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RESEARCH ARTICLE

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The Bianchi models: Then and now

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1 The Bianchi universes

The Bianchi universe models are spatially homogeneous cosmological models that in general are anisotropic. Such cosmologies provide interesting generalisations of the standard Friedmann-Lemaître models of cosmology (which are based on the spatially homogeneous and isotropic Robertson-Walker geometries, with spatial sections of constant curvature). The Bianchi models are defined to be the family of cosmologies in which there is a 3-dimensional group of isometries G_3 acting on spacelike 3-surfaces; making these surfaces of homogeneity in space-time (all physical quantities are necessarily constant on them). They are characterised in terms of their specific 3-dimensional symmetry groups, originally classified by L Bianchi [2, 3]. To be of cosmological interest, the space-times must be non-empty ($T_{\alpha\beta} \neq 0$) with a preferred 4-velocity field u^a determining the world lines of fundamental observers [17]. The simply transitive group of isometries G_3 is generated by Killing vectors $\xi_\nu (\nu = 1, 2, 3)$ with structure constants $C^\gamma_{\alpha\beta}$, defined by

$$[\xi_\alpha, \xi_\beta] = C^\gamma_{\alpha\beta}\, \xi_\gamma,\ C^\gamma_{\alpha\beta} = C^\gamma_{[\alpha\beta]} \tag{1}$$

The group G_3 might be a subgroup of a larger multiply transitive symmetry group G_4 or G_6, in which case in general there will be several different simply transitive subgroups G_3. The focus in Bianchi models is on the existence of simply transitive groups because in that case there are simple representations of the

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universe models based on the existence of basis vector fields $\mathbf{e}_a$ ($a = 0, 1, 2, 3$) pagebreak invariant under all transformations of this group:

$$[\mathbf{e}_a, \xi_\kappa] = 0. \tag{2}$$

Such vector fields do not exist in the multiply transitive case ($G_r, r > 3$). The only spatially homogeneous cosmological models that are not Bianchi universes are the Kantowski-Sachs locally rotationally symmetric family with $K = +1$ (these are invariant under a group of symmetries G_4 with no simply transitive subgroup [16]).

1.1 Historical notes

Already in 1921 Kasner [32] looked at the Bianchi (Abelian) spatially homogeneous anisotropic models in the vacuum case and in 1933 Lemaître [34] examined the matter case, but neither did so from a group theory viewpoint – these are the simplest anisotropic generalisations of the Robertson-Walker models, with obvious metric

$$ds^2 = -dt^2 + X^2(t)dx^2 + Y^2(t)dy^2 + Z^2(t)dz^2, \;\; u^a = \delta^a_0, \tag{3}$$

One does not need group theory for their derivation. Study of their dynamics was later picked up by E. Schücking, B. B. Robinson, A. K. Raychaudhuri, K. S. Thorne, and others, but usually without specific group theory characterisation.

Robertson's article on cosmology [48] refers to the paper by Bianchi [3] which discusses homogeneous 3-dimensional Riemannian manifolds from a group theory viewpoint, and so lays the foundation for the study of Bianchi models. The key point here is that the structure constants $C^{\gamma}_{\alpha\beta}$ have to satisfy the integrability conditions

$$C^{\alpha}{}_{[\beta\gamma}C^{\delta}{}_{\varepsilon]\alpha} = 0 \tag{4}$$

established by Lie [35–37] (actually the Jacobi identities for the Killing vector fields). The complication is that one can choose a new basis ξ'_ν by making an arbitrary constant linear transformation of the original basis vectors, hence changing the form of the structure constants by a tensor transformation; so the same Lie algebra can be represented in many different ways. Bianchi gave a complete solution to these integrability equations in the case of 3-dimensional Lie algebras, determining nine different group types and giving canonical forms for the structure constants in each case.

In 1935–1936, Robertson [49] and Walker [64] in discussing the Robertson-Walker models gave generic Lie group prescriptions for obtaining these solutions, in effect treating them as Bianchi geometries, but apparently never used specific Lie algebra types in their calculations. In particular they did not identify the group types that applied to these models, nor explain how the simply transitive and multiply transitive subgroups were related to each other. Their actual derivation of the metrics rather relied directly on their spatial homogeneity and isotropy, as in [48].

