Gen. Relativ. Gravit. (2006) 38(6): 1069–1082
DOI 10.1007/s10714-006-0289-y

RESEARCH ARTICLE

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Mining metrics for buried treasure

Received: 12 December 2005 / Published online: 18 March 2006

Abstract The same but different: That might describe two metrics. On the surface CLASSI may show two metrics are locally equivalent, but buried beneath may be a wealth of further structure. This was beautifully described in a paper by Malcolm MacCallum in 1998. Here I will illustrate the effect with two flat metrics – one describing ordinary Minkowski spacetime and the other describing a three-parameter family of Gal'tsov-Letelier-Tod spacetimes. I will dig out the beautiful hidden classical singularity structure of the latter (a structure first noticed by Tod in 1994) and then show how quantum considerations can illuminate the riches. I will then discuss how quantum structure can help us understand classical singularities and metric parameters in a variety of exact solutions mined from the Exact Solutions book.

Keywords Singularity · Exact solution

1 Introduction

I am very happy to be here to talk at Malcolm's 60th birthday celebration. This talk is a belated 60th birthday present. It will focus on cylindrically symmetric spacetimes [1] whose metrics contain buried treasure: essential parameters that DO NOT appear in the Cartan scalars, i.e., the scalars that feature in the Cartan equivalence method for spacetime classification (see, e.g., [2]). As the exact solutions book [1] says, "The method ...due to Cartan ...gives sets of scalars

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providing a unique *local* characterization, and thus leads to a procedure for comparing metrics." The emphasis on "local" is mine; it is the key to the possibility of buried treasure. As Malcolm says in his 1998 paper [3] "the metric may have parameters which are important globally but do not appear in the Cartan scalars" and "The parameters cannot change the values of the Cartan scalars defined by the Riemann tensor and its derivatives at a point, and this directs attention to the possible global holonomy found by taking suitable closed curves..."

The essential (buried) parameters:

- are unique to the characterization of the geometry,
- do not appear in the Cartan scalars (i.e., the spacetimes are locally equivalent),
- do appear in expressions of the (linear or affine) holonomy (i.e., the spacetimes are globally inequivalent), and
- may be due to a singular axis (requiring a relaxation of the usual definition of cylindrical symmetry – see, e.g., Mars and Senovilla [4], Carot et al [5], and The Exact Solutions Book [1]) or may be necessary to match to a regular source in the interior.

There is a famous "flat" example: the three-parameter family of Gal'tsov-Letelier-Tod (GLT) spacetimes [6, 7],

$$ds^2 = -(dt + \alpha d\phi)^2 + dr^2 + \beta^2 r^2 d\phi^2 + (dz + \gamma d\phi)^2. \tag{1}$$

The coordinate ranges are the usual: $-\infty < t < \infty$, $0 \le r < \infty$, $0 \le \phi \le 2\pi$ and $-\infty < z < \infty$. Here the constants α, β and γ are the "essential parameters". These spacetimes will be used as the key examples in the first half of this talk: a local analysis, as in CLASSI, does not distinguish them from Minkowski spacetime, although, as we shall see, there is a wealth of global structure hidden in this three-parameter family.

The plan of this talk is the following: After the Introduction (1), the classical structure of the GLT spacetimes will be described (2). The global structure of spacetimes will be reviewed (3), as this is necessary to understand classical and quantum singularities. The quantum singularity of GLT spacetimes (4) and special cases (5) will be discussed. General cylindrically symmetric static spacetimes with disclinations and dislocations will be considered (6) and followed by studies of generalized Levi-Civita spacetimes with dislocations (7), Chitre et al. spacetimes (8), the Melvin universes (9) and (10) generalized Raychaudhuri spacetimes with disclinations and dislocations (10). A final discussion will conclude the talk (11).

2 GLT spacetimes – classical aspects

Gal'tsov and Letelier in 1993 [6] and Tod in 1994 [7] completely analyzed the spacetimes described in Eq. (1). As Tod noted, this three-parameter family of spacetimes describe a multitude of physically interesting cases. If $\alpha = 0, \gamma = 0, \beta^2 \neq 1$, they describe the idealized cosmic string [12–15]. If $\alpha = 0$ and the final term is absent, they describe the "point source" in $2 + 1$ gravity [16–18]. If $\alpha \neq 0$ and the final term is absent, they describe the "rotating point source" of $2+1$ gravity [18, 19]. And, if $\alpha = 0, \gamma \neq 0$, the GLT spacetime is the asymptotic metric [20] at a large spatial separation from a cylindrically symmetric gravitational wave.

Here we will specialize our discussion to the static case ($\alpha = 0$) [8, 9] where the metric takes the form,

$$ds^2 = -dt^2 + dr^2 + \beta^2 r^2 d\phi^2 + (dz + \gamma d\phi)^2. \tag{2}$$

This metric is classically singular if $\beta^2 \neq 1$ and/or $\gamma \neq 0$; in these cases there is a quasi-regular singularity at $r = 0$. For clarity we will consider two special cases: an idealized cosmic string and a screw dislocation spacetime.

