variables in the limit $a_2 \to a_1$. Instead one should implement $a_1 \equiv a_2$ already in the Lagrangian level. In that case the Lagrangian (13) simplifies to

$$L = \frac{\mu}{2}\dot{r}^2 + \frac{\mu r^2}{2}\dot{\phi}^2 + \frac{GMM'}{r} + \frac{GMM'}{10r^3}\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right]$$
$$+\frac{\kappa_n M}{5}a_1^2\dot{\beta}^2 + \frac{\kappa_n' M'}{5}a_1'^2\dot{\beta}'^2 + \frac{\kappa_n M}{10}\left(2\dot{a}_1^2 + \dot{a}_3^2\right) + \frac{\kappa_n' M'}{10}\left(2\dot{a}_1'^2 + \dot{a}_3'^2\right)$$
$$-k_1 K M \rho_c^{1/n} - k_1' K' M' \rho_c'^{1/n'} + \frac{3}{5-n}\frac{GM^2}{2R^3}\mathcal{J} + \frac{3}{5-n'}\frac{GM'^2}{2R'^3}\mathcal{J}', \quad (21)$$

where we introduced a new variable $\beta := \gamma - \psi$. The number of degrees of freedom of the binary system is thus reduced from 12 to 8, resulting in an enormeous simplification of the numerical calculations. Using Eq. (94), the equations of motion for binary systems with axially symmetric stars become

$$\ddot{a}_1 = \frac{GM'}{2r^3}a_1 + a_1\dot{\beta}^2 + \left(\frac{5k_1}{\kappa_n n}\frac{P_c}{\rho_c}\right)\frac{1}{a_1} - \frac{3GM}{\kappa_n\left(1-\frac{n}{5}\right)}\frac{a_1 A_1}{2R^3}, \quad (22)$$

$$\ddot{a}_3 = -\frac{GM'}{r^3}a_3 + \left(\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c}\right)\frac{1}{a_3} - \frac{3GM}{\kappa_n\left(1-\frac{n}{5}\right)}\frac{a_3 A_3}{2R^3}, \quad (23)$$

$$\ddot{r} = r\dot{\phi}^2 - \frac{G\mathcal{M}}{r^2} - \frac{3}{10}\frac{G\mathcal{M}}{r^4}\left\{\kappa_n(a_1^2 - a_3^2) + \kappa_n'(a_1'^2 - a_3'^2)\right\}, \quad (24)$$

$$\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}, \quad (25)$$

$$\ddot{\beta} = -2\frac{\dot{a}_1}{a_1}\dot{\beta}. \quad (26)$$

To derive this result from Eqs. (14)–(20) one has to introduce new variables $(a_1 + a_2)/2$ and $\gamma - \psi$ before taking the limit $a_2 \to a_1$.

Later on we shall study the gravitational waves emitted by binaries with axially symmetric components in detail, but first we shall derive the Hamiltonian formalism for general Riemann-S binaries.

2.2 Riemann-S binary systems in Hamiltonian formalism

In the previous section we have revisited the derivation of the Lagrangian equations of motion for a Riemann-S binary. For the following analysis it is, however, more suitable to apply the Hamiltonian formulation of the problem. The generalized momenta $p_i = \partial L/\partial \dot{q}_i$ derived from the Lagrangian (13) are given by

$$
\begin{aligned}
p_r &= \mu \dot{r}, \quad p_\phi = \mu r^2 \dot{\phi}, \quad p_{a_i} = \frac{\kappa_n M}{5} \dot{a}_i, \\
p_\psi &= \frac{\kappa_n M}{5} \left[\left(a_1^2 + a_2^2 \right) \dot{\psi} - 2 a_1 a_2 \dot{\gamma} \right], \\
p_\gamma &= \frac{\kappa_n M}{5} \left[\left(a_1^2 + a_2^2 \right) \dot{\gamma} - 2 a_1 a_2 \dot{\psi} \right].
\end{aligned} \tag{27}
$$

For $a_1 \neq a_2$ it is possible to invert these equations in order to express the generalized velocities in terms of generalized momenta:

$$
\begin{aligned}
\dot{r} &= \frac{p_r}{\mu}, \quad \dot{\phi} = \frac{p_\phi}{\mu r^2}, \quad \dot{a}_i = \frac{5}{\kappa_n M} p_{a_i}, \\
\dot{\gamma} &= \frac{5}{\kappa_n M} \left[\frac{a_1^2 + a_2^2}{\left(a_1^2 - a_2^2\right)^2} p_\gamma + 2 \frac{a_1 a_2}{\left(a_1^2 - a_2^2\right)^2} p_\psi \right], \\
\dot{\psi} &= \frac{5}{\kappa_n M} \left[\frac{a_1^2 + a_2^2}{\left(a_1^2 - a_2^2\right)^2} p_\psi + 2 \frac{a_1 a_2}{\left(a_1^2 - a_2^2\right)^2} p_\gamma \right].
\end{aligned} \tag{28}
$$

The corresponding equations for a_i', ψ' and γ' are obtained by replacing unprimed variables by primed ones. For $a_1 \equiv a_2$ the generalized momenta p_ψ and p_γ are not independent variables, as can be easily seen from Eqs. (27). This is reflected by the observation that the inversion problem $p_i(\dot{q}_i) \rightarrow \dot{q}_i(p_i)$ is ill defined in this case. We shall turn to this later on, but for now we focus on the most general case, assuming $a_1 \neq a_2$. A Legendre transformation of the Lagrangian (13) leads to the Hamiltonian

$$
\begin{aligned}
H = & \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu r^2} + \frac{5}{2\kappa_n M} \sum_{i=1}^{3} p_{a_i}^2 + \frac{5}{2\kappa_n' M'} \sum_{i=1}^{3} p_{a_i}'^2 \\
& + \frac{5}{2\kappa_n M} \frac{a_1^2 + a_2^2}{\left(a_1^2 - a_2^2\right)^2} \left(p_\gamma^2 + p_\psi^2 \right) + \frac{10}{\kappa_n M} \frac{a_1 a_2}{\left(a_1^2 - a_2^2\right)^2} p_\gamma p_\psi
\end{aligned}
$$

$$+\frac{5}{2\kappa_n' M'}\frac{a_1'^2+a_2'^2}{\left(a_1'^2-a_2'^2\right)^2}\left(p_\gamma'^2+p_\psi'^2\right)+\frac{10}{\kappa_n' M'}\frac{a_1' a_2'}{\left(a_1'^2-a_2'^2\right)^2}p_\psi' p_\gamma'$$
$$+k_1 K M \rho_c^{1/n}-\frac{3}{5-n}\frac{GM^2}{2R^3}\mathcal{J}+k_1' K' M' \rho_c'^{1/n'}-\frac{3}{5-n'}\frac{GM'^2}{2R'^3}\mathcal{J}'$$
$$-\frac{G\mathcal{M}\mu}{r}-\frac{G\mathcal{M}\mu\kappa_n}{10r^3}\left[a_1^2\left(3\cos^2\alpha-1\right)+a_2^2\left(3\sin^2\alpha-1\right)-a_3^2\right]$$
$$-\frac{G\mathcal{M}\mu\kappa_n'}{10r^3}\left[a_1'^2\left(3\cos^2\alpha'-1\right)+a_2'^2\left(3\sin^2\alpha'-1\right)-a_3'^2\right]. \tag{29}$$

Once the Hamiltonian is given, the corresponding equations of motion

$$\dot{q}_i=\frac{\partial H}{\partial p_i},\quad \dot{p}_i=-\frac{\partial H}{\partial q_i}$$

are easy to calculate using the relation for $\partial\mathcal{J}/\partial a_i$ derived in the appendix. With the equations for the $\dot{q}_i$ already given in Eqs. (28) the remaining equations read

$$\dot{p}_{a_1}=\frac{5}{\kappa_n M}\left[\frac{a_1(3a_2^2+a_1^2)}{\left(a_1^2-a_2^2\right)^3}\left(p_\gamma^2+p_\psi^2\right)+2a_2\frac{a_2^2+3a_1^2}{\left(a_1^2-a_2^2\right)^3}p_\psi p_\gamma\right]$$
$$+\frac{G\mathcal{M}\mu}{5r^3}\kappa_n a_1\left(3\cos^2\alpha-1\right)+\frac{k_1 M}{na_1}\frac{P_c}{\rho_c}-\frac{3GM^2}{5-n}\frac{a_1 A_1}{2R^3}, \tag{30}$$
$$\dot{p}_{a_2}=-\frac{5}{\kappa_n M}\left[\frac{a_2\left(3a_1^2+a_2^2\right)}{\left(a_1^2-a_2^2\right)^3}\left(p_\psi^2+p_\gamma^2\right)+2a_1\frac{a_1^2+3a_2^2}{\left(a_1^2-a_2^2\right)^3}p_\psi p_\gamma\right]$$
$$+\frac{G\mathcal{M}\mu}{5r^3}\kappa_n a_2\left(3\sin^2\alpha-1\right)+\frac{k_1 M}{na_2}\frac{P_c}{\rho_c}-\frac{3GM^2}{5-n}\frac{a_2 A_2}{2R^3}, \tag{31}$$
$$\dot{p}_{a_3}=\frac{k_1 M}{na_3}\frac{P_c}{\rho_c}-\frac{3GM^2}{5-n}\frac{a_3 A_3}{2R^3}-\frac{G\mathcal{M}\mu}{5r^3}\kappa_n a_3, \tag{32}$$
$$\dot{p}_\psi=0, \tag{33}$$
$$\dot{p}_\gamma=\frac{3}{10}\frac{G\mathcal{M}\mu}{r^3}\kappa_n(a_1^2-a_2^2)\sin 2\alpha, \tag{34}$$
$$\dot{p}_r=-\frac{G\mathcal{M}\mu}{r^2}+\frac{p_\phi^2}{\mu r^3}-\frac{3}{10}\frac{G\mathcal{M}\mu\kappa_n}{r^4}\left[a_1^2(3\cos^2\alpha-1)+a_2^2(3\sin^2\alpha-1)-a_3^2\right]$$
$$-\frac{3}{10}\frac{G\mathcal{M}\mu\kappa_n'}{r^4}\left[a_1'^2\left(3\cos^2\alpha'-1\right)+a_2'^2\left(3\sin^2\alpha'-1\right)-a_3'^2\right], \tag{35}$$
$$\dot{p}_\phi=-\frac{3}{10}\frac{G\mathcal{M}\mu}{r^3}\left[\kappa_n\left(a_1^2-a_2^2\right)\sin 2\alpha+\kappa_n'\left(a_1'^2-a_2'^2\right)\sin 2\alpha'\right]. \tag{36}$$

Together with the corresponding equations for the primed variables this system of differential equations describes the dynamics of the binary completely.

Let us now assume $a_1\equiv a_2$, $a_1'\equiv a_2'$. As we have already pointed out in this case the dynamical equations (28) and (30)–(36) do not apply. The generalized momenta derived from Lagrangian (21) read

$$p_{a_1} = \frac{2}{5}\kappa_n M\dot{a}_1, \quad p_{a_3} = \frac{\kappa_n M}{5}\dot{a}_3, \quad p_\beta = \frac{2}{5}\kappa_n M a_1^2\dot{\beta},$$
$$p_r = \mu\dot{r}, \quad p_\phi = \mu r^2\dot{\phi}, \tag{37}$$

which can be easily inverted, giving

$$\dot{a}_1 = \frac{5}{2\kappa_n M}p_{a_1}, \quad \dot{a}_3 = \frac{5}{\kappa_n M}p_{a_3}, \quad \dot{\beta} = \frac{5}{2\kappa_n M}\frac{p_\beta}{a_1^2},$$
$$\dot{r} = \frac{p_r}{\mu}, \quad \dot{\phi} = \frac{p_\phi}{\mu r^2}. \tag{38}$$

As before a Legendre transformation of the Lagrangian (21) leads to the corresponding Hamiltonian

$$H = \frac{5}{2\kappa_n M}\left[\frac{p_{a_1}^2}{2} + p_{a_3}^2 + \frac{p_\beta^2}{2a_1^2}\right] + \frac{5}{2\kappa_n' M'}\left[\frac{p_{a_1}'^2}{2} + p_{a_3}'^2 + \frac{p_\beta'^2}{2a_1'^2}\right] + \frac{p_r^2}{2\mu}$$
$$+\frac{p_\phi^2}{2\mu r^2} + k_1 K M\rho_c^{1/n} - \frac{3GM^2}{5-n}\frac{\mathcal{J}}{2R^3} + k_1' K' M'\rho_c'^{1/n'}$$
$$-\frac{3GM'^2}{5-n'}\frac{\mathcal{J}'}{2R'^3} - \frac{G\mathcal{M}\mu}{r} - \frac{G\mathcal{M}\mu}{10r^3}\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right]. \tag{39}$$

Note that β, β' and ϕ are cyclic variables, i.e. the corresponding generalized momenta are constants. The Hamiltonian equations are then given by Eqs. (38) and

$$\dot{p}_{a_1} = \frac{5}{2\kappa_n M}\frac{p_\beta^2}{a_1^3} + \frac{G\mathcal{M}\mu\kappa_n}{5r^3}a_1 + 2\frac{k_1 M}{na_1}\frac{P_c}{\rho_c} - \frac{3GM^2}{5-n}\frac{a_1 A_1}{R^3}, \tag{40}$$

$$\dot{p}_{a_3} = -\frac{G\mathcal{M}\mu\kappa_n}{5r^3}a_3 + \frac{k_1 M}{na_3}\frac{P_c}{\rho_c} - \frac{3GM^2}{5-n}\frac{a_3 A_3}{2R^3}, \tag{41}$$

$$\dot{p}_r = -\frac{G\mathcal{M}\mu}{r^2} + \frac{p_\phi^2}{\mu r^3} - \frac{3}{10}\frac{G\mathcal{M}\mu}{r^4}\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right], \tag{42}$$

$$\dot{p}_\phi = \dot{p}_\beta = 0, \tag{43}$$

and, of course, the corresponding equations for p_{a_i}' and β'.