The first systematic use of Lie group theory to study Bianchi geometries was by Kurt Gödel, who developed the study of the geometry and dynamics of Bianchi IX cosmological models in 1952 [23] as a follow up to his stationary space-time homogeneous model (the Gödel universe of 1949 [22]). Thus he was the first to introduce these anisotropic models explicitly characterised by their group symmetries, but did so in an elusive and enigmatic way, to a considerable degree relying on the special properties of Bianchi IX symmetries; i.e. the SO(3) symmetry group. According to Schücking (who visited him in Princeton) he carried out many calculations for more general group types, but never published them. Then Taub [56] derived the dynamic equations for the generic vacuum Bianchi geometries, explaining the techniques needed to develop these equations in a non-holonomic and non-orthogonal basis, thus giving the first easily available systematic exposition of methods usable for all the spatially homogeneous Bianchi models. The 3-dimensional Lie algebras were classified in terms of the reduction to the canonical forms for their structure constants determined by Bianchi [2]. Somewhat later Schücking developed a similar version suitable for cosmology (he derived the equations with a fluid source term) and published a brief note on it in the 1958 Solvay conference proceedings [26], and a detailed account in the Appendix to the Heckmann and Schücking article in the Witten volume (1962) [27]. He emphasized the role of the automorphism group of each Lie algebra, i.e. the homogeneous linear transformations which leave the structure constants fixed, in conjunction with use of a non-orthogonal basis in simplifying the equations. This was taken up much later by Jantzen and others. The Heckmann and Schücking article however did not give the specific structure constant decomposition that is used today, even though it was known to Schücking at the time. At about this time, Petrov also carried out an independent systematic examination of space-times invariant under Lie groups, resulting in a major book [47], but he did not focus specifically on cosmological models.

The present streamlined understanding of the relationship between the Lie algebra structure constants and the Bianchi group types, needed to study systematically the dynamics of spatially homogeneous universe models, was developed by Engelbert Schücking in Hamburg in the years 1956 to 1957, and became general knowledge through the papers of Estabrook, Wahlquist, and Behr [21] on the one hand and of Ellis and MacCallum [19] on the other. The key element is the representation of the structure constants developed by Schücking and given in W Kundt's notes on Schücking's Hamburg seminars, 'obvious' once one has raised the lower two indices on the structure constants by use of the antisymmetric 3-symbol to give a quantity with two upstairs indices. That is, one can define $N^{\alpha\delta}$ and A_γ by

$$C^{\alpha}_{\beta\gamma}\eta^{\delta\beta\gamma} = C^{(\alpha\delta)} + C^{[\alpha\delta]} = N^{\alpha\delta} + \epsilon^{\alpha\delta\gamma}A_\gamma,\ \ A_c = C^{c}_{ac}. \tag{5}$$

1.2 The classification

The importance of this representation arises from the fact that this made possible the immediate solution of the Lie identities (4), which become:

$$N^{ab}A_b = 0 \tag{6}$$

and consequently gave a simple classification of the Lie Algebras on using the obvious diagonal bases for the symmetric tensor N^{ab}. It is as follows: One can always diagonalise N_{ab}, choosing A_a in the 1-direction when it is non-zero. Then we get the types:

$$\begin{bmatrix} Class & Type & N_1 & N_2 & N_3 & A & \\ A & I & 0 & 0 & 0 & 0 & Abelian \\ & II & 1 & 0 & 0 & 0 & \\ & VI_0 & 0 & 1 & -1 & 0 & \\ & VII_0 & 0 & 1 & 1 & 0 & \\ & VIII & -1 & 1 & 1 & 0 & \\ & IX & 1 & 1 & 1 & 0 & \\ B & V & 0 & 0 & 0 & 1 & \\ & IV & 0 & 0 & 1 & 1 & \\ & VI_h & 0 & 1 & -1 & \sqrt{-h} & h < 0 \\ & III & 0 & 1 & -1 & 1 & same\ as\ VI_{-1} \\ & VIII_h & 0 & 1 & 1 & \sqrt{h} & h > 0 \end{bmatrix}$$

The Ellis-MacCallum paper [19] gave this classification, introduced the Class A/Class B notation that is now in common use [Class A: $A_a = 0$, Class B: $A_a \neq 0$], determined the field equations using as variables the rotation coefficients for an invariant orthonormal basis e_a, showed the exceptional status of the Bianchi VI_h models with $h = -1/9$, and determined all the multiply transitive cases. Further papers by MacCallum with others developed observational properties of these models [43], gave a potential formalism (generalising Misner's work) that gave a good guide to the dynamics of the various classes of models [39], and discussed the problems with variational principles for the class B models [44].