An idealized cosmic string is described if $\beta^2 \neq 1$ and $\gamma = 0$ in Eq. (2). In this case there are incomplete geodesics which hit $r = 0$; this is a quasiregular singularity, a disclination in crystallographic terminology (see, e.g., [21]). There is non-trivial linear holonomy and $r = 0$ is a δ-function in curvature [7, 22].

The second special case is the screw dislocation spacetime, where $\gamma \neq 0$ and $\beta^2 = 1$ in Eq. (2). Incomplete geodesics hit $r = 0$. A curve of bounded acceleration goes to $r = 0, z = \infty$ in finite affine length. There is again a quasiregular singularity at $r = 0$; this is called a dislocation in crystallographic terminology. There is non-trivial affine holonomy and $r = 0$ is a δ-function in torsion [7].

3 Global structure – singularities

In a maximal spacetime,

- A *classical singularity* exists if there are incomplete geodesics or incomplete paths of bounded acceleration [23–25].
- A *quantum singularity* exists if the evolution of a test wave packet is not uniquely defined by the initial wave packet, without specifying extra information not present in the wave operator, spacetime metric and manifold alone (i.e., one must add boundary conditions at the singularity) [8, 9, 26, 27].

Given the two categories of singularities, various questions arise: (1) Are all classically singular spacetimes quantum mechanically singular as well? (2) Since the topological parameters (e.g., α, β, γ in GLT spacetime) affect the existence of a classical singularity, do they affect the existence of a quantum singularity as well? The answer to the first question is negative (see Horowitz and Marolf [27]). We will consider the second question after briefly discussing classical and quantum singularities in more detail.

3.1 Classical singularities

In classical general relativity singularities are not part of the spacetime (the manifold is smooth): they are boundary points in an otherwise maximal spacetime [23]. For the timelike and null geodesics (or curves of bounded acceleration) that hit these boundary points, there is an incompleteness, an abrupt ending to the classical particle paths. The classical singularities which occur in otherwise maximal spacetimes have been divided into three types by Ellis and Schmidt [23]:

- *quasiregular* (e.g., the 2D cone, the idealized cosmic string)

- *non-scalar curvature* (e.g. whimper cosmologies)
- *scalar curvature* (e.g., the center of a Schwarzschild black hole, the classical Big Bang.)

What if quantum wave packets are used instead of classical particles to test for a singularity? A quantum singularity would have to be described by ill-posed wave propogation. We'll see next how this has been defined.

3.2 Quantum singularities

According to Horowitz and Marolf [27], a static spacetime is quantum mechanically singular if the spatial portion of the Klein-Gordon wave operator is not essentially self-adjoint [28] on a C_0^∞ domain in $\mathcal{L}^2$, a Hilbert space of square integrable functions. In this case the evolution of the test quantum wave packet is not uniquely determined by the initial wavefunction, the spacetime metric and the manifold.

An operator, A, is called self-adjoint if

(i) $A = A^\dagger$
(ii) $Dom(A) = Dom(A^\dagger)$

where $A^\dagger$ is the adjoint of A. An operator is essentially self-adjoint if (i) is met and (ii) can be met by expanding the domain of the operator or its adjoint so that it is true [28].

A relativistic scalar quantum particle with mass M can be described by the positive frequency solution to the Klein-Gordon equation

$$\frac{\partial^2 \Psi}{\partial t^2} = -A\Psi \tag{3}$$

in a static spacetime, where the spatial operator is

$$A = -VD^i(VD_i) + V^2M^2 \tag{4}$$

with $V = -\xi_\nu \xi^\nu$. Here ξ^ν is the timelike Killing field and D_i is the spatial covariant derivative on the static slice Σ. The Hilbert space is $\mathcal{L}^2(\Sigma)$, the space of square integrable functions on Σ.

If we initially define the domain of A to be $C_0^\infty(\Sigma)$, A is a real, positive, symmetric operator and self-adjoint extensions always exist [28]. If there is only a single, unique extension A_E, then A is essentially self-adjoint. In this case, the Klein-Gordon equation for a free scalar particle takes the form [27]:

$$i\frac{d\Psi}{dt} = A_E^{1/2}\Psi \tag{5}$$

with

$$\Psi(t) = \exp(-it(A_E)^{1/2})\Psi(0). \tag{6}$$

These equations are ambiguous if A is not essentially self-adjoint. This fact led Horowitz and Marolf to define quantum mechanically singular spacetimes as those in which A is not essentially self-adjoint. Examples are considered by Horowitz and Marolf [27], Kay and Studer [30], Helliwell and Konkowski [8], Helliwell et al. [9], Konkowski et al. [10], and Konkowski et al. [11].