3 Leading order radiation reaction in Riemann-S binaries

On the Newtonian level, tidally coupled binaries form a conservative system. However, according to the theory of General Relativity binary systems loose energy due to the emission of gravitational waves, the first nonvanishing dissipative terms appearing at 2.5 PN approximation. The following section is devoted to the calculation of the leading order radiation reaction terms for Riemann-S binaries. This can be done by virtue of the Burke-Thorne radiation reaction potential $\Phi_{\rm reac}$, given e.g. in [17],

$$\Phi_{\rm reac} = \frac{G}{5c^5} I\!\!\!\!-^{(5)}_{\bar{a}\bar{b}} x_{\bar{a}} x_{\bar{b}}. \tag{44}$$

As before the $x_{\bar{a}}$ denote coordinates in the corotating coordinate frame. Thus the computation of the radiation reaction potential reduces in principle to the calculation of the 5th time derivative of the STF mass quadrupole tensor *in the corotating system*. The mass quadrupole tensor being additive at Newtonian order, it is possible to consider orbital and stellar contributions separately.

3.1 Time derivatives of the stellar mass quadrupole tensor

According to Eq. (44) the leading order gravitational wave emission is governed by the time variations of the STF mass quadrupole tensor. Thus even an isolated, oscillating star represents a source of gravitational waves. In a binary system tidal coupling between stellar and orbital degrees of freedom leads to a change in the gravitational wave pattern. Though the orbital contribution in general clearly dominates the gravitational waveforms emitted by the binary, the stellar contributions cannot be neglected. Let us consider a coordinate transformation such that the origin of the corotating system is centered at M. The relation between the coordinates X_a of an inertial system centered at M, the coordinates x_a of the body-framed system and the coordinates $x_{\bar{a}}$ can be read off from Fig. 1,

$$x_a = T_{ab}(\gamma) X_b, \quad x_{\bar{a}} = T_{\bar{a}b}(\phi) X_b, \tag{45}$$

where

$$T_{ab}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{46}$$

As has already been mentioned, the STF mass quadrupole tensor takes a particular simple form in the body-fixed system,

$$I\!\!\!\!-_{ab} = I_{ab} - \frac{1}{3}\delta_{ab} I_{cc}, \quad I_{ab} = \int \rho x_a x_b \mathrm{d}V = \frac{\kappa_n M}{5} a_a^2 \delta_{ab}. \tag{47}$$

To calculate the time derivatives of the mass quadrupole tensor in the corotating system we could in a first step consider the time derivatives of the star's mass quadrupole tensor in the inertial system, the later one being related to $I\!\!\!{}^{-}_{ab}$ by the transformation

$$I\!\!\!{}^{-(\mathrm{IS})}_{\alpha\beta} = T^{\dagger}(\gamma)_{\alpha i} T^{\dagger}_{\beta j}(\gamma) I\!\!\!{}^{-}_{ij},$$

and only in a second step apply the transformation into the corotating system according to

$$I\!\!\!{}^{-(5)}_{\bar{a}\bar{b}} = T_{\bar{a}\alpha}(\phi) T_{\bar{b}\beta}(\phi) I\!\!\!{}^{-(5,\mathrm{IS})}_{\alpha\beta}. \tag{48}$$

This calculation is straightforward, but successively inserting the Newtonian equations of motion leads to rather complicated expressions. In our approach another strategy is more suitable. Following Lai and Shapiro (see also [18]) we combine the steps mentioned above according to

$$\begin{aligned} I\!\!\!{}^{-(5)}_{\bar{a}\bar{b}} &= T_{\bar{a}\alpha}(\phi) T_{\bar{b}\beta}(\phi) \frac{\mathrm{d}^5}{\mathrm{d}t^5}\left[T^{\dagger}_{\alpha i}(\gamma) T^{\dagger}_{\beta j}(\gamma) I\!\!\!{}^{-}_{ij}\right] \\ &= \sum_{m=0}^{5} \binom{5}{m} \left[\frac{\mathrm{d}^{5-m}}{\mathrm{d}t^{5-m}} I\!\!\!{}^{-}_{ij}\right] \sum_{p=0}^{m} \binom{m}{p} \left[T_{\bar{a}\alpha}(\phi) \frac{\mathrm{d}^{m-p}}{\mathrm{d}t^{m-p}} T^{\dagger}_{\alpha i}(\gamma)\right] \left[T_{\bar{b}\beta}(\phi) \frac{\mathrm{d}^{p}}{\mathrm{d}t^{p}} T^{\dagger}_{\beta j}(\gamma)\right] \\ &= \sum_{m=0}^{5} \sum_{p=0}^{m} \binom{5}{m} \binom{m}{p} \left[\frac{\mathrm{d}^{5-m}}{\mathrm{d}t^{5-m}} I\!\!\!{}^{-}_{ij}\right] R^{m-p}_{\bar{a}i} R^{p}_{\bar{b}j}, \end{aligned} \tag{49}$$

where

$$R^{p}_{\bar{a}i} := T_{\bar{a}\alpha}(\phi) \frac{\mathrm{d}^{p}}{\mathrm{d}t^{p}} T^{\dagger}_{\alpha i}(\gamma).$$

The calculation of the matrices $R^{p}_{\bar{a}i}$ is straightforward and given in the Appendix. However, the resulting expressions for $I\!\!\!{}^{-(5)}_{\bar{a}\bar{b}}$ being rather complicated, we should imply further assumptions on the internal stellar motion. In particular, it is reasonable to consider all internal velocities and accelerations to be small, thus applying the quasi-static approximation to the stellar degrees of freedom. This strategy was already followed by Lai and Shapiro [13], but they applied the quasi-static assumption to the orbital motion, too. While this is justified for circular orbits or if one is interested in a few cycles only, for elliptical orbits this leads to a growing phase error in the gravitational waveforms. Since we are interested in the long-term evolution of the system we shall apply the quasi-static approximation to the stars only. Neglecting all terms of order $O(\ddot{a}_i)$ and $O(\ddot{\Omega})$, keeping only terms linear in $\dot{a}_i$ and $\dot{\Omega}$ and using that $|\dot{a}_i| \ll |\Omega a_i|$, Eq. (49) simplifies to

$$\mathfrak{I}^{(5)}_{\bar{a}\bar{b}} = \mathfrak{I}_{ij}\sum_{p=0}^{5}\binom{5}{p} R^{p}_{\bar{a}i}R^{5-p}_{\bar{b}j} + \dot{\mathfrak{I}}_{ij}\sum_{p=0}^{4}\binom{4}{p} R^{p}_{\bar{a}i}R^{4-p}_{\bar{b}j}. \tag{50}$$

Note that $\ddot{\mathfrak{I}}_{ab} \approx 0$ in this approximation. After some algebra we end up with

$$\begin{aligned}\mathfrak{I}^{(5)}_{\bar{a}\bar{b}} &= 16\Omega^5(I_{11}-I_{22})\begin{pmatrix}\sin 2\alpha & \cos 2\alpha & 0\\ \cos 2\alpha & -\sin 2\alpha & 0\\ 0 & 0 & 0\end{pmatrix}\\ &\quad -\left[80\Omega^3\dot{\Omega}(I_{11}-I_{22}) + 40\Omega^4(\dot{I}_{11}-\dot{I}_{22})\right]\begin{pmatrix}-\cos 2\alpha & \sin 2\alpha & 0\\ \sin 2\alpha & \cos 2\alpha & 0\\ 0 & 0 & 0\end{pmatrix}.\end{aligned} \tag{51}$$

It is remarkable that the components of $\mathfrak{I}^{(5)}_{\bar{a}\bar{b}}$ are nonvanishing only for $a_1 \neq a_2$. In other words, the quasi-static approximation does not allow for gravitational wave emission of an isolated, axially symmetric, polytropic star.[2]

3.2 The orbital contribution to $\mathfrak{I}^{(5)}_{\bar{a}\bar{b}}$

As has been outlined before, the quasi-static approximation does not apply to the orbital motion in general. To calculate $\mathfrak{I}^{(5),orb}_{\bar{a}\bar{b}}$ we compute $\mathfrak{I}^{(5),orb}_{ij}$ in the inertial system by successively inserting the Newtonian equations of motion

$$\ddot{r} = -\frac{G\mathcal{M}}{r^2} + r\dot{\phi}^2, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r}. \tag{52}$$

More precisely, one should insert the equations of motion of the tidally coupled system (cf. Eqs. (19) and (20)). This would complicate the problem enormeously. However, these corrections, being of higher order in $1/r$, can easily be neglected.

In the inertial system the nonvanishing components of $\mathfrak{I}^{(\mathrm{orb})}_{ij}$ read

$$\begin{aligned}&\mathfrak{I}^{(\mathrm{orb})}_{11} = \frac{\mu r^2}{6}(1+3\cos 2\phi), \quad \mathfrak{I}^{(\mathrm{orb})}_{22} = \frac{\mu r^2}{6}(1-3\cos 2\phi), \quad \mathfrak{I}^{(\mathrm{orb})}_{33} = -\frac{\mu r^2}{3},\\ &\mathfrak{I}^{(\mathrm{orb})}_{12} = \frac{\mu r^2}{2}\sin 2\phi.\end{aligned} \tag{53}$$

[2] Without the quasi-approximation there exist nonvanishing contributions to $\mathfrak{I}^{(5)}_{\bar{a}\bar{b}}$ for axially symmetric polytropes, too.

Using Eqs. (52) to calculate $\text{I}_{ij}^{(5)}$ in the inertial system and finally transfering to the corotating system centered at the center of mass according to

$$\text{I}_{\bar{a}\bar{b}}^{(5,\,\mathrm{orb})} = T_{\bar{a}\alpha}(\phi)T_{\bar{b}\beta}(\phi)\text{I}_{\alpha\beta}^{(5),(\mathrm{orb},\,IS)}, \tag{54}$$

we end up with

$$\begin{aligned}
\text{I}_{\bar{1}\bar{1}}^{(5),\,\mathrm{orb}} &= -\frac{8G\mathcal{M}\mu}{3}\frac{\dot{r}}{r^4}\left[4\,\frac{G\mathcal{M}}{r} + 3\dot{r}^2 + 18r^2\dot{\phi}^2\right],\\
\text{I}_{\bar{2}\bar{2}}^{(5),\,\mathrm{orb}} &= \frac{2G\mathcal{M}\mu}{3}\frac{\dot{r}}{r^4}\left[8\,\frac{G\mathcal{M}}{r} + 6\dot{r}^2 + 81r^2\dot{\phi}^2\right],\\
\text{I}_{\bar{3}\bar{3}}^{(5),\,\mathrm{orb}} &= \frac{2G\mathcal{M}\mu}{3}\frac{\dot{r}}{r^4}\left[8\,\frac{G\mathcal{M}}{r} + 6\dot{r}^2 - 9r^2\dot{\phi}^2\right],\\
\text{I}_{\bar{1}\bar{2}}^{(5),\,\mathrm{orb}} &= -4\,\frac{G\mathcal{M}\mu}{r^3}\,\dot{\phi}\left[\frac{2G\mathcal{M}}{r} + 9\dot{r}^2 - 6r^2\dot{\phi}^2\right].
\end{aligned} \tag{55}$$

3.3 Leading order gravitational radiation: general case

In a recent paper we discussed the dynamics and gravitational wave emission of a binary system consisting of a rotating, oscillating dusty disk and a point mass [10]. There we incorporated the leading order radiation reaction into the Hamiltonian equations by adding a dissipative part to the Hamiltonian. The Hamiltonian equations can then be applied in the usual way. Here we choose an alternative approach, leaving the Lagrangian or Hamiltonian unchanged but modifying the Euler-Lagrangian equations according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \mathcal{F}_{q_i}. \tag{56}$$