2 Geometry

2.1 Variables

Given a simply transitive group of isometries G_3 on a spacelike family of hypersurfaces, we choose a tetrad e_a that obeys Eq. (2) by transporting the tetrad from some initial point in each surface to all other points in the surface by use of the group of isometries G_3. This gives a basis at each point with commutator coefficients $\gamma^c_{ab}(t)$,defined by

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c_{ab}\mathbf{e}_c, \tag{7}$$

depending only on time. It follows from the symmetries that the metric tensor components are also only functions of time; that is we have a simply transitive group of isometries acting on a spacelike G_3, if and only if there is a basis $\mathbf{e}_a$ such that

$$\gamma^a_{bc} = \gamma^a_{bc}(t), \quad g_{ab} = g_{ab}(t). \tag{8}$$

If the tetrad $\mathbf{e}_a$ has components ${e_a}^i(x^\mu, t)$: that is,

$$\mathbf{e}_a = {e_a}^i(\partial/\partial x^i), \tag{9}$$

then the metric has the form:

$$ds^2 = -dt^2 + \gamma_{ab}(t)\big(e^a_i dx^i\big)\big(e^b_j dx^j\big) \tag{10}$$

where the $e^a{}_i = e^a{}_i(x^\nu, t)$ are inverse to $e_a{}^i(x^\mu, t)$: that is, $e^a{}_i e_b{}^i = \delta^a_b,\ e^a{}_i e_a{}^j = \delta^j_i$. A key feature is that from (2) it follows [15, 19] that on each surface of constant time, the quantities γ^a_{bc} are the same as the structure constants C^a_{bc} of the symmetry group G_3, up to a (generally time-dependent) linear transformation.

We can shift the time dependence between $\gamma^a_{bc}(t)$ and $g_{ab}(t)$ by changing the time-dependence of the basis $\mathbf{e}_a$. One choice is to put all the time dependence in the metric components $\gamma_{ab}(t)$, with the commutator coefficients γ^a_{bc} constants; then by appropriate choice of bases they can indeed be made identical to the canonical structure constants of the group: $\gamma^a_{bc} = C^a_{bc}$. Alternatively, one can use an orthonormal tetrad $\mathbf{e}_a$, so that $g_{ab} = \eta_{ab} = diag(-1, 1, 1, 1)$, putting all the time variation in the commutators $\gamma^a_{bc}(t)$ of the basis vectors. These quantities (which are then linear combinations of the rotation coefficients of the basis vectors) then become the essential geometrical variables of the theory. Defining $\gamma^\alpha_{\beta\gamma}\eta^{\delta\beta\gamma} = n^{\alpha\delta} + \epsilon^{\alpha\delta\gamma}a_\gamma$, the Bianchi type will follow from the canonical form for these quantities, which must satisfy the identities

$$n^{\alpha\beta}a_\beta = 0 \tag{11}$$

One can diagonalise $n^{\alpha\beta}$ in both classes A and B, choosing a_α in the 1-direction when it is non-zero. Then one gets the Bianchi types as above, but no longer with the diagonal components normalised to ± 1. A key distinction is between *Orthogonal* and *Tilted* models, characterised by whether the matter in the solution moves orthogonal to the spatially homogeneous surfaces or not. Rotating models must be tilted, and are much more complex than non-rotating models.

2.2 Higher symmetries

As mentioned above, the simply transitive group may be a subgroup of a multiply transitive group. Our paper [19] gave a complete classification of the relation of the Bianchi models to the multiply transitive spatially homogeneous models The isotropic models (all spatial directions are equivalent) are the Friedmann-Lemaître-Robertson-Walker (FLRW) models as follows:

$$\begin{bmatrix} FLRW\ \ k=+1 & Bianchi\ IX & two\ commuting\ groups \\ FLRW\ \ k=0 & Bianchi\ I & Bianchi\ VII_0 \\ FLRW\ \ k=-1 & Bianchi\ V & Bianchi\ VII_h \end{bmatrix}$$