The definition of quantum singularity as originally stated by Horowitz and Marolf [27] applies only to the Klein-Gordon scalar field wave operator; however, it is easily extended to Maxwell and Dirac fields [9]. We say that a spacetime is quantum mechanically singular with respect to a Maxwell or Dirac field if the spatial portion of any component of the field operator fails to be essentially self-adjoint. We take the Hilbert space to be $\mathcal{L}^2$ and the original domain to be C_0^∞. To test for essential self-adjointness of the spatial portion A of a component of the operator, we use the von Neumann [31] criterion. It involves setting $A^*\Psi = \pm i\Psi$ and determining the number of solutions that belong to $\mathcal{L}^2$ for each i. If the deficiency indices are $(0, 0)$, so that no solutions are square integrable, then the operator is essentially self-adjoint.

4 GLT spacetimes – quantum aspects

In this section we will consider various wave operators in GLT spacetime, determine for which modes the operators are essentially self-adjoint and show that the GLT spacetimes are generically quantum mechanically singular.

4.1 Scalar particles

The Klein-Gordon equation $\Box\Phi = M^2\Phi$ can be separated in GLT spacetime [8, 9]. Here

$$\Phi \sim e^{im\phi} e^{ikz} e^{-i\omega t} R(r). \tag{7}$$

The spatial derivative operator fails to be essentially self-adjoint for Φ modes with

$$-1 < \frac{m - \gamma k}{\beta} < 1 \tag{8}$$

where the separation constants m and k are the azmuthal quantum number and the momentum, respectively.

4.2 Null vector particles

The classical source-free Maxwell equations $A_{\mu;\nu}^{;\nu} = 0$ in the Lorentz gauge $A_{;\mu}^{\mu} = 0$ can be separated in the GLT spacetime by taking linear combinations of modes [9]. Here

$$A_\mu \sim e^{im\phi} e^{ikz} e^{-i\omega t} R_\mu(r). \tag{9}$$

The spatial derivative operator fails to be essentially self-adjoint for A^μ modes with

$$-1 < \frac{m - \gamma k}{\beta} < 1, \tag{10}$$

the same as for scalar particles.

4.3 Free spin-1/2 particles

The Dirac equation $i\gamma^{\alpha}\Psi_{;\alpha} = M\Psi$ for spin-1/2 particles can be separated in the GLT spacetime [9]. Here

$$\Psi \sim \begin{pmatrix} \sqrt{(E+M)}R_1(r) \\ i\sqrt{(E+M)}R_2(r)e^{i\phi} \end{pmatrix} e^{-iEt} e^{im\phi} e^{ikz}. \tag{11}$$

The spatial derivative operator is essentially self-adjoint for Ψ modes with

$$-\frac{3}{2} < \frac{m - \gamma k + 1/2}{\beta} < \frac{3}{2}. \tag{12}$$

4.4 Summary

It is therefore clear that no matter which type of quantum particle is used (scalar, null vector or spinor), the GLT spacetimes are generically quantum mechanically singular. This is due to the fact that specific wave modes are not usually chosen to make the spatial wave operator essentially self-adjoint and with general modes the operators are not essentially self-adjoint.

5 Special cases – quantum aspects

Here we consider special cases of GLT spacetime and test each for quantum singularity using a Klein-Gordon field.

5.1 Minkowski spacetime

GLT spacetime reduces to Minkowski spacetime if $\beta^2 = 1$ and $\gamma = 0$. Both $m = 0$ modes (with Bessel function $J_0 \sim 1$ and Neumann function $N_0 \sim \ln(r)$) are $\mathcal{L}^2$, but $r = 0$ is a regular surface within the spacetime so the N_0 mode is excluded. Therefore the spatial Klein-Gordon wave operator A is essentially self-adjoint and the spacetime is quantum mechanically nonsingular. (A well-known result presented here for completeness.)

5.2 Idealized cosmic string

GLT spacetime reduces to the idealized cosmic string spacetime if $\beta^2 \neq 1$ and $\gamma = 0$. Both $m = 0$ modes (with Bessel function $J_0 \sim 1$ and Neumann function $N_0 \sim \ln(r)$) are $\mathcal{L}^2$, but $r = 0$ is NOT a regular surface within the spacetime and N_0 mode cannot be excluded. Therefore the spatial Klein-Gordon wave operator A is not essentially self-adjoint and the spacetime is quantum mechanically singular. (For details, see [8, 9, 27, 30].)

5.3 Screw dislocation spacetime

GLT spacetime reduces to the screw dislocation spacetime if $\beta^2 = 1$ and $\gamma \neq 0$. There is a continuous infinity of double square-integrable modes for each m with $-1 < m - \gamma k < 1$. Therefore the spatial Klein-Gordon operator is not essentially self-adjoint and the spacetime is quantum mechanically singular. (For details, see [8]).

6 General cylindrically symmetric spacetimes with a disclination and a dislocation

A particularly convenient way to establish essential self-adjointness in the spatial operator of the Klein-Gordon equation is to use the concepts of limit circle and limit point behavior. The approach is as follows. The Klein-Gordon equation for the spacetimes considered in this section can be separated in the coordinates t, ρ, θ, z. Only the radial equation is non-trivial. With changes in both dependent and independent variables, the radial equation can be written as a one-dimensional Schrödinger equation:

$$H\Psi(x) = E\Psi(x) \ , \tag{13}$$

where $x \in (0, \infty)$ and the operator $H = -d^2/dx^2 + V(x)$.