The generalized dissipative forces $\mathcal{F}_{q_i}$ are calculated from the energy dissipation rate

$$\mathcal{W} = -\int \mathbf{v}\cdot\nabla\Phi_{\mathrm{reac}}\rho\,\mathrm{d}V \tag{57}$$

as $\mathcal{F}_{q_i} = \partial W/\partial \dot{q}_i$. Let us consider the contribution $\mathcal{W}_M$ of star M to the energy dissipation rate of the binary. The velocity $\mathbf{v}$ of a fluid element of M can be separated according to $\mathbf{v} = \mathbf{u} + \mathbf{u}_{\mathrm{orb}}$, where

$$\mathbf{u} = \left(\frac{a_1}{a_2}\Lambda - \Omega\right)x_2\mathbf{e}_1 + \left(-\frac{a_2}{a_1}\Lambda + \Omega\right)x_1\mathbf{e}_2 + \frac{\dot{a}_1}{a_1}x_1\mathbf{e}_1 + \frac{\dot{a}_2}{a_2}x_2\mathbf{e}_2 + \frac{\dot{a}_3}{a_3}x_3\mathbf{e}_3$$

is the velocity of a fluid element relative to the center of M and

$$\mathbf{u}_{\mathrm{orb}} = -\dot{r}_M\mathbf{e}_{\bar{1}} - r_M\dot{\phi}\mathbf{e}_{\bar{2}} \tag{58}$$

is the orbital velocity of the star's center of mass. As before the $\mathbf{e}_i$ denote the unit vectors in the body-fixed coordinate system, while the $\mathbf{e}_{\bar a}$ represent unit vectors in the corotating system. As one can easily read off from Fig. 1, the coordinates are related according to

$$x_{\bar 1} = x_1 \cos\alpha + x_2 \sin\alpha - r_M, \quad x_{\bar 2} = -x_1 \sin\alpha + x_2 \cos\alpha, \quad x_{\bar 3} = x_3. \tag{59}$$

Inserting this into Eq. (57) and using Eq. (5) the contribution of the star M to the energy dissipation rate yields

$$\begin{aligned}\mathcal{W}_M = &-\frac{2G}{5c^5}\frac{\kappa_n M}{5}\Big[a_1\dot a_1\left(\text{Ɨ}^{(5)}_{\bar1\bar1}\cos^2\alpha + \text{Ɨ}^{(5)}_{\bar2\bar2}\sin^2\alpha - \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha\right)\\
&+a_2\dot a_2\left(\text{Ɨ}^{(5)}_{\bar1\bar1}\sin^2\alpha + \text{Ɨ}^{(5)}_{\bar2\bar2}\cos^2\alpha + \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha\right) + \text{Ɨ}^{(5)}_{\bar3\bar3}a_3\dot a_3\\
&+\Omega\left(a_1^2 - a_2^2\right)\left(\text{Ɨ}^{(5)}_{\bar1\bar2}\cos 2\alpha + \frac{1}{2}\left(\text{Ɨ}^{(5)}_{\bar1\bar1} - \text{Ɨ}^{(5)}_{\bar2\bar2}\right)\sin 2\alpha\right)\Big]\\
&-\frac{2GM}{5c^5}\left[r_M\dot r_M \text{Ɨ}^{(5)}_{\bar1\bar1} + r_M^2\dot\phi\,\text{Ɨ}^{(5)}_{\bar1\bar2}\right].\end{aligned} \tag{60}$$

The corresponding contribution of M' is obtained by replacing unprimed quantities by primed ones. Note that according to Fig. 1 the orbital velocity of M' is given by $\mathbf{u}'_{\text{orb}} = \dot r'_M \mathbf{e}_{\bar 1} + r'_M\dot\phi\,\mathbf{e}_{\bar 2}$. Adding up both contributions and using $Mr_M\dot r_M + M'r'_M\dot r'_M = \mu r\dot r$, the energy dissipation rate of the binary system reads

$$\begin{aligned}\mathcal{W} = &-\frac{2G}{5c^5}\frac{\kappa_n M}{5}\Big[a_1\dot a_1\left(\text{Ɨ}^{(5)}_{\bar1\bar1}\cos^2\alpha + \text{Ɨ}^{(5)}_{\bar2\bar2}\sin^2\alpha - \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha\right)\\
&+a_2\dot a_2\left(\text{Ɨ}^{(5)}_{\bar1\bar1}\sin^2\alpha + \text{Ɨ}^{(5)}_{\bar2\bar2}\cos^2\alpha + \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha\right) + \text{Ɨ}^{(5)}_{\bar3\bar3}a_3\dot a_3\\
&+\Omega\left(a_1^2 - a_2^2\right)\left(\text{Ɨ}^{(5)}_{\bar1\bar2}\cos 2\alpha + \frac{1}{2}\left(\text{Ɨ}^{(5)}_{\bar1\bar1} - \text{Ɨ}^{(5)}_{\bar2\bar2}\right)\sin 2\alpha\right)\Big]\\
&-\frac{2G}{5c^5}\frac{\kappa'_n M'}{5}\Big[a'_1\dot a'_1\left(\text{Ɨ}^{(5)}_{\bar1\bar1}\cos^2\alpha' + \text{Ɨ}^{(5)}_{\bar2\bar2}\sin^2\alpha' - \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha'\right)\\
&+a'_2\dot a'_2\left(\text{Ɨ}^{(5)}_{\bar1\bar1}\sin^2\alpha' + \text{Ɨ}^{(5)}_{\bar2\bar2}\cos^2\alpha' + \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha'\right) + \text{Ɨ}^{(5)}_{\bar3\bar3}a'_3\dot a'_3\\
&+\Omega\left(a_1'^2 - a_2'^2\right)\left(\text{Ɨ}^{(5)}_{\bar1\bar2}\cos 2\alpha' + \frac{1}{2}\left(\text{Ɨ}^{(5)}_{\bar1\bar1} - \text{Ɨ}^{(5)}_{\bar2\bar2}\right)\sin 2\alpha'\right)\Big]\\
&-\frac{2G\mu}{5c^5}\left(r\dot r\text{Ɨ}^{(5)}_{\bar1\bar1} + r^2\dot\phi\,\text{Ɨ}^{(5)}_{\bar1\bar2}\right).\end{aligned} \tag{61}$$

It is now straightforward to derive the generalized forces

$$\begin{aligned}\mathcal{F}_{a_1} &= -\frac{2G}{5c^5}\frac{\kappa_n M}{5}\left[\text{Ɨ}^{(5)}_{\bar1\bar1}\cos^2\alpha + \text{Ɨ}^{(5)}_{\bar2\bar2}\sin^2\alpha - \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha\right]a_1,\\
\mathcal{F}_{a_2} &= -\frac{2G}{5c^5}\frac{\kappa_n M}{5}\left[\text{Ɨ}^{(5)}_{\bar1\bar1}\sin^2\alpha + \text{Ɨ}^{(5)}_{\bar2\bar2}\cos^2\alpha + \text{Ɨ}^{(5)}_{\bar1\bar2}\sin 2\alpha\right]a_2,\end{aligned}$$

$$
\begin{aligned}
\mathcal{F}_{a_3} &= -\frac{2G}{5c^5}\frac{\kappa_n M}{5}\text{Ɨ}^{(5)}_{\bar{3}\bar{3}}a_3,\\
\mathcal{F}_{\gamma} &= -\frac{2G}{5c^5}\frac{\kappa_n M}{5}\left(a_1^2-a_2^2\right)\left(\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}\cos 2\alpha+\frac{1}{2}(\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}-\text{Ɨ}^{(5)}_{\bar{2}\bar{2}})\sin 2\alpha\right),\\
\mathcal{F}_{\psi} &= 0,\\
\mathcal{F}_{r} &= -\frac{2G\mu}{5c^5}\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}\,r,\\
\mathcal{F}_{\phi} &= -\frac{2G\mu}{5c^5}\,\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}r^2,
\end{aligned}
\tag{62}
$$

the corresponding generalized forces for M' being obtained by replacing unprimed variables by primed ones. The Euler–Lagrangian equations governing the dynamics of the system read now

$$
\begin{aligned}
\ddot{a}_1 &= [\text{Eq. (14)}]-\frac{2G}{5c^5}\left(\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}\cos^2\alpha+\text{Ɨ}^{(5)}_{\bar{2}\bar{2}}\sin^2\alpha-\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha\right)a_1,\\
\ddot{a}_2 &= [\text{Eq. (15)}]-\frac{2G}{5c^5}\left(\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}\sin^2\alpha+\text{Ɨ}^{(5)}_{\bar{2}\bar{2}}\cos^2\alpha+\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}\sin 2\alpha\right)a_2,\\
\ddot{a}_3 &= [\text{Eq. (16)}]-\frac{2G}{5c^5}\text{Ɨ}^{(5)}_{\bar{3}\bar{3}}a_3,\\
\dot{\Lambda} &= \left(\frac{a_1}{a_2}-\frac{a_2}{a_1}\right)^{-1}\left[-2\left(\frac{\dot{a}_1}{a_2}-\frac{\dot{a}_2}{a_1}\right)\Lambda-2\left(\frac{\dot{a}_1}{a_1}-\frac{\dot{a}_2}{a_2}\right)\Omega+\frac{3GM'}{r^3}\sin 2\alpha\right.\\
&\quad\left.-\frac{2G}{5c^5}\left(2\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}\cos 2\alpha+\left(\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}-\text{Ɨ}^{(5)}_{\bar{2}\bar{2}}\right)\sin 2\alpha\right)\right],\\
\dot{\Omega} &= \left(\frac{a_1}{a_2}-\frac{a_2}{a_1}\right)^{-1}\left[-2\left(\frac{\dot{a}_1}{a_1}-\frac{\dot{a}_2}{a_2}\right)\Lambda-2\left(\frac{\dot{a}_1}{a_2}-\frac{\dot{a}_2}{a_1}\right)\Omega\right.\\
&\quad\left.+\left(\frac{a_1}{a_2}+\frac{a_2}{a_1}\right)\left\{\frac{3}{2}\frac{GM'}{r^3}\sin 2\alpha-\frac{2G}{5c^5}\left(\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}\cos 2\alpha+\frac{1}{2}\left(\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}-\text{Ɨ}^{(5)}_{\bar{2}\bar{2}}\right)\sin 2\alpha\right)\right\}\right],\\
\ddot{r} &= r\dot{\phi}^2-\frac{G\mathcal{M}}{r^2}-\frac{3}{10}\frac{G\mathcal{M}}{r^4}\left[\kappa_n\left(a_1^2\left(3\cos^2\alpha-1\right)+a_2^2\left(3\sin^2\alpha-1\right)-a_3^2\right)\right.\\
&\quad\left.+\kappa_n'\left(a_1'^2\left(3\cos^2\alpha'-1\right)+a_2'^2\left(3\sin^2\alpha'-1\right)-a_3'^2\right)\right]-\tfrac{2G}{5c^5}\,\text{Ɨ}^{(5)}_{\bar{1}\bar{1}}\,r,\\
\ddot{\phi} &= -2\,\frac{\dot{r}\dot{\phi}}{r}-\frac{3G\mathcal{M}}{10r^5}\left[\kappa_n\left(a_1^2-a_2^2\right)\sin 2\alpha+\kappa_n'\left(a_1'^2-a_2'^2\right)\sin 2\alpha'\right]-\frac{2G}{5c^5}\,\text{Ɨ}^{(5)}_{\bar{1}\bar{2}}.
\end{aligned}
\tag{63}
$$

Together with the corresponding equations for M' these equations describe the evolution of the binary system including leading order gravitational reaction. Note that in Eqs. (63) all contributions to the leading order gravitational reaction are included. The quasi-static approximation for the stellar degrees of freedom enters into the explicit calculation of the time derivatives of the STF mass quadrupole tensor.