In particular, this shows (see also [24]) that the $k = -1$ FLRW models correspond to Bianchi VII_h (and not Bianchi VIII as many had believed till that time). The Locally Rotationally Symmetric (anisotropic) models, with a single preferred spatial direction, are

$$\begin{bmatrix} \text{Orthogonal} & c = 0 & c \neq 0 \\ Taub-NUT\ I & [none] & Bianchi\ IX \\ Taub-NUT\ 3 & Bianchi\ I,\ VII_0 & Bianchi\ II \\ Taub-NUT\ 2 & Bianchi\ III\ [KS\ -1] & Bianchi\ VII_h,\ III \\ \text{Tilted} & & \\ & Bianchi\ V, VII_h & \end{bmatrix}$$

The parameter $c = 0$ if and only if the preferred spatial direction is hypersurface-orthogonal. The Kantowski-Sachs universe with $K = +1$ is not a Bianchi model, as it has no simply transitive subgroup of isometries.

3 Dynamics

To study dynamics of these models, one must choose a suitable equation of state for the matter, usually taken to be either a perfect fluid with suitable equations of state, or a scalar field with specified potential. The fluid flow velocity, or normal vector to the surfaces of constant scalar field, defines a unique 4-velocity field in the space-time enabling a unique 1+3 splitting of variables and equations. One can also use kinetic theory models of matter and include electromagnetic fields. As mentioned above, in the fluid case a key distinction is between orthogonal models and tilted models [33]. Both fluid vorticity and acceleration can occur only in tilted models.

To derive the homogeneous models from a simplified variational problem, the idea is to integrate over a right- or left-invariant volume of the homogeneous submanifolds and vary the Lagrangian without disturbing the symmetry. One should then get a one-dimensional variational problem. However this works only for vanishing vector of the structure constants (that is, class A). It does not work for class B because the surface term does not vanish in this case, see [44].

The Energy-Momentum conservation equations are integrability conditions for the full field equations. These split into four constraint equations and six evolution equations. It is crucial that one check consistency of all the equations once any kinematic restrictions are placed on the matter. For example, it turns out that for both the cases of pressure-free matter and radiation, expanding and rotating solutions must have non-zero shear [23]. Furthermore the exceptional orthogonal case of type VI_h, $h = -1/9$ is found by from such a consistency check [19]. Having checked the consistency of the equations, there are various ways of understanding the resulting dynamics.

3.1 Exact and approximate solutions

One can in some cases obtain exact solutions for simple matter types; in particular this has been done for Bianchi I models (see e.g. [20]) and some Locally Rotationally Symmetric models [16]. A comprehensive current summary is given in [55], Chapter 14. As well as perfect fluids, the effects of magnetic fields and neutrinos (described by a kinetic theory model) have been considered. One can also analyse the equations for asymptotic behaviour near the initial singularity and at late times. Particularly interesting is the study by Misner [45] of isotropisation of the universe, if it starts off anisotropically, due to the effects of neutrino viscosity.

3.2 Automorphism group

In obtaining such solutions the use of the automorphism group of the Lie algebra can result in considerably simplified equations, in effect by separating out the true dynamical degrees of freedom from frame degrees of freedom. This method was mentioned by Schücking [27] and utilised for example by Collins and Hawking in their classic studies [11, 12] of late time behaviours, showing that late time isotropy is a special case in Bianchi models . The method was developed systematically by Jantzen [30, 31] for all Bianchi types, see also [54] giving a powerful way to search for power law solutions. The drawback of the method is that its application is very particular to each group type, see e.g. [50], so that relations between the behaviours of related group types is obscured. Furthermore the translation from geometrically obvious variables to the automorphism group based variables can be complex, so that the physical meaning of the solution obtained finally in terms of those new variables [51, 52] is rather obscure.

3.3 Potential formalism

Misner introduced a Hamiltonian based potential formalism for type IX models [46] that throws light on their dynamics by allowing visualisation in terms of motion of a particle in a triangular potential well, bouncing between the potential walls. Belinskii et al. [1] used as similar formalism to study the motion of these universes and to show that they have chaotic-like behaviour at very early times, characterised by Mixmaster as a Mixmaster Universe. MacCallum developed this potential formalism systematically for all group types [39–41], see also [53].