Here we will use this technique to study the general cylindrically symmetric static spacetime with a disclination and a dislocation. The metric is given by

$$ds^2 = e^{-2U}[e^{2K}(d\rho^2 - dt^2) + \rho^2 B^2 d\phi^2] + e^{2U}[dz + Ad\phi]^2 \ , \tag{14}$$

where $U, K, B.A$ are functions of ρ alone. (This metric form is taken from the Exact Solutions Book, Sects. 22.1 and 22.3 with a slight change in notation [1]; if $B^2 = 1$ this metric agrees with Eq. (1.1) in Malcolm's 1998 paper [3]). Here we will further restrict B to be a positive constant. The coordinate ranges are the usual ones.

The classical singularity structure depends on U, K, B, A and can be determined using the usual tests for each particular case under consideration. The quantum singularity structure will be tested using Weyl's limit point-limit circle criterion [32] and applying applicable theorems taken from Reed and Simon [28]. The Klein-Gordon wave equation $\Box\Phi = M^2\Phi$ has mode solutions given by

$$\Phi \sim e^{-i\omega t} e^{ikz} e^{im\phi} H(\rho) \ , \tag{15}$$

where

$$H,_{\rho\rho} + \frac{1}{\rho} H,_{\rho} + \{\omega^2 - M^2 e^{-2U} e^{2K} - k^2 e^{-4U} e^{2K} - \rho^{-2} e^{2K} B^{-2}(m - kA)^2\} H = 0. \tag{16}$$

Here square integrability is judged by

$$\int d\rho \sqrt{\frac{-g_3}{g_{00}}} H^* H = \int d\rho \rho B H^* H. \tag{17}$$

If we change variables by letting $H = x^{-1/2}\psi$ and $x = \sqrt{B}\rho$, then square integrability is judged by $\int \psi^* \psi dx$ and the radial equation takes one-dimensional Schrödinger form of Eq. (13). Explicitly,

$$\psi,_{xx} + (E - V(x))\psi = 0 \,, \tag{18}$$

where $E = \omega^2/B$ and

$$V(x) = \frac{M^2}{B} e^{-2U} e^{2K} + \frac{k^2}{B} e^{-4U} e^{2K} + \frac{1}{B^2 x^2} e^{2K} (m - kA)^2 - \frac{1}{4x^2}. \tag{19}$$

We can now study the limit point-limit circle behavior and determine the essential self-adjointness of the spatial operator [1]:

Definition 1 *The potential $V(x)$ is in the limit circle case at $x = 0$ if for some E, and therefore for all E, all solutions of Eq. (18) are square integrable at zero. If $V(x)$ is not in the limit circle case, it is in the limit point case.*

A similar definition pertains to $x = \infty$. The potential $V(x)$ is in the limit circle case at $x = \infty$ if all solutions of Eq. (18) are square integrable at infinity; otherwise, $V(x)$ is in the limit point case at infinity.

There are of course two linearly independent solutions of the Schrödinger equation for given E. If $V(x)$ is in the limit circle case at zero, both solutions are $\mathcal{L}^2$ at zero, so all linear combinations are $\mathcal{L}^2$ as well. We would therefore need a boundary condition at $x = 0$ to establish a unique solution. If $V(x)$ is in the limit *point* case, the $\mathcal{L}^2$ requirement eliminates one of the solutions, leaving a unique solution without the need of establishing a boundary condition at $x = 0$. This is the whole idea of testing for quantum singularities; there is no singularity if the solution is unique, as it is in the limit point case. The critical theorem is due to Weyl [28, 32].

Theorem 1 (The Weyl limit point-limit circle criterion) *If $V(x)$ is a continuous real-valued function on $(0, \infty)$, then $H = -d^2/dx^2 + V(x)$ is essentially self-adjoint on $C_0^\infty(0, \infty)$ if and only if $V(x)$ is in the limit point case at both zero and infinity.*

The following theorem can be used to establish the limit circle-limit point behavior at infinity [28].

Theorem 2 (Theorem X.8 of Reed and Simon [28]) *If $V(x)$ is continuous and real-valued on $(0, \infty)$, then $V(x)$ is in the limit point case at infinity if there exists a* positive *differentiable function $M(x)$ such that*

(i) $V(x) \geq -M(x)$
(ii) $\int_1^\infty [M(x)]^{-1/2} dx = \infty$
(iii) $M'(x)/M^{3/2}(x)$ *is bounded near* ∞.

Then $V(x)$ is in the limit point case (complete) at ∞.

[1] This section is based on Appendix to X.1 in Reed and Simon [28].

A sufficient choice of the $M(x)$ function for our purposes is the power law function $M(x) = cx^2$ where $c > 0$. Then (ii) and (iii) are satisfied, so if $V(x) \geq -cx^2$, $V(x)$ is in the limit point case at infinity.

A theorem useful near zero is the following.