3.4 Specializing to binary systems with $a_1 = a_2,\ a'_1 = a'_2$

In the previous section we derived the Lagrangian equations of motion for arbitrary Riemann-S binaries. From now on we shall impose an additional constraint on the stellar degrees of freedom, requiring $a_1 = a_2,\ a'_1 = a'_2$. On Newtonian level the equations of motion governing the dynamics of those systems have been derived in Sect. 2. Now let us consider the radiation reaction part of the equations of motion. In principle we follow the same strategy as in the previous section. The velocity of a fluid element of star M relative to the star's center reads now

$$\mathbf{u} = -\dot{\beta} x_2 \mathbf{e}_1 + \dot{\beta} x_1 \mathbf{e}_2 + \frac{\dot{a}_1}{a_1}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \frac{\dot{a}_3}{a_3} x_3 \mathbf{e}_3,$$

and the orbital motion is $\mathbf{u}_{orb} = -\dot{r}_M \mathbf{e}_{\bar{1}} - r_M \dot{\phi} \mathbf{e}_{\bar{2}}$. Calculating $\mathcal{W}_M$ and $\mathcal{W}'_M$ according to Eq. (57), the gravitational energy dissipation rate of the binary system yields

$$\begin{aligned}\mathcal{W} = &-\frac{2G}{5c^5}\frac{\kappa_n M}{5}\left[a_1\dot{a}_1\left(\mathbf{I}^{(5)}_{\bar{1}\bar{1}} + \mathbf{I}^{(5)}_{\bar{2}\bar{2}}\right) + a_3\dot{a}_3 \mathbf{I}^{(5)}_{\bar{3}\bar{3}}\right] \\ &-\frac{2G}{5c^5}\frac{\kappa'_n M'}{5}\left[a'_1\dot{a}'_1\left(\mathbf{I}^{(5)}_{\bar{1}\bar{1}} + \mathbf{I}^{(5)}_{\bar{2}\bar{2}}\right) + a'_3\dot{a}'_3 \mathbf{I}^{(5)}_{\bar{3}\bar{3}}\right] - \frac{2G\mu}{5c^5}\left(r\dot{r}\,\mathbf{I}^{(5)}_{\bar{1}\bar{1}} + r^2\dot{\phi}\,\mathbf{I}^{(5)}_{\bar{1}\bar{2}}\right). \qquad (64)\end{aligned}$$

The generalized dissipative forces take on a particular simple form:

$$\begin{aligned}\mathcal{F}_{a_1} &= -\frac{2G}{5c^5}\frac{\kappa_n M}{5}\left(\mathbf{I}^{(5)}_{\bar{1}\bar{1}} + \mathbf{I}^{(5)}_{\bar{2}\bar{2}}\right)a_1 = \frac{2G}{5c^5}\frac{\kappa_n M}{5}\mathbf{I}^{(5)}_{\bar{3}\bar{3}} a_1, \\ \mathcal{F}_{a_3} &= -\frac{2G}{5c^5}\frac{\kappa_n M}{5}\mathbf{I}^{(5)}_{\bar{3}\bar{3}} a_3, \\ \mathcal{F}_{\beta} &= 0, \\ \mathcal{F}_r &= -\frac{2G\mu}{5c^5}\,\mathbf{I}^{(5)}_{\bar{1}\bar{1}} r, \\ \mathcal{F}_{\phi} &= -\frac{2G\mu}{5c^5}\,\mathbf{I}^{(5)}_{\bar{1}\bar{2}} r^2. \end{aligned} \qquad (65)$$

So the Lagrangian equations of motion including leading order radiation reaction terms are given by

$$\begin{aligned}\ddot{a}_1 &= \frac{GM'}{2r^3}a_1 + a_1\dot{\beta}^2 + \left(\frac{5k_1}{\kappa_n n}\frac{P_c}{\rho_c}\right)\frac{1}{a_1} - \frac{3GM}{\kappa_n\left(1-\frac{n}{5}\right)}\frac{a_1 A_1}{2R^3} + \frac{G}{5c^5}\mathbf{I}^{(5)}_{\bar{3}\bar{3}} a_1, \\ \ddot{a}_3 &= -\frac{GM'}{r^3}a_3 + \left(\frac{5k_1}{\kappa_n n}\frac{P_c}{\rho_c}\right)\frac{1}{a_3} - \frac{3GM}{\kappa_n\left(1-\frac{n}{5}\right)}\frac{a_3 A_3}{2R^3} - \frac{2G}{5c^5}\,\mathbf{I}^{(5)}_{\bar{3}\bar{3}} a_3, \\ \ddot{\beta} &= -2\,\frac{\dot{a}_1}{a_1}\,\dot{\beta}, \end{aligned} \qquad (66)$$

$$\ddot{r} = r\dot{\phi}^2 - \frac{G\mathcal{M}}{r^2} - \frac{3G\mathcal{M}}{10r^4}\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right] - \frac{2G}{5c^5}\,\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}\,r,$$
$$\ddot{\phi} = -2\,\frac{\dot{r}\dot{\phi}}{r} - \frac{2G}{5c^5}\,\mathfrak{I}^{(5)}_{\bar{1}\bar{2}}.$$

At this point we should emphasize that in the quasi-static approximation the stellar contribution to $\mathfrak{I}^{(5)}_{\bar{a}\bar{b}}$ vanishes, as can be easily seen from Eq. (51). This means, in that approximation only the time varying orbital mass quadrupole tensor gives rise to the emission of gravitational waves. However, coupling the internal dynamics of the stars with the orbital dynamics both, the orbital dynamics as well as the gravitational waveforms are affected by the internal dynamics.

For our purposes it is more suitable to describe the dynamics of the binary system in Hamiltonian formalism. In the presence of dissipative forces the well-known Hamiltonian equations are modified according to

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + \mathcal{F}_{q_i}, \tag{67}$$

where the Hamiltonian H is given by Eq. (29) and Eq. (39), respectively. The generalized momenta are defined in the usual way according to $p_i = \partial L/\partial\dot{q}_i$. The Hamiltonian equations derived in Sect. 2.2. (cf. Eqs. (40)–(43)) are thus altered according to

$$\begin{aligned}
\dot{p}_{a_1} &= \frac{5}{2\kappa_n M}\frac{p_\beta^2}{a_1^3} + \frac{G\mathcal{M}\mu\kappa_n}{5r^3}\,a_1 + 2\frac{k_1 M}{na_1}\frac{P_c}{\rho_c} - \frac{3GM^2}{5-n}\frac{a_1A_1}{R^3} + \frac{2G}{5c^5}\frac{\kappa_n M}{5}\mathfrak{I}^{(5)}_{\bar{3}\bar{3}}a_1,\\
\dot{p}_{a_3} &= -\frac{G\mathcal{M}\mu\kappa_n}{5r^3}\,a_3 + \frac{k_1 M}{na_3}\frac{P_c}{\rho_c} - \frac{3GM^2}{5-n}\frac{a_3A_3}{2R^3} - \frac{2G}{5c^5}\frac{\kappa_n M}{5}\,\mathfrak{I}^{(5)}_{\bar{3}\bar{3}}a_3,\\
\dot{p}_\beta &= 0,\\
\dot{p}_\phi &= -\frac{2G\mu}{5c^5}\,\mathfrak{I}^{(5)}_{\bar{1}\bar{2}}r^2,\\
\dot{p}_r &= -\frac{G\mathcal{M}\mu}{r^2} + \frac{p_\phi^2}{\mu r^3} - \frac{3}{10}\frac{G\mathcal{M}\mu}{r^4}\left[\kappa_n(a_1^2 - a_3^2) + \kappa_n'(a_1'^2 - a_3'^2)\right] - \frac{2G\mu}{5c^5}\,\mathfrak{I}^{(5)}_{\bar{1}\bar{1}}r,
\end{aligned} \tag{68}$$

while Eqs. (38) remain unchanged. In the quasi-static approximation it is the time varying orbital STF mass quadrupole tensor alone, which contributes to $\mathfrak{I}^{(5)}_{\bar{a}\bar{b}}$. Explicitly, the components of $\mathfrak{I}^{(5)}_{\bar{a}\bar{b}}$ read

$$\begin{aligned}
\mathfrak{I}^{(5)}_{\bar{1}\bar{1}} &= -\frac{8G\mathcal{M}}{3}\frac{p_r}{r^4}\left[4\,\frac{G\mathcal{M}}{r} + 3\,\frac{p_r^2}{\mu^2} + 18\,\frac{p_\phi^2}{\mu^2 r^2}\right],\\
\mathfrak{I}^{(5)}_{\bar{2}\bar{2}} &= \frac{2G\mathcal{M}}{3}\frac{p_r}{r^4}\left[8\,\frac{G\mathcal{M}}{r} + 6\frac{p_r^2}{\mu^2} + 81\,\frac{p_\phi^2}{\mu^2 r^2}\right],
\end{aligned} \tag{69}$$

$$\mathrm{I}^{(5)}_{\bar{3}\bar{3}} = \frac{2G\mathcal{M}}{3}\frac{p_r}{r^4}\left[8\,\frac{G\mathcal{M}}{r} + 6\,\frac{p_r^2}{\mu^2} - 9\,\frac{p_\phi^2}{\mu^2 r^2}\right],$$
$$\mathrm{I}^{(5)}_{\bar{1}\bar{2}} = -\frac{4G\mathcal{M}}{r^5}\,p_\phi\left[2\,\frac{G\mathcal{M}}{r} + 9\frac{p_r^2}{\mu^2} - 6\,\frac{p_\phi^2}{\mu^2 r^2}\right].$$

For numerical calculations it is useful to apply the following scaling:

$$t = \frac{G\mathcal{M}}{c^3}\tau, \quad p_r = \mu c\tilde{p}_r, \quad p_{a_i} = \mu c\tilde{p}_{a_i}, \quad p_\phi = \tfrac{G\mathcal{M}\mu}{c}\tilde{p}_\phi,$$
$$p_\beta = \tfrac{G\mathcal{M}\mu}{c}\tilde{p}_\beta, \quad a_i = \tfrac{G\mathcal{M}}{c^2}\tilde{a}_i, \quad r = \tfrac{G\mathcal{M}}{c^2}\tilde{r}. \tag{70}$$

Introducing the equilibrium radius R_0 of a nonrotating (spherical) polytrope of mass M and polytropic index n the terms containing P_c/ρ_c can be expressed as [19]

$$\frac{k_1 M}{n}\frac{P_c}{\rho_c} = \frac{GM^2}{(5-n)R_0}\left(\frac{R_0}{R}\right)^{3/n}. \tag{71}$$

The full set of differential equations governing the dynamics of the binary system is then given by

$$\begin{aligned}
\dot{\tilde{p}}_{a_1} &= \frac{5}{2\kappa_n C_1}\frac{\tilde{p}_\beta^2}{\tilde{a}_1^3} + \frac{\kappa_n}{5}\frac{\tilde{a}_1}{\tilde{r}^3} + \frac{1}{5-n}\frac{C_1}{C_2}\left[\frac{2}{\tilde{a}_1\tilde{R}_0}\left(\frac{\tilde{R}_0}{\tilde{R}}\right)^{3/n} - 3\,\frac{\tilde{a}_1 A_1}{\tilde{R}^3}\right]\\
&\quad + \frac{4\kappa_n}{75}\frac{\tilde{a}_1}{C_2}\frac{\tilde{p}_r}{\tilde{r}^4}\left[\frac{8}{\tilde{r}} + 6\tilde{p}_r^2 - 9\,\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right],\\
\dot{\tilde{p}}'_{a_1} &= \frac{5}{2\kappa'_n C_2}\frac{\tilde{p}_\beta'^2}{\tilde{a}_1'^3} + \frac{\kappa'_n}{5}\frac{\tilde{a}'_1}{\tilde{r}^3} + \frac{1}{5-n'}\frac{C_2}{C_1}\left[\frac{2}{\tilde{a}'_1\tilde{R}'_0}\left(\frac{\tilde{R}'_0}{\tilde{R}'}\right)^{3/n'} - 3\,\frac{\tilde{a}'_1 A'_1}{\tilde{R}'^3}\right]\\
&\quad + \frac{4\kappa'_n}{75}\frac{\tilde{a}'_1}{C_1}\frac{\tilde{p}_r}{\tilde{r}^4}\left[\frac{8}{\tilde{r}} + 6\tilde{p}_r^2 - 9\,\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right],\\
\dot{\tilde{p}}_{a_3} &= -\frac{\kappa_n}{5}\frac{\tilde{a}_3}{\tilde{r}^3} + \frac{C_1}{C_2}\frac{1}{5-n}\left[\frac{1}{\tilde{a}_3\tilde{R}_0}\left(\frac{\tilde{R}_0}{\tilde{R}}\right)^{3/n} - \frac{3}{2}\frac{\tilde{a}_3 A_3}{\tilde{R}^3}\right]\\
&\quad - \frac{4\kappa_n}{75}\frac{\tilde{a}_3}{C_2}\frac{\tilde{p}_r}{\tilde{r}^4}\left[\frac{8}{\tilde{r}} + 6\tilde{p}_r^2 - 9\,\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right],\\
\dot{\tilde{p}}'_{a_3} &= -\frac{\kappa'_n}{5}\frac{\tilde{a}'_3}{\tilde{r}^3} + \frac{C_2}{C_1}\frac{1}{5-n'}\left[\frac{1}{\tilde{a}'_3\tilde{R}_0}\left(\frac{\tilde{R}'_0}{\tilde{R}'}\right)^{3/n'} - \frac{3}{2}\frac{\tilde{a}'_3 A'_3}{\tilde{R}'^3}\right]\\
&\quad - \frac{4\kappa'_n}{75}\frac{\tilde{a}'_3}{C_1}\frac{\tilde{p}_r}{\tilde{r}^4}\left[\frac{8}{\tilde{r}} + 6\tilde{p}_r^2 - 9\,\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right],
\end{aligned}$$

$$
\begin{aligned}
\dot{\tilde{p}}_\beta &= \dot{\tilde{p}}'_\beta = 0,\\
\dot{\tilde{p}}_\phi &= \frac{8\nu}{5}\frac{\tilde{p}_\phi}{\tilde{r}^3}\left[\frac{2}{\tilde{r}} + 9\tilde{p}_r^2 - 6\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right],\\
\dot{\tilde{p}}_r &= -\frac{1}{\tilde{r}^2} + \frac{\tilde{p}_\phi^2}{\tilde{r}^3} - \frac{3}{10\tilde{r}^4}\left[\kappa_n(\tilde{a}_1^2 - \tilde{a}_3^2) + \kappa'_n(\tilde{a}_1'^2 - \tilde{a}_3'^2)\right]\\
&\quad + \frac{16\nu}{15}\frac{\tilde{p}_r}{\tilde{r}^3}\left[\frac{4}{\tilde{r}} + 3\tilde{p}_r^2 + 18\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right],\\
\dot{\tilde{a}}_1 &= \frac{5}{2\kappa_n C_1}\tilde{p}_{a_1},\\
\dot{\tilde{a}}'_1 &= \frac{5}{2\kappa'_n C_2}\tilde{p}'_{a_1},\\
\dot{\tilde{a}}_3 &= \frac{5}{\kappa_n C_1}\tilde{p}_{a_3},\\
\dot{\tilde{a}}'_3 &= \frac{5}{\kappa'_n C_2}\tilde{p}'_{a_3},\\
\dot{\beta} &= \frac{5}{2\kappa_n C_1}\frac{\tilde{p}_\beta}{\tilde{a}_1^2},\\
\dot{\beta}' &= \frac{5}{2\kappa'_n C_2}\frac{\tilde{p}'_\beta}{\tilde{a}_1'^2},\\
\dot{\tilde{r}} &= \tilde{p}_r,\\
\dot{\phi} &= \frac{\tilde{p}_\phi}{\tilde{r}^2},
\end{aligned}
\tag{72}
$$

where $C_1 = \frac{\mathcal{M}}{M'} = \frac{M}{\mu}$ and $C_2 = \frac{\mathcal{M}}{M} = \frac{M'}{\mu}$. For the numerical calculations shown below we shall assume that integration starts at periastron, i.e. the initial values for the orbit are given by

$$
\begin{aligned}
\tilde{r}(0) &= \tilde{d}(0)(1-\epsilon(0)), \quad \phi(0) = 0, \quad \tilde{p}_r(0) = 0,\\
\tilde{p}_\phi(0) &= \sqrt{\tilde{d}(0)(1-\epsilon(0)^2)},
\end{aligned}
$$

$\tilde{d}$ being the semi-major axis of the orbit. The quasi-static approximation requires all velocities inside the stars to be small. In particular, the mean radius $\tilde{R}$ varies only slowly with time. Hence we impose $\dot{\tilde{R}}(0) \overset{!}{=} 0$, i.e. $2\dot{\tilde{a}}_1(0)\tilde{a}_1(0)\tilde{a}_3(0) + \tilde{a}_1(0)^2\dot{\tilde{a}}_3(0) = 0$. This yields immediately a relation between the generalized momenta corresponding to a_1 and a_3 respectively,

$$
\dot{\tilde{p}}_{a_3}(0) = -\frac{\tilde{a}_3(0)}{\tilde{a}_1(0)}\tilde{p}_{a_1}(0), \quad \dot{\tilde{p}}'_{a_3}(0) = -\frac{\tilde{a}'_3(0)}{\tilde{a}'_1(0)}\tilde{p}'_{a_1}(0).
$$

Given suitable (i.e. small) values of $\dot{\tilde{a}}_i(0)$, $\dot{\tilde{a}}'_i(0)$ and $\dot{\beta}(0)$ respectively $\dot{\beta}'(0)$, we still have to fix the radii R_0 an R'_0, respectively. R_0 represents the equilibrium radius of a spherically symmetric (i.e. non-rotating) polytrope of mass M and polytropic index n. Assuming that the rotating star is in equilibrium at $t=0$, we can apply the equilibrium relation between R and R_0, which has been derived in [19]:[3]

$$R_0 = R(0)\left[\frac{3\ \arcsin\ e(0)}{e(0)}\,(1-e(0)^2)^{1/6}\left(1-\frac{1}{e(0)^2}+\frac{\sqrt{1-e(0)^2}}{e(0)\arcsin\ e(0)}\right)\right]^{n/(3-n)}, \tag{73}$$

where $e(0)=\sqrt{1-(a_3(0)/a_1(0))^2}$ is the eccentricity of the ellipsoid.

4 Gravitational waveforms

In suitable coordinates the gravitational field, observed in an asymptotically flat space, can be expressed as [20]

$$h_{ij}^{\mathrm{rad}} = \frac{G}{Dc^4}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[\left(\frac{1}{c}\right)^{l-2}\ {}^{(l)}\mathcal{I}^{lm}\left(t-\frac{D}{c}\right)T_{ij}^{E2,lm}(\Theta,\Phi)\right.$$
$$\left.+\left(\frac{1}{c}\right)^{l-1}\ {}^{(l)}\mathcal{S}^{lm}\left(t-\frac{D}{c}\right)T_{ij}^{B2,lm}(\Theta,\Phi)\right], \tag{74}$$

where D is the source-observer distance and the indices i,j refer to Cartesian coordinates in the asymptotic space. $\mathcal{I}^{lm}$ and $\mathcal{S}^{lm}$ are the spherical radiative mass and current multipole moments, respectively, while $T_{ij}^{E2,lm}$ and $T_{ij}^{B2,lm}$ represent the so called pure-spin tensor harmonics of electric and magnetic type, respectively. Finally, the upper index l denotes the number of time derivatives.[4] In leading order approximation only the $l=2$ terms contribute to the gravitational field, which reflects the quadrupole character of the gravitational radiation, i.e.

$$h_{ij}^{\mathrm{rad}} = \frac{G}{Dc^4}\sum_{m=-2}^{2}\ddot{\mathcal{I}}^{2m}\left(t-\frac{D}{c}\right)T_{ij}^{E2,2m}(\Theta,\Phi). \tag{75}$$

---

[3] Of course this does not mean that the star is in equilibrium once the integration has started.

[4] Exploiting the transverse traceless character of the gravitational radiation it is possible to introduce two polarization vectors **p** and **q** in the plane orthogonal to the direction of propagation. This leads to the more familiar $h_+, h_\times$ notation,

$$h_+ = \frac{p_ip_j - q_iq_j}{2}h_{ij}^{\mathrm{rad}}, \quad h_\times = \frac{p_iq_j + p_jq_i}{2}h_{ij}^{\mathrm{rad}},$$

where h_+ and $h_\times$ represent the two polarization states.

However, one might chose to use the more familiar STF mass and current multipole moments rather than calculating the spherical radiative ones which appear in Eq. (75). In particular, the STF mass multipole moments are related to the spherical radiative ones by

$$\mathcal{I}^{lm}(t) = \frac{16\pi}{(2l+1)!!}\left[\frac{(l+1)(l+2)}{2l(l-1)}\right]^{1/2} \mathrm{I}_{A_l} Y^{*\,lm}_{A_l}, \tag{76}$$

where $A_l = i_1 \cdots i_l$ is a multi-index, while the $Y^{lm}_{A_l}$ are defined as

$$Y^{lm} := (-1)^m (2l-1)!! \sqrt{\frac{2l+1}{4\pi(l-m)!(l+m)!}} \\ \times \left(\delta^1_{\langle i_1} + i\delta^2_{\langle i_1}\right) \cdots \left(\delta^1_{i_m} + i\delta^2_{i_m}\right) \delta^3_{i_{m+1}} \cdots \delta^3_{i_l \rangle}.$$

The brackets $\langle \ldots \rangle$ denote symmetrization. Inserting this into Eq. (76) the spherical radiative mass quadrupole moments contributing to the leading order gravitational waveform read

$$\mathcal{I}^{20} = 4\sqrt{\frac{3}{5\pi}} \mathrm{I}^{(\mathrm{IS})}_{33}, \tag{77}$$

$$\mathcal{I}^{21} = -2\sqrt{\frac{8\pi}{5}} \left(\mathrm{I}^{(\mathrm{IS})}_{13} - i\mathrm{I}^{(\mathrm{IS})}_{23}\right), \tag{78}$$

$$\mathcal{I}^{22} = \sqrt{\frac{8\pi}{5}} \left(\mathrm{I}^{(\mathrm{IS})}_{11} - \mathrm{I}^{(\mathrm{IS})}_{22} - 2i\mathrm{I}^{(\mathrm{IS})}_{12}\right). \tag{79}$$

In our model only the $\mathcal{I}^{20}$ and $\mathcal{I}^{22}$-components are present in the leading order gravitational wave field, since $\mathcal{I}^{21}$ vanishes due to the symmetry of the binary system. To calculate the gravitational waveforms explicitly we need to know the second time derivatives of the STF mass quadrupole moment in the inertial frame. As before we shall compute the contributions of the star's quadrupole moments and the orbital terms separately.

4.1 Orbital contribution

Consider an inertial center of mass system. In this system the nonvanishing components of the orbital STF mass quadrupole tensor are given by Eq. (53). Taking the second time derivatives and inserting the Newtonian equations of motion (24) and (25) we end up with

$$\ddot{\mathrm{I}}^{(\mathrm{IS})}_{11} = -2\mu r\dot{r}\dot{\phi}\sin 2\phi + \frac{\mu}{3}(1+3\cos 2\phi)\left[\dot{r}^2 - \frac{G\mathcal{M}}{r}\right] + \frac{\mu r^2}{3}(1-3\cos 2\phi)\dot{\phi}^2 \\ -\frac{G\mathcal{M}\mu}{10r^3}(1+3\cos 2\phi)\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa'_n\left(a_1'^2 - a_3'^2\right)\right]$$

$$= -2\frac{p_r p_\phi}{\mu r}\sin 2\phi + \frac{1}{3\mu}(1+3\cos 2\phi)\left[p_r^2 - \frac{G\mathcal{M}\mu^2}{r}\right] + \frac{p_\phi^2}{3\mu r^2}(1-3\cos 2\phi)$$
$$- \frac{G\mathcal{M}\mu}{10r^3}(1+3\cos 2\phi)\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right], \qquad (80)$$

$$\ddot{\mathrm{I}}_{22}^{(\mathrm{IS})} = 2\mu r\dot{r}\dot{\phi}\sin 2\phi + \frac{\mu}{3}(1-3\cos 2\phi)\left[\dot{r}^2 - \frac{G\mathcal{M}}{r}\right] + \frac{\mu r^2}{3}(1+3\cos 2\phi)\,\dot{\phi}^2$$
$$- \frac{G\mathcal{M}\mu}{10r^3}(1-3\cos 2\phi)\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right]$$
$$= 2\frac{p_r p_\phi}{\mu r}\sin 2\phi + \frac{1}{3\mu}(1-3\cos 2\phi)\left[p_r^2 - \frac{G\mathcal{M}\mu^2}{r}\right] + \frac{p_\phi^2}{3\mu r^2}(1+3\cos 2\phi)$$
$$- \frac{G\mathcal{M}\mu}{10r^3}(1-3\cos 2\phi)\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right], \qquad (81)$$

$$\ddot{\mathrm{I}}_{33}^{(\mathrm{IS})} = \frac{2\mu}{3}\left[\frac{G\mathcal{M}}{r} - r^2\dot{\phi}^2 - \dot{r}^2\right] + \frac{G\mathcal{M}\mu}{5r^3}\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right]$$
$$= \frac{2}{3\mu}\left[\frac{G\mathcal{M}\mu^2}{r} - \frac{p_\phi^2}{r^2} - p_r^2\right] + \frac{G\mathcal{M}\mu}{5r^3}\left[\kappa_n\left(a_1^2 - a_3^2\right) + \kappa_n'\left(a_1'^2 - a_3'^2\right)\right], \qquad (82)$$

$$\ddot{\mathrm{I}}_{12}^{(\mathrm{IS})} = 2\mu r\dot{r}\dot{\phi}\cos 2\phi + \left[\dot{r}^2 - \frac{G\mathcal{M}}{r} - r^2\dot{\phi}^2 - \frac{3G\mathcal{M}\kappa_n}{10r^3}\left(a_1^2 - a_3^2\right)\right.$$
$$\left. - \frac{3G\mathcal{M}\kappa_n'}{10r^3}(a_1'^2 - a_3'^2)\right]\mu\sin 2\phi$$
$$= 2\frac{p_r p_\phi}{\mu r}\cos 2\phi + \left[\frac{p_r^2}{\mu} - \frac{G\mathcal{M}\mu}{r} - \frac{p_\phi^2}{\mu r^2} - \frac{3G\mathcal{M}\mu\kappa_n}{10r^3}\left(a_1^2 - a_3^2\right)\right.$$
$$\left. - \frac{3G\mathcal{M}\mu\kappa_n'}{10r^3}\left(a_1'^2 - a_3'^2\right)\right]\sin 2\phi. \qquad (83)$$