Theorem 3 (Theorem X.10 of Reed and Simon [28]) *Let $V(x)$ be continuous and positive near zero. If $V(x) \geq \frac{3}{4}x^{-2}$ near zero then $V(x)$ is in the limit point case. If for some $\epsilon > 0$, $V(x) \leq (\frac{3}{4} - \epsilon)x^{-2}$ near zero, then $V(x)$ is in the limit circle case.*

Here we can write our $V(x)$ (Eq. (19)) as

$$V(x) = V_1(x) - \frac{1}{4x^2}. \tag{20}$$

Near zero we therefore have the following results:

- If $V_1(x) < \frac{1}{4x^2}$, the theorem does not apply.
- If $V_1(x) \geq x^{-2}$, $V(x)$ is in the limit point case at 0.
- If $\frac{1}{4x^2} \leq V_1(x) \leq \frac{(1-\epsilon)}{x^2}$ for some $\epsilon > 0$, $V(x)$ is in the limit circle case at 0.

Usually, however, it is easiest just to solve the Schrödinger equation near zero and test the resulting approximate solutions for square integrability.

7 Generalized Levi-Civita spacetimes with dislocations

Here we will consider a Levi-Civita (LC) metric that has been generalized with the addition of a timelike dislocation ($\alpha \neq 0$) and a spacelike dislocation ($\gamma \neq 0$):

$$ds^2 = -r^{4\sigma}(dt + \alpha d\theta)^2 + r^{8\sigma^2 - 4\sigma}dr^2 + r^{8\sigma^2 - 4\sigma}(dz + \gamma d\theta)^2 + \frac{r^{2-4\sigma}}{C^2}d\theta^2. \tag{21}$$

Here σ and C are the usual Levi-Civita parameters and the coordinate ranges are the usual ones. The constant σ is related to the mass per unit length of the infinite line mass that the Levi-Civita metric can describe, whereas the constant $C^2 \neq 1$ represents a disclination in the spacetime. For a fuller discussion of Levi-Civita spacetimes see, for example, Bonnor [34], Konkowski, Helliwell and Wieland [10], and the papers by Herrera et al [35, 36]. The generalized Levi-Civita metric is static if $\alpha = 0$, it reduces to the ordinary Levi-Civita metric if $\alpha = 0$ and $\gamma = 0$ (see 22.7 of the Exact Solutions Book [1]), and it reduces to the GLT metric if $\alpha \neq 0, \gamma \neq 0, C^2 \neq 1$ and $\sigma = 0$.

The analysis here will be restricted to the static case:

$$ds^2 = -r^{4\sigma}dt^2 + r^{8\sigma^2 - 4\sigma}dr^2 + r^{8\sigma^2 - 4\sigma}(dz + \gamma d\theta)^2 + \frac{r^{2-4\sigma}}{C^2}d\theta^2. \tag{22}$$

The classical singularity structure depends on the parameter values:

- $\sigma \neq 0, 1/2$ – scalar curvature singularity.
- $\sigma = 0, \gamma = 0, C^2 = 1$ – Minkowski spacetime – non-singular.

- $\sigma = 0, \gamma = 0, C^2 \neq 1$ – Idealized cosmic string – quasi-regular, disclination singularity.
- $\sigma = 0, \gamma \neq 0, C^2 = 1$ – Screw dislocation spacetime – quasi-regular, dislocation singularity.
- $\sigma = 1/2$ – Minkowski spacetime in accelerated coordinates – non-singular.

What about the quantum singularity structure? That too depends on the parameter values. We will consider two distinct cases.

7.1 Generalized LC spacetimes with $\sigma = 1/2$

We will first consider the $\sigma = 1/2$ case, which is Minkowski spacetime in accelerated coordinates. The Klein-Gordon equation is separable and the radial equation can be written in Schrödinger form,

$$\psi_{,xx} + (E - V(x))\psi = 0\ , \tag{23}$$

where $E = C^2\omega^2$,

$$V(x) = C^2(k^2 + M^2 + m^2C^2)\exp(2Cx), \tag{24}$$

and $x = \frac{1}{C}\ln(r)$ with $x \in (-\infty, \infty)$. As $x \to \pm\infty$, $V(x) > -cx^2$, so the potential is limit point at $\pm\infty$. Therefore, Minkowski spacetime in accelerated coordinates is clearly and unambiguously quantum mechanically non-singular.