Note that tidal coupling introduces an additional contribution to $\ddot{\mathrm{I}}_{ij}^{(\mathrm{IS})}$. Since this contribution is very small it can be neglected for elliptical orbits and for $\ddot{\mathcal{I}}^{22}$. However, it must be taken into account when considering $\ddot{\mathcal{I}}^{20}$ for circular orbits, where $\ddot{\mathcal{I}}^{20}$ would vanish identically for a point-particle system.[5]

4.2 Stellar contributions

To calculate the contribution of, say, star M, let us assume the origin of the inertial system to coincide with the center of the star. The coordinates X_i in the inertial system and coordinates x_i in the body-fixed system are then related by the $O(3)$-transformation given in Eq. (45), and the elements of the star's STF mass quadrupole tensor read

[5] Due to tidal interaction effects the orbit is modified which in turn leads to a small, but nonvanishing $\ddot{\mathcal{I}}^{20}$-component of the gravitational radiation field.

$$\mathcal{I}_{ij}^{(\mathrm{IS}),\,\mathrm{star}} = T^{\dagger}_{i\alpha}(\gamma) T^{\dagger}_{j\beta}(\gamma) \mathcal{I}^{\mathrm{star}}_{\alpha\beta}.$$

$\mathcal{I}^{\mathrm{star}}_{\alpha\beta}$ refers to the body-fixed system, where the STF mass quadrupole tensor takes a particularly simple form. For the most general case when all semi-major axes of the star are different, the transformation to the inertial system yields[6]

$$\begin{aligned}
\mathcal{I}_{11}^{(\mathrm{IS})} &= \frac{1}{2}(I_{11} - I_{22}) \cos 2\gamma + \frac{1}{6}(I_{11} + I_{22} - 2I_{33}),\\
\mathcal{I}_{22}^{(\mathrm{IS})} &= -\frac{1}{2}(I_{11} - I_{22}) \cos 2\gamma + \frac{1}{6}(I_{11} + I_{22} - 2I_{33}),\\
\mathcal{I}_{33}^{(\mathrm{IS})} &= -\frac{1}{3}(I_{11} + I_{22} - 2I_{33}),\\
\mathcal{I}_{12}^{(\mathrm{IS})} &= \frac{1}{2}(I_{11} - I_{22}) \sin 2\gamma.
\end{aligned} \tag{84}$$

It is easy to see that this reduces to $I_{11}^{(\mathrm{IS})} = I_{22}^{(\mathrm{IS})} = -I_{33}^{(\mathrm{IS})}/2 = (I_{11} - I_{33})/3 = 1/3(I_{11} - I_{33})$ for $a_1 \equiv a_2$. A straightforward calculation yields

$$\begin{aligned}
\ddot{\mathcal{I}}_{11}^{(\mathrm{IS})} &= \left[\frac{1}{2}(\ddot{I}_{11} - \ddot{I}_{22}) - 2(I_{11} - I_{22})\dot{\gamma}^2\right] \cos 2\gamma\\
&\quad - \left[2(\dot{I}_{11} - \dot{I}_{22})\dot{\gamma} + (I_{11} - I_{22})\ddot{\gamma}\right] \sin 2\gamma + \frac{1}{6}(\ddot{I}_{11} + \ddot{I}_{22} - 2\ddot{I}_{33}),\\
\ddot{\mathcal{I}}_{22}^{(\mathrm{IS})} &= -\left[\frac{1}{2}(\ddot{I}_{11} - \ddot{I}_{22}) - 2(I_{11} - I_{22})\dot{\gamma}^2\right] \cos 2\gamma\\
&\quad + \left[2(\dot{I}_{11} - \dot{I}_{22})\dot{\gamma} + (I_{11} - I_{22})\ddot{\gamma}\right] \sin 2\gamma + \frac{1}{6}(\ddot{I}_{11} + \ddot{I}_{22} - 2\ddot{I}_{33}),\\
\ddot{\mathcal{I}}_{33}^{(\mathrm{IS})} &= -\frac{1}{3}(\ddot{I}_{11} + \ddot{I}_{22} - 2\ddot{I}_{33}),\\
\ddot{\mathcal{I}}_{12}^{(\mathrm{IS})} &= \left[\frac{1}{2}(\ddot{I}_{11} - \ddot{I}_{22}) - 2(I_{11} - I_{22})\dot{\gamma}^2\right] \sin 2\gamma\\
&\quad + \left[2(\dot{I}_{11} - \dot{I}_{22})\dot{\gamma} + (I_{11} - I_{22})\ddot{\gamma}\right] \cos 2\gamma.
\end{aligned}$$

In the quasi-static approximation the 2nd time derivative of the components of the stellar mass quadrupole tensor in the body fixed system vanishes ($I_{ij} \approx 0$) and we are left with

$$\begin{aligned}
\ddot{\mathcal{I}}_{11}^{(\mathrm{IS})} &= -2(I_{11} - I_{22})\Omega^2 \cos 2\gamma - \left[2(\dot{I}_{11} - \dot{I}_{22})\Omega + (I_{11} - I_{22})\dot{\Omega}\right] \sin 2\gamma,\\
\ddot{\mathcal{I}}_{22}^{(\mathrm{IS})} &= -\ddot{\mathcal{I}}_{11}^{(\mathrm{IS})},\\
\ddot{I}_{12}^{(\mathrm{IS})} &= -2(I_{11} - I_{22})\Omega^2 \sin 2\gamma + \left[2(\dot{I}_{11} - \dot{I}_{22})\Omega + (I_{11} - I_{22})\dot{\Omega}\right] \cos 2\gamma,
\end{aligned} \tag{85}$$

where $\Omega = \dot{\gamma}$. We should emphasize that Eqs. (85) imply that $\ddot{I}_{ij}^{(\mathrm{IS})}$ vanishes for axially symmetric polytropic stars, i.e. for $a_1 \equiv a_2$. In other words, in the

[6] From now on we submit the index *star*.

quasi-static approximation there is no direct contribution of the stars to the components of the gravitational field. Nevertheless the internal degrees of freedom give rise to modifications of the binary's gravitational wave forms due to tidally driven modifications of the orbital motion and hence of $\ddot{\mathcal{I}}^{2m}$.

4.3 Gravitational waveforms of the binary system

As it has been pointed out before, in the case of axially symmetric stellar components of the binary only the orbital mass quadrupole tensor contributes to the components of h_{ij}, the actual gravitational waveforms being modified due to tidal coupling. Inserting Eqs. (80) into the expressions for $\mathcal{I}^{2m}$ given by Eqs. (77)–(79), the components of the leading order gravitational wave field read

$$\ddot{\mathcal{I}}^{20} = 4\sqrt{\frac{3\pi}{5}}\mu c^2\left[\frac{2}{3}\left(\frac{1}{\tilde{r}}-\tilde{p}_r^2-\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right)+\frac{1}{5\tilde{r}^3}\left(\kappa_n\left(\tilde{a}_1^2-\tilde{a}_3^2\right)+\kappa_n'\left(\tilde{a}_1'^2-\tilde{a}_3'^2\right)\right)\right], \quad (86)$$

$$\Re(\ddot{\mathcal{I}}^{22}) = \sqrt{\frac{8\pi}{5}}\mu c^2\left[-4\frac{\tilde{p}_r\tilde{p}_\phi}{\tilde{r}}\sin 2\phi \right.$$
$$\left. +\left\{2\left(\tilde{p}_r^2-\frac{1}{\tilde{r}}-\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right)-\frac{3}{5\tilde{r}^3}\left(\kappa_n\left(\tilde{a}_1^2-\tilde{a}_3^2\right)+\kappa_n'\left(\tilde{a}_1'^2-\tilde{a}_3'^2\right)\right)\right\}\cos 2\phi\right], \quad (87)$$

$$\Im\left(\ddot{\mathcal{I}}^{22}\right) = \sqrt{\frac{8\pi}{5}}\mu c^2\left[-4\frac{\tilde{p}_r\tilde{p}_\phi}{\tilde{r}}\cos 2\phi \right.$$
$$\left. -\left\{2\left(\tilde{p}_r^2-\frac{1}{\tilde{r}}-\frac{\tilde{p}_\phi^2}{\tilde{r}^2}\right)-\frac{3}{5\tilde{r}^3}\left(\kappa_n\left(\tilde{a}_1^2-\tilde{a}_3^2\right)+\kappa_n'\left(\tilde{a}_1'^2-\tilde{a}_3'^2\right)\right)\right\}\sin 2\phi\right]. \quad (88)$$

Note the presence of an additional quadrupole-quadrupole coupling term in above equations. This term can be easily neglected for the 22-component of the gravitational radiation, but it will significantly modify the 20-component. This is due to the symmetry of the binary, the spins of the stars being aligned perpendicular to the orbital plane.

5 Numerical results and discussion

In the last few years there has been a growing interest in investigating the influence of tidal interaction onto the inspiral process and the actual leading order gravitational waveforms emitted by a binary system. Within the framework of linear perturbation theory of stellar oscillations, close binary systems of nonrotating neutron stars where studied by Kokkotas and Schäfer [7] and later on by Lai and Ho [8]. In these papers it was shown that tidal interaction may draw energy from the orbital motion, thus speeding up the inspiral process. This effect is strongest in the case of a tidal resonance when the orbital frequency is the *m*th fraction of a star's eigenfrequency, m being an integer number. Resonant tidal oscillations of a nonrotating white dwarf-compact object binary were investigated by Rathore et al. [21], but in this paper dissipation due to gravitational waves emission was not included. On the contrary our model incorporates not only dissipation due to gravitational wave emission, but also stellar rotation. Moreover, we do not restrict ourselves to small, linear perturbations of thermal equilibrium, but allow for oscillations with arbitrary large amplitude. However, there is only one oscillation mode per star present in our analysis (this mode corresponding roughly to the f-mode), this means we truncate all the higher eigenmodes which are incorporated in the linear perturbation scenarios. If much of the oscillation energy is stored in these higher eigenmodes the gravitational waveforms calculated from our model will fail to give results of high accuracy. Therefore we restrict ourselves to moderate orbital separations, when the dominant contribution of the oscillation energy comes from the f-mode. In Fig. 2 the oscillation of the semi-major axis a_1 is shown for a particular example.

The most prominent feature of the model is that it allows us to study the long term evolution of the binary system. In fact, in our numerical calculations we were able to follow the orbital evolution over hundreds of periods. In particular we studied the influence of the equation of state on the binary's dynamics and thus onto the leading order gravitational wave pattern emitted during the time evolution. In the past it was argued that the influence of the polytropic index will not be reflected in the gravitational waveforms almost until the final plunge down [3]. While this is certainly true if one considers only a few orbits this suggestion has to be carefully checked for long term evolutions. Given suitable initial conditions our model allows for the investigation of the orbital evolution over hundreds of periods. It is thus an ideal tool to study the influence of the polytropic index on the dynamics as well as on the gravitational waveform in great detail. Note that the equations of motion given in Eqs. (72) do not apply

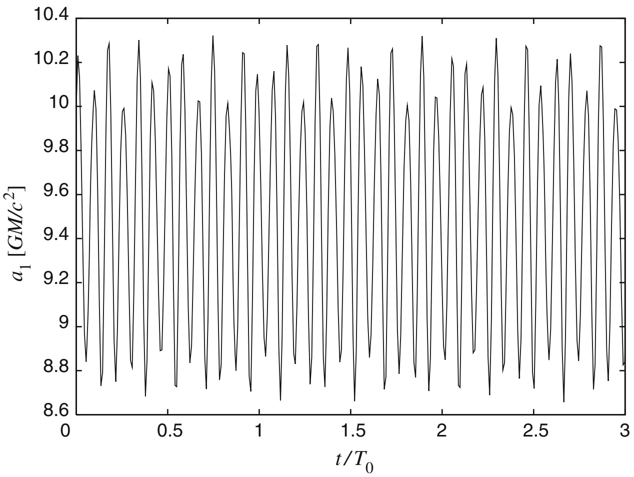

Fig. 2 Example for the stellar oscillation within the affine model. Plotted is the oscillation of the semi-major axis a_1 of star 1 in a tidally coupled binary system

to the limit of incompressible fluids ($n = 0$) and to the relativistic limit, which is represented by $\Gamma = 4/3$, i.e. $n = 3$. For $n = 0$ Eq. (71) has to be replaced by

$$\frac{5k_1}{n\kappa_n}\frac{P_c}{\rho_c} = 2\frac{P_c}{\rho_c}.$$

(For an explicit expression of P_c/ρ_c in terms of the stellar degrees of freedom see [13].) For $n \to 3$ numerical integration becomes increasingly instable. Moreover, Eq. (73), which gives a relation between the equilibrium radius R_0 of a nonrotating polytrope and the equilibrium mean radius R of a rotating polytrope, becomes singular for $n = 3$.