7.2 Generalized LC spacetimes with $\sigma \neq 1/2$

All other cases can be considered together. Again the Klein-Gordon equation is separable and the radial equation can be written in Schrödinger form,

$$\psi_{,xx} + (E - V(x))\psi = 0\ , \tag{25}$$

where $E = C\omega^2/\beta$, $\beta = (2\sigma - 1)^2$ and

$$\begin{aligned} V(x) = &(Ck^2/\beta)(\beta Cx^2)^{(-\beta+1)/\beta} + (CM^2/\beta)(\beta Cx^2)^{2\sigma/\beta} \\ &+(m - \gamma k)^2\frac{C^2}{\beta}(\beta Cx^2)^{(4\sigma-1)/\beta} - \frac{1}{4x^2}\ , \end{aligned} \tag{26}$$

with

$$x = \frac{1}{C}\frac{r^{(2\sigma-1)^2}}{2\sigma - 1} \tag{27}$$

for $x \in (0, \infty)$. As $x \to \infty$, $V(x) > -cx^2$, so the potential is limit point at infinity. As $x \to 0$, the behaviour is as follows:

- $\sigma \neq 0$ $V(x) \to -\frac{1}{4x^2}$. The asymptotic forms of the two independent solutions to the Schrödinger equation are $\psi_1 \sim x^{1/2}$ and $\psi_2 \sim x^{1/2}\ln(x)$. Both are $\mathcal{L}^2$ so the potential $V(x)$ is limit circle at zero.

- $\sigma = 0$ $V(x) \to -\frac{1/4-(m-\gamma k)^2C^2}{x^2}$. The asymptotic forms of the two independent solutions to the Schrödinger equation are $\psi_1 \sim x^{1/2+|m-\gamma k|C}$, and $\psi_2 \sim x^{1/2}\ln(x)$ if $m = \gamma k$ or $\psi_1 \sim x^{1/2-|m-\gamma k|C}$ if $m \neq \gamma k$. Therefore $V(x)$ is limit circle at zero if $|m - \gamma k|C < 1$ (except for the special case $\gamma = 0$ of Minkowski spacetime, where the irregular ψ_2 solution is discarded at zero because $x = 0$ is a regular hypersurface in the spacetime) and $V(x)$ is limit point at zero if $|m - \gamma k|C \geq 1$.

7.3 Results

The following results were obtained:

- $\sigma = 0, C^2 = 1, \gamma = 0$ Minkowski spacetime. Here $x = 0$ is a regular hypersurface in the spacetime so the ψ_2 modes are discarded and the potential is limit point. Minkowski spacetime is quantum mechanically non-singular (a well-known result repeated for completeness).
- $\sigma = 1/2$ Minkowski spacetime in accelerated coordinates. The potential $V(x)$ is limit point. Minkowski spacetime in accelerated coordinates is quantum mechanically nonsingular.
- $\sigma = 0$ ($C^2 \neq 1$ and/or $\gamma \neq 0$) and $\sigma \neq 0$ The potential $V(x)$ is limit circle. These generalized LC spacetimes are quantum mechanically singular.

These agree when $\gamma = 0$ with the results obtained by Konkowski et al. [10] for ordinary LC spacetimes.

8 Chitre et al. spacetimes

Under consideration next are a family of spacetimes discovered by Chitre et al. [37]. Their metric is

$$ds^2 = \rho^{-4/9}\exp(a^2\rho^{2/3})(d\rho^2 - dt^2) + \rho^{4/3}d\phi^2 + \rho^{2/3}(dz + a\rho^{2/3}d\phi)^2. \quad (28)$$

They are described in the Exact Solutions Book, Sect. 22.12 [1]. Here a is a constant. The coordinate ranges are the usual ones.

The Chitre et al. spacetimes are classically singular with a scalar curvature singularity at $\rho = 0$. Are they quantum mechanically singular? The Klein-Gordon equation is separable and the radial wave equation can be written in one-dimensional Schrödinger form,

$$\psi_{,xx} + (E - V(x))\psi = 0\,, \quad (29)$$

with $\rho = x$, $E = \omega^2$ and

$$V(x) = M^2x^{-4/9}e^{a^2x^{2/3}} + k^2x^{-10/9}e^{a^2x^{2/3}} + x^{-16/9}e^{a^2x^{2/3}}(m - kax^{2/3})^2 - \frac{1}{4x^2}. \quad (30)$$

As $x \to \infty$, $V(x) > -cx^2$, so the potential is limit point at infinity. As $x \to 0$, $V(x) \to -\frac{1}{4x^2}$. The asymptotic forms of the two independent solutions to the Schrödinger equation are $\psi_1 \sim x^{1/2}$ and $\psi_2 \sim x^{1/2}\ln(x)$. Both are $\mathcal{L}^2$ so the potential $V(x)$ is limit circle at zero. Thus, for all a values, the Chitre et al spacetimes are quantum mechanically singular.

9 Melvin universes

Next we look at Melvin spacetimes [38] which are given in 22.13 of the Exact Solutions Book [1]. This is a one-parameter family of spacetimes with metric,

$$ds^2 = -\alpha^2(1+R^2)^2(dt^2 - dR^2) + \frac{\alpha^2 R^2}{(1+R^2)^2}d\theta^2 + (1+R^2)^2 dz^2, \tag{31}$$

where α is a constant and the coordinate ranges are the usual ones.