In Fig. 3 we compare the orbital evolution of a slightly elliptic ($\epsilon(0) = 0.4$) equal mass binary system for different choices of n'. As expected tidal coupling

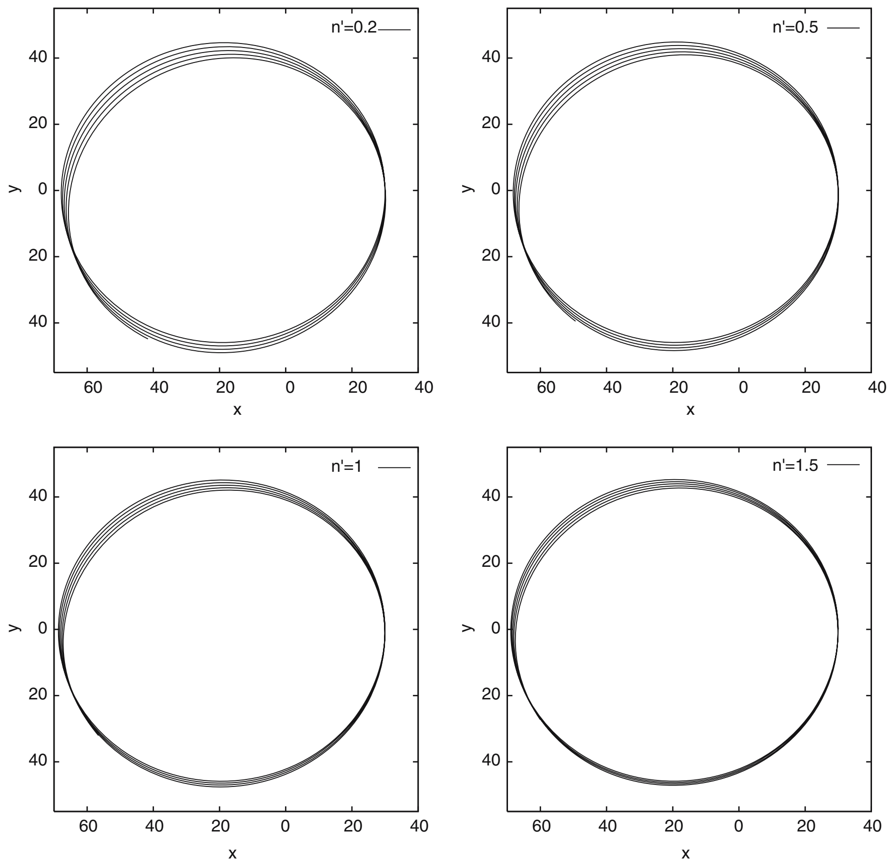

Fig. 3 Orbital evolution and periastron advance due to tidal coupling for an elliptic, equal mass binary with initial values $\epsilon(0) = 0.4, \tilde{d}(0) = 50, \tilde{a}_1(0) = \tilde{a}'_1(0) = 10, \tilde{a}_3(0) = \tilde{a}'_3(0) = 8, \dot{\beta}(0) = \dot{\beta}'(0) = 0.01, \dot{\tilde{a}}_1(0) = 0.02, \dot{\tilde{a}}'_1(0) = 0.01$. All systems have polytropic index $n = 1$, while star 2 obeys a polytropic equation of state with $n' = 0.2$ (*upper left*), $n' = 0.5$ (*upper right*), $n' = 1$ (*lower left*) and $n' = 1.5$ (*lower right*). Star 1 has polytropic index $n = 1$ in all scenarios

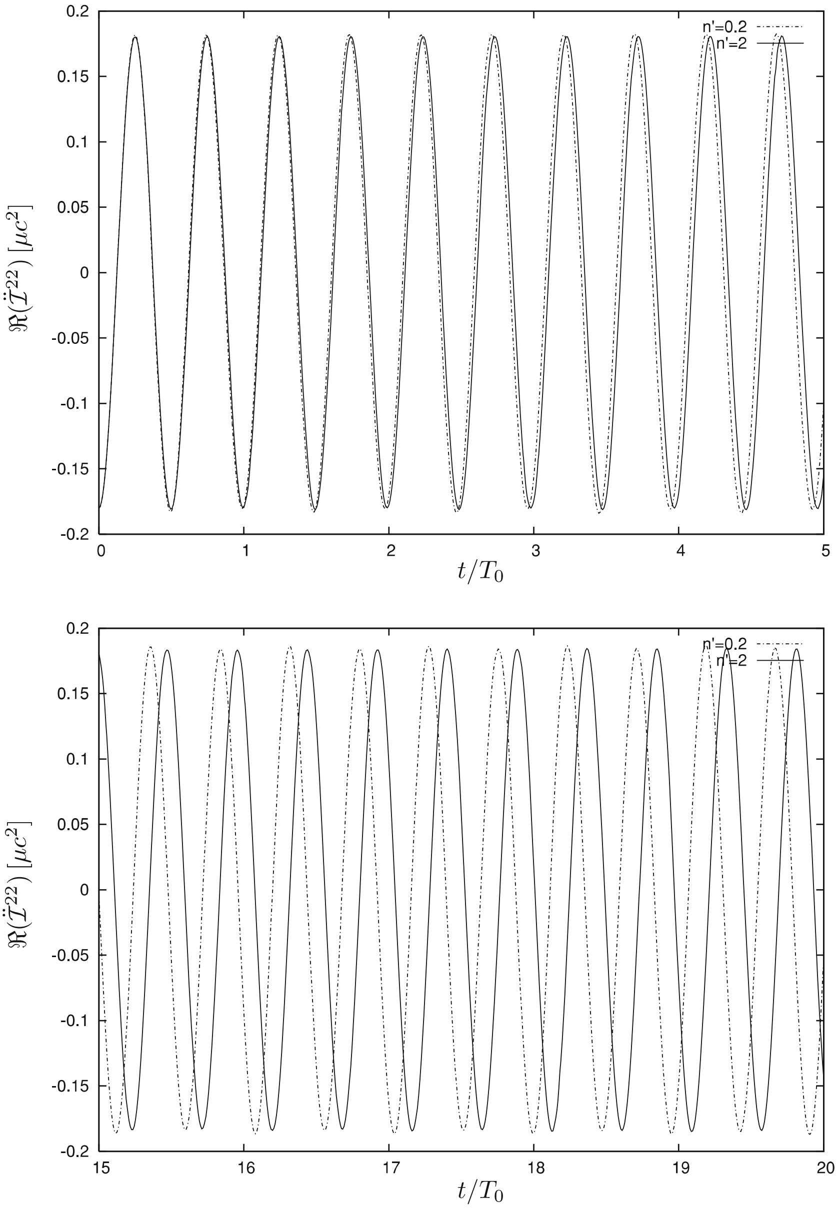

Fig. 4 The $\Re(\ddot{\mathcal{I}}^{22})$-component of the leading order gravitational waveform for a circular orbit ($\epsilon(0) = 0$) for polytropic indices $n' = 0.2$ (*dotted line*) and $n' = 2$ (*solid line*) at different time stages. The polytropic index of star 1 is assumed to be $n = 1$, the initial parameters are the same as in Fig. 3

induces a periastron advance, but it is clearly visible that the periastron advance increases with decreasing values of n'. Not too surprisingly, the polytropic index n' also affects the inspiral process. For fixed value of n the inspiral process speeds up the smaller the values of n', i.e. the larger the value of Γ'. As it is shown in Figs. 4 and 6 the effect of the equation of state on the gravitational waveforms emitted by the binary *cannot* be neglected even for moderate orbital distances if one considers the long-term evolution of the binary system. For circular orbits, the tidally induced modification of the gravitational wave pattern is strongest in the $\ddot{\mathcal{I}}^{20}$-component of h_{ij} (see Fig. 5), but also in the 22-component there is an significant phase shift due to different polytropic indizes n'. This is demonstrated in Fig. 4 where we compare the $\Re(\ddot{\mathcal{I}}^{22})-$component of the gravitational wave field for different values of n'. In these particular examples the orbit is assumed to be circular in the absence of tidal perturbation and the initial orbital separation is taken to be $\tilde{d}(0) = 50$. Even in this case a phase shift in the $\Re(\ddot{\mathcal{I}}^{22})-$component is already obvious after 10 orbital periods.

As already mentioned before, for a circular orbit the influence of the equation of state is strongest reflected by the $\ddot{\mathcal{I}}^{20}$-component of h_{ij}. For a point particle binary, where the leading order gravitational waveform is known analytically, this contribution to h_{ij} would vanish for a circular orbit. This is not the case if the orbital motion is tidally coupled to the internal motion of the stars (see Fig. 5).

For elliptic binaries the influence of the polytropic index n' on the $\ddot{\mathcal{I}}^{22}$-component is demonstrated in Figs. 6 and 7. Note that even for large orbital distances different polytropic indices n' manifest themselves in a remarkable phase shift

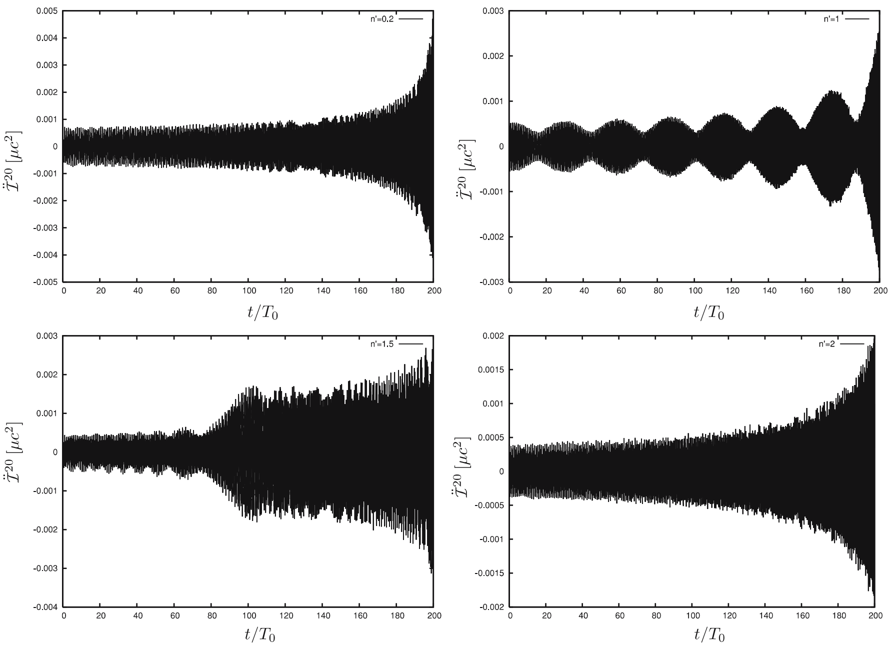

Fig. 5 $\ddot{\mathcal{I}}^{20}$-component of the leading order gravitational waves for a binary with polytropic index $n = 1$ for star 1 and $n' = 0.2$ (*upper left*), $n' = 1$ (*upper right*), $n' = 1.5$ (*bottom left*), $n' = 2$ (*bottom right*). The eccentricity at $t = 0$ is assumed to be zero, the initial parameters are the same as in Fig. 4

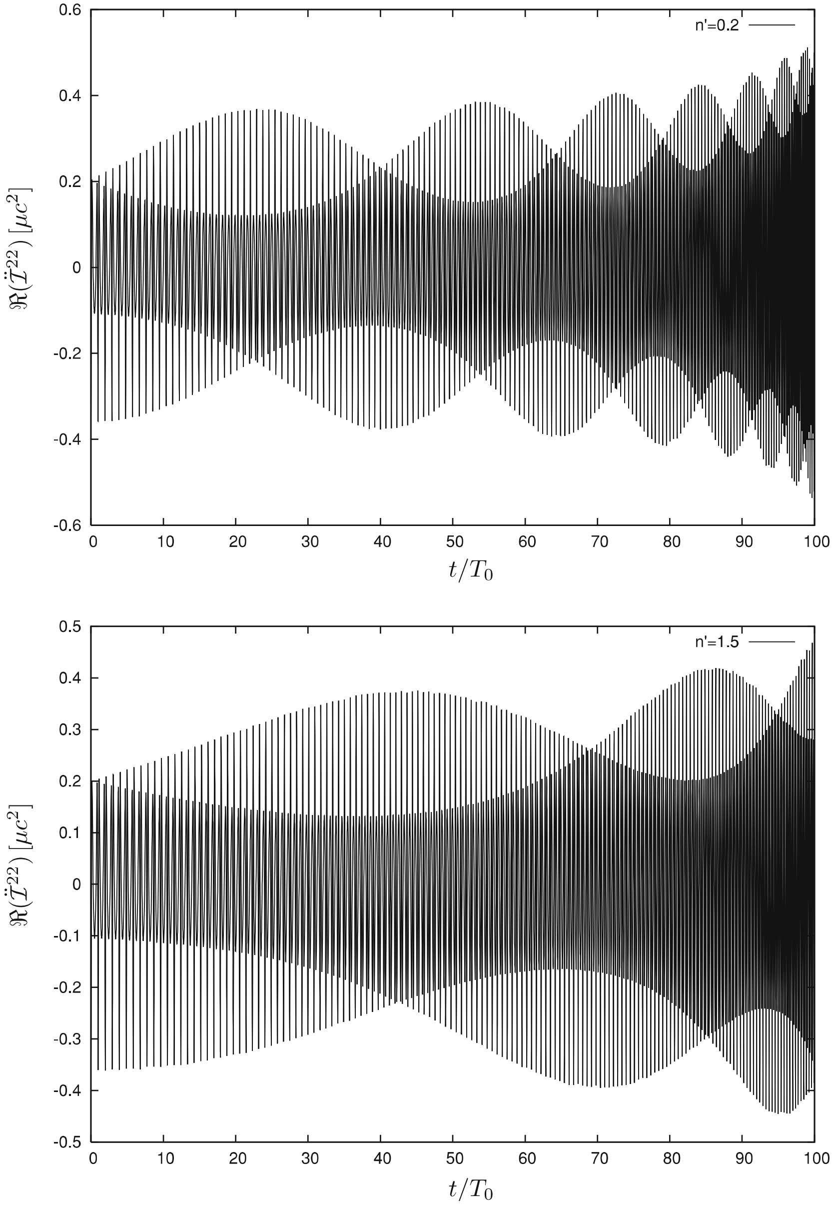

Fig. 6 Influence of the polytropic index n' on the real part of $\ddot{\mathcal{I}}^{22}$ for an elliptic binary with $\epsilon(0) = 0.4$. The initial parameters are the same as in Fig. 4. *Left*: $n' = 0.2$, *right*: $n' = 1.5$

of the $\ddot{\mathcal{I}}^{22}$-component of h_{ij} already after a few orbital cycles. This is demonstrated in Fig. 7, where we compare $\Re(\ddot{\mathcal{I}}^{22})$ for different values of n' at different stages of the inspiral process.

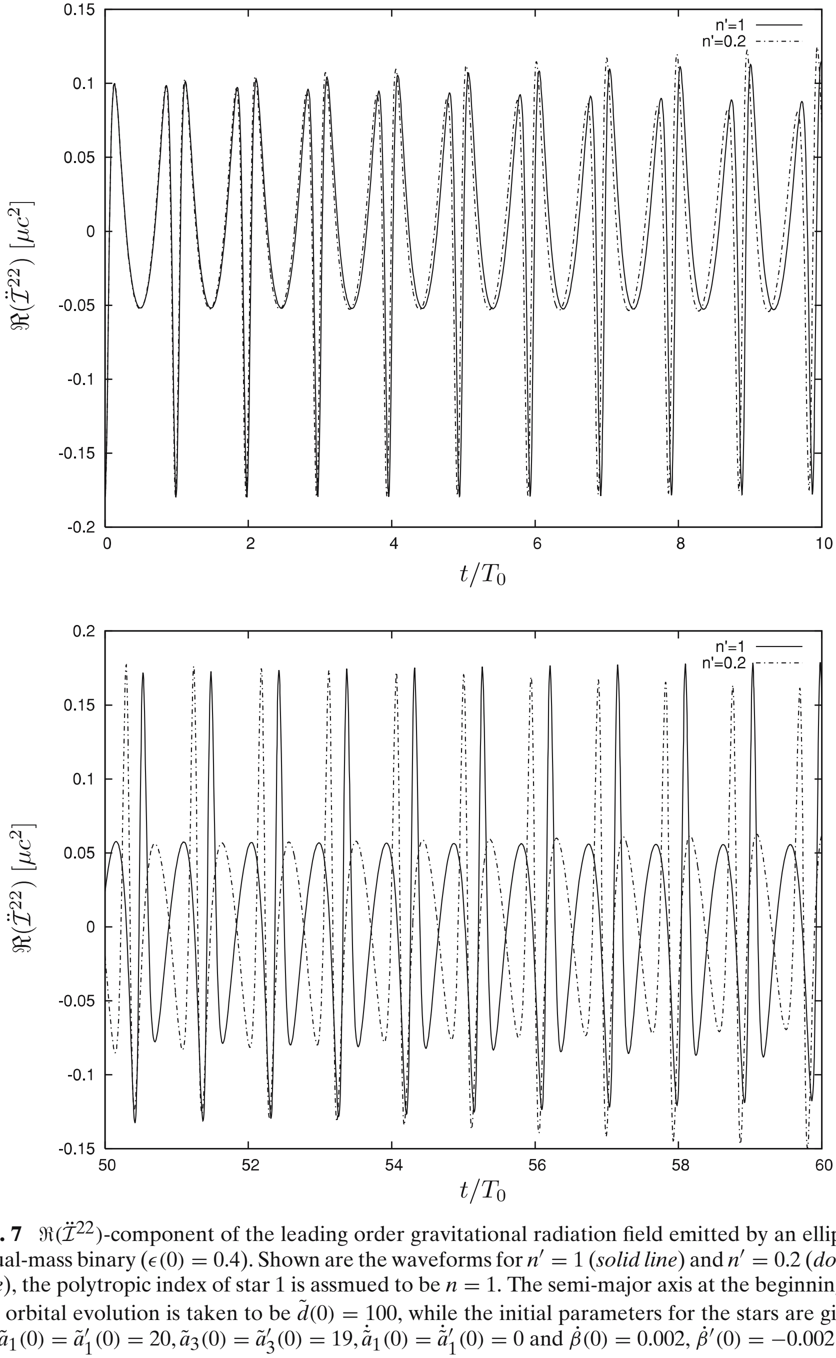

Fig. 7 $\Re(\ddot{\mathcal{I}}^{22})$-component of the leading order gravitational radiation field emitted by an elliptic, equal-mass binary ($\epsilon(0) = 0.4$). Shown are the waveforms for $n' = 1$ (*solid line*) and $n' = 0.2$ (*dotted line*), the polytropic index of star 1 is assmued to be $n = 1$. The semi-major axis at the beginning of the orbital evolution is taken to be $\tilde{d}(0) = 100$, while the initial parameters for the stars are given by $\tilde{a}_1(0) = \tilde{a}'_1(0) = 20, \tilde{a}_3(0) = \tilde{a}'_3(0) = 19, \dot{\tilde{a}}_1(0) = \dot{\tilde{a}}'_1(0) = 0$ and $\dot{\beta}(0) = 0.002$, $\dot{\beta}'(0) = -0.002$

To summarize, we have shown that including the internal structure of the stars will lead to significant changes of the leading order gravitational wave pattern compared to a point particle binary. Moreover, even for moderate relative distances of the stars there is a notable phase shift in the $\ddot{\mathcal{I}}^{22}$-component of the gravitational radiation field after some orbital periods. Of course our model does not respect all features of stellar dynamics. In particular it does not account for higher order oscillation modes which are present in linear perturbation theory. Nevertheless, our model could give an approximative description of a neutron star-neutron star binary. Although the internal structure of a neutron star is still not understood very good (the best known equations of motion are only given in a tabulated form), astrophysical observations indicate that the thermodynamics of neutron stars can be approximately described by a polytropic equation of state with polytropic index $n \approx 0.5-1.0$ (see e.g. [22]). Moreover, all low-mass white dwarfs can be modelled with an effective polytropic index $n \approx 1.5$. Thus our model might be applicable to study the long term evolution of NS-NS binaries, NS-White Dwarf binaries or binaries consisting of either a neutron star or a White Dwarf and a compact object.

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Appendix

A The Lane–Emden equation reviewed

For spherically symmetric polytropes des equations of hydrodynamics are given by

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm(r)\rho(r)}{r^2}, \quad \frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi\rho(r)r^2$$

which can be combined to a single equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{89}$$

Introducing dimensionless variables ξ and θ as

$$\rho := \rho_c\theta^n, \quad r = a_0\xi,$$

with

$$a_0 = \left[\frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]^{1/2},$$

we obtain the well known Lane–Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n, \quad \theta(0) = 1, \quad \theta'(0) = 0. \tag{90}$$

For $n < 5$ the solution of Eq. (90) decreases monotonically and becomes zero at a finite value of ξ. This value, denoted by ξ_1, is characterized by vanishing pressure and density and hence represents the star's surface.

B Chandrasekhar's coefficients A_i and $\mathcal{J}$

The quantity $\mathcal{J}$ is defined in [16] (chapter 3) according to

$$\mathcal{J} := a_1a_2a_3\int_0^\infty \frac{\mathrm{d}u}{\sqrt{\left(a_1^2+u\right)\left(a_2^2+u\right)\left(a_3^2+u\right)}} \tag{91}$$

while the coefficients A_i are given by

$$A_i := a_1a_2a_3\int_0^\infty \frac{\mathrm{d}u}{\left(a_i^2+u\right)\sqrt{\left(a_1^2+u\right)\left(a_2^2+u\right)\left(a_3^2+u\right)}}. \tag{92}$$

One can easily show that

$$\mathcal{J} = a_1^2A_1 + a_2^2A_2 + a_3^2A_3.$$

Another useful relation that can be easily derived from Eq. (91) is

$$\frac{\partial\mathcal{J}}{\partial a_i} = \frac{1}{a_i}\left(\mathcal{J} - a_i^2A_i\right) \quad \text{(no sum)} \tag{93}$$

and

$$\frac{\partial\mathcal{J}}{\partial a_1} = \frac{2}{a_1}\left(\mathcal{J} - a_1^2A_1\right), \quad \frac{\partial\mathcal{J}}{\partial a_3} = \frac{1}{a_3}\left(\mathcal{J} - a_3^2A_3\right) \qquad \text{for } a_1 = a_2. \tag{94}$$

In the special case $a_1 = a_2 > a_3$ it is possible to find an analytical expression for A_1 and A_3. Solving (92) yields

$$A_1 = A_2 = \frac{\sqrt{1-e^2}}{e^3}\,\text{arc sin}\,e - \frac{1-e^2}{e^2} \tag{95}$$

$$A_3 = \frac{2}{e^2} - \frac{2\sqrt{1-e^2}}{e^3}\,\text{arc sin}\,e \tag{96}$$

where we defined

$$e := \sqrt{1 - \left(\frac{a_3}{a_1}\right)^2}. \tag{97}$$

C $\mathbf{I}^{(5),\,\mathrm{star}}_{\bar{a}\bar{b}}$ beyond the quasi-static approximation

Throughout the paper we applied the quasi-static approximation to calculate the radiation reaction terms of the stellar degrees of freedom. This approximation is justified if all internal velocities and accelerations inside the star are small. Dropping this assumption, the result is much more complicated. In particular, the 5th time derivative of the STF mass quadrupole tensor of M in the corotating coordinate frame reads

$$\begin{aligned}
I^{(5)}_{\bar{a}\bar{b}} = (I_{11} - I_{22}) \Bigg[&\left(16\Omega^5 - 40\Omega^2\ddot{\Omega} - 60\Omega\dot{\Omega}^2 + \Omega^{(4)}\right) \begin{pmatrix} \sin 2\alpha & \cos 2\alpha & 0 \\ \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&- \left(80\Omega^3\dot{\Omega} - 20\dot{\Omega}\ddot{\Omega} - 10\Omega\Omega^{(3)}\right) \begin{pmatrix} -\cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] \\
&+ (\dot{I}_{11} - \dot{I}_{22}) \Bigg[\left(40\Omega^4 - 30\dot{\Omega}^2 - 40\Omega\ddot{\Omega}\right) \begin{pmatrix} \cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & -\cos 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \left(120\Omega^2\dot{\Omega} - 5\Omega^{(3)}\right) \begin{pmatrix} -\sin 2\alpha & \cos 2\alpha & 0 \\ \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] \\
&+ (\ddot{I}_{11} - \ddot{I}_{22}) \Bigg[(10\ddot{\Omega} - 40\Omega^3) \begin{pmatrix} \sin 2\alpha & \cos 2\alpha & 0 \\ \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} + 60\Omega\dot{\Omega} \begin{pmatrix} -\cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] \\
&+ 10(I^{(3)}_{11} - I^{(3)}_{22}) \Bigg[2\Omega^2 \begin{pmatrix} -\cos 2\alpha & \sin 2\alpha & 0 \\ \sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} + \dot{\Omega} \begin{pmatrix} \sin 2\alpha & \cos 2\alpha & 0 \\ \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg] \\
&+ 5\Omega\,(I^{(4)}_{11} - I^{(4)}_{22}) \begin{pmatrix} \sin 2\alpha & \cos 2\alpha & 0 \\ \cos 2\alpha & -\sin 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&+ \frac{1}{2}(I^{(5)}_{11} - I^{(5)}_{22}) \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha & 0 \\ -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} I^{(5)}_{11} & & \\ & I^{(5)}_{22} & \\ & & 2I^{(5)}_{33} \end{pmatrix}.
\end{aligned}$$

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