The Melvin spacetimes are classically non-singular for all values of α. They are also quantum mechanically non-singular. This is easily seen by writing the radial portion of the Klein-Gordon equation in Schrödinger form,

$$\psi_{,xx} + (E - V(x))\psi = 0. \tag{32}$$

Here $E = \omega^2$, $R = x$, and

$$V(x) = \alpha^2 k^2 + M^2 + \frac{(m^2 - 1/4)}{x^2}. \tag{33}$$

As $x \to \infty$, $V(x) > -cx^2$, so the potential is limit point at infinity. As $x \to 0$, $V(x) \to \frac{(m^2-1/4)}{x^2}$. The asymptotic forms of the two independent solutions to the Schrödinger equation are $\psi_1 \sim x^{1/2+|m|}$, and $\psi_2 \sim x^{1/2}\ln(x)$ if $m = 0$ and $\psi_2 \sim x^{1/2-|m|}$ if $m \neq 0$. The $\psi_2(m = 0)$ solution is $\mathcal{L}^2$ but it is not allowed as $x = 0$ is a regular hypersurface of the spacetime. The potential is thus also limit point at zero and the Melvin universes are quantum mechanically non-singular for all parameter values.

10 Generalized Raychaudhuri spacetimes with a disclination and dislocations

The last spacetimes under consideration are generalized Raychaudhuri spacetimes with a disclination and two dislocations. These are generalizations of the ordinary Raychaudhuri spacetimes [39] described in 22.16 of the Exact Solutions book [1]. Their metric is

$$\begin{aligned} ds^2 = &-a^2(\ln(b\rho))^2(dt + \alpha d\phi)^2 + a^2(\ln(b\rho))^2 d\rho^2 + a^2B^2\rho^2(\ln(b\rho))^2 d\phi^2 \\ &+ a^{-2}(\ln(b\rho))^{-2}(dz + Ad\phi)^2 \ , \end{aligned} \tag{34}$$

where a, b, α, A, and B are constants and the coordinate ranges are the usual ones. If α is not equal to zero, there is a timelike dislocation; if A is not equal to zero, there is a spacelike dislocation; and if $B^2 \neq 1$, there is a disclination. If $\alpha = 0$ the spacetimes are static. If $\alpha = 0$, $A = 0$, $B^2 = 1$ then the ordinary two-parameter Raychaudhuri spacetimes are recovered with a and b as the only parameters.

These generalized Raychaudhuri spacetimes all have a scalar curvature singularity at $\rho = 0$. What about quantum singularities? Here we will consider only the static case,

$$ds^2 = -a^2(\ln(b\rho))^2 dt^2 + a^2(\ln(b\rho))^2 d\rho^2 + a^2 B^2 \rho^2 (\ln(b\rho))^2 d\phi^2 \\ +a^{-2}(\ln(b\rho))^{-2}(dz + A d\phi)^2 \tag{35}$$

which has a disclination and spacelike dislocation. For simplicity, assume B is positive in the following analysis. The Klein-Gordon equation is separable in the metric coordinates and the radial equation can be put into Schrödinger form,

$$\psi_{,xx} + (E - V(x))\psi = 0\ , \tag{36}$$

where $\rho = x$, $E = \omega^2/B^2$, and

$$V(x) = \frac{M^2 a^2}{B}(\ln(bx))^2 + \frac{k^2 a^4}{B}(\ln(bx))^4 \\ + \frac{1}{B^2 x^2}(m - kA)^2 - \frac{1}{4x^2}. \tag{37}$$

As $x \to \infty$, $V(x) > -cx^2$, so the potential is limit point at infinity. As $x \to 0$, $V(x) \sim [-\frac{1}{4} + \frac{(m-kA)^2}{B^2}]x^{-2}$ and the two independent asymptotic solutions to the Schrödinger equation are $\psi_1 \sim x^{1/2+|m-ka|/B}$ and $\psi_2 \sim x^{1/2}\ln(x)$ if $m = kA$ or $\psi_2 \sim x^{1/2-|m-ka|/B}$ if $m \neq kA$. Therefore, $V(x)$ is limit circle at zero if $\frac{|m-kA|}{B} < 1$ and $V(x)$ is limit point at zero if $\frac{|m-kA|}{B} \geq 1$. The static generalized Raychaudhuri spacetimes are thus quantum mechanically singular for Klein-Gordon modes $-1 < \frac{m-kA}{B} < 1$. If generic modes are allowed, all static generalized Raychaudhuri spacetimes are generically quantum mechanically singular.

11 Conclusions

The essential (buried) parameters in the spacetimes considered are not evident in a local analysis of the metrics as is done by CLASSI. They are evident, however, in a global analysis as one finds when examining the spacetimes for classical and quantum singularities. In such analyses there is a wealth of information that can be mined from the metric structure.

I end with a quote that seems apropos to the buried treasure of globally essential parameters. It is from Lewis Carroll's *Through the Looking Glass*:

> "I see nobody on the road," says Alice. "I only wish I had such eyes," the King remarked in a fretful tone. "To be able to see nobody. And at that distance, too! Why, it's as much as I can do to see real people, by this light."

Acknowledgements One of us (DAK) was partially funded by NSF grants PHY-9988607 and PHY- 0241384 to the U.S. Naval Academy. She also thanks Queen Mary, University of London, where some of this work was carried out.

References

1. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C., Herlt, E.: Exact Solutions of Einstein's Field Equations, vol. 2. Cambridge: Cambridge University Press (2003)
2. MacCallum, M.A.H., Skea, J.E.F.: SHEEP: a computer algebra system for general relativity. In: Reboucas, M.J., Roque, W.L. (eds.), Algebraic Computing in General Relativity. Proceedings of the first Brazilian School on Computer Algebra vol. 2. Oxford, Oxford University Press p. 1 (1994)
3. MacCallum, M.A.H.: Gen. Relativ. Grav. **30**, 131 (1998)
4. Mars, M., Senovilla, J.M.M.: Class. Quant. Grav. **10**, 1633 (1993)
5. Carot, J., Senovilla, J.M.M., Vera, R.: Class. Quant. Grav. **16**, 3025 (1999)
6. Gal'tsov, D.V., Letelier, P.S.: Phys. Rev. D **47**, 4273 (1993)
7. Tod, K.P.: Class. Quantum Grav. **11**, 1331 (1994)
8. Konkowski, D.A., Helliwell, T.M.: Gen. Relativ. Grav. **33**, 1131 (2001)
9. Helliwell, T.M., Konkowski, D.A., Arndt, V.: Gen. Relativ. Grav. **35**, 79 (2003)
10. Konkowski, D.A., Helliwell, T.M., Wieland, C.: Class. Quant. Grav. **21**, 265 (2004)
11. Konkowski, D.A., Reese, C., Helliwell, T.M., Wieland, C.: Classical and Quantum Singularities of Levi-Civita Spacetimes with and without a Cosmological Constant. To appear in Proceedings of the Workshop on the Dynamics and Thermodynamics of Black Holes and Naked Singularities (2004) [gr-qc/0410114]
12. Ford, L.H., Vilenkin, A.: J. Phys. A: Math. Gen. **14**, 2353 (1981)
13. Gott, J.R.: Astrophys. J. **288**, 422 (1985)
14. Hiscock, W.A.: Phys. Rev. D **31**, 3288 (1985)
15. Linet, B.: Gen. Relativ. Grav. **17**, 1109 (1985)
16. Clement, G.: Gen. Relativ. Grav. **16**, 477 (1984)
17. Deser, S., Jackiw, R., t'Hooft, G.: Ann. Phys., N.Y. **152**, 220 (1984)
18. Brown, J.D.: Lower Dimensional Gravity Singapore, World Scientific (1988)
19. Clement, G.: Int. J. Theo. Phys. **24**, 267 (1985)
20. Berger, B.K., Chrusciel, P.T., Moncrief, V.: Ann. Phys., N. Y. **237**, 322 (1995)
21. Puntigam, R.A., Soleng, H.H.: Class. Quant. Grav. **14**, 1129 (1997)
22. Garfinkle, D.: Class. Quant. Grav. **16**, 4101 (1999)
23. Ellis, G.F.R., Schmidt, B.G.: Gen. Relativ. Grav **8**, 915 (1977)
24. Tipler, F., Clarke, C.J.S., Ellis, G.: Singularities and horizons – a review article. In: Held, A. (ed.), General Relativity and Gravitation. New York, Plenum Press (1980)
25. Hawking, S.W. Ellis, G.F.R.: The Large-Scale Structure of Spacetime. Cambridge, Cambrdge University Press (1973)
26. Wald, R.M.: J. Math Phys. **21**, 2802 (1980)
27. Horowitz, G.T., Marolf, D.: Phys. Rev. D **52**, 5670 (1995)
28. Reed, M., Simon, B.: Functional Analysis. New York, Academic Press; Reed, M., Simon, B. 1972 Fourier Analysis and Self-Adjointness. New York, Academic Press (1972)
29. Richmyer, R.-D.: Principles of Advanced Mathematical Physics, vol. I. New York, Springer (1978)
30. Kay, B.S., Studer, U.M.: Commun. Math. Phys. **139**, 103 (1991)
31. von Neumann, J.: Math. Ann. **102**, 49 (1929)
32. Weyl, H.: Math. Ann. **68**, 220 (1910)
33. Levi-Civita, T.: Rend. Acc. Lincei **28**, 101 (1919)
34. Bonnor, W.B.: The static cylinder in general relativity. In: Harvey, A. (ed.), On Einstein's Path New York, Springer p. 113 (1999)
35. Herrera, L., Ruifernández, J., Santos, N.O.: Gen. Relativ. Grav. **33**, 515 (2001)
36. Herrera, L., Santos, N.O., Teixeira, A.F.F., Wang, A.Z.: Class. Quantum Grav. **18**, 3847 (2001)
37. Chitre, D.M., Guven, R., Nutku, Y.: J. Math. Phys. **16**, 475 (1975)
38. Melvin, M.A.: Phys. Lett. **8**, 65 (1964)
39. Raychaudhuri, A.K.: Ann. Phys. **11**, 501 (1960)
40. Carroll, L.: Through the Looking Glass and What Alice Found There New York, Morrow (1933)