3.4 Nature of singularities

These studies showed that quite new possibilities occur for the singularities in anisotropic universes: cigar and pancake singularities (which occur in Bianchi I models [57]) and oscillatory singularities (which occur in Bianchi IX [46]). The return maps developed by Belinskii et al for the Bianchi IX models [1] may represent true chaos in the mathematical sense as one approaches the singularity; and ongoing discussion has debated whether this is indeed the case, and whether these singularities are generic. It seems this may indeed be the case [59]. However there can be non-scalar (“whimper”) singularities when the nature of the group action changes at a horizon where the surfaces of homogeneity change from spacelike to timelike [10, 33] (and analytic continuation is needed to extend the solution across the horizon). Thus the isotropic singularities of the FLRW models are seen as a very special case, emphasizing the geometric speciality of these models: indeed the universe could be very anisotropic at early times and still appear very anisotropic at later times. An important paper by Wald [63] showed that anisotropies necessarily die away as the universe expands from the initial singularity if there is a positive cosmological constant present, which is effectively the case in an inflationary era in the early universe. However inflation only occurs in Bianchi models if there is not too much anisotropy to begin with, so that the scalar field can in fact dominate the early dynamics. Thus not all Bianchi models have an inflationary early era, even if effective scalar fields are present in the early universe.

3.5 Horizons and causality

In the special Bianchi I solutions with pancake initial singularities, the particle horizon is broken in the preferred orthogonal direction: there are no causal limits on communication in that direction [20, 57]. Misner proposed that in Bianchi IX models, the chaotic-like initial behaviour, with successive Bianchi I epochs occurring but with different axes each time, the horizon might successively be broken in all three spatial directions, thus solving the horizon problem in the early universe [46]. However it eventually turned out that this would not work.

3.6 Late time behaviour

As already mentioned, Collins and Hawking [11, 12, 25] showed isotropy is unstable at late times in some Bianchi models with a usual fluid matter source. A later series of studies regarding isotropisation are discussed in [61], Sect. 15.2.4. However Wald's Theorem [63], mentioned above, shows that if a cosmological constant is present at late times, it isotropises the universe at late times, so that the Collins and Hawking anisotropic modes are damped out. Current observations imply there is indeed an effective cosmological constant dominating the universe's dynamics at present. Whether the Wald or Collins and Hawking result applies at much later times depends on whether this effective cosmological constant ('quintessence') remains positive or dies away in the far future. We can only tell which is the case when we identify its physical nature, which is at present unknown.

3.7 Dynamical systems

Collins [9, 13] and Bogoyavlensky [6] used dynamical systems methods, introducing phase planes with compactified boundaries, to characterise the evolution of particular Bianchi classes of universe models. The Ellis-MacCallum orthonormal formalism for Bianchi models became much more powerful through introduction of an expansion-normalised version of the variables by John Wainwright, providing the basis for systematic use of dynamical-systems methods to illuminate the dynamics of all the orthogonal Bianchi models, relating the behaviour of different types to each other in an illuminating way (see [61] for an exposition and [62] for a current survey). The resulting phase space enable one to determine generic behaviour, find attractors and vacuum asymptotes, and to determine exact power-law self-similar solutions that are fixed points in the phase space [29].

These studies have been used to examine the generic nature of the initial singularities, suggesting that BLK were right about an oscillatory behaviour in general in the very early universe [59]. They also revealed the phenomenon of intermediate isotropisation, discussed below, as well as confirming that in the absence of a cosmological constant, late-time isotropy is the exception; generic models eventually become anisotropic.

One should note here that most of these studies relate to orthogonal models, hence they do not include rotating universes. There is still useful work to be done investigating the tilted cases comprehensively.

4 Observations

Cosmological models only attain relevance by predicting testable observational relations. These have been studied extensively in the case of Bianchi models.

4.1 Galaxy observations

Firstly, there will be anisotropies in the standard cosmological observational relations for galaxies and other discrete sources. In the Bianchi I case, one can determine the redshift and area distances for arbitrary directions [58], and hence the usual observational relations: (M, z), (N, z), for example, for arbitrary directions. In more general Class A cosmologies one can analytically find these relations down the principle axes [43], and also show that all observations in the universe will be invariant under discrete symmetries of a discrete group of isotropies (depending on the group type). Since the work of Godel [23] it has been known that an effect of tilt is number count anisotropies – the universe looks inhomogeneous, despite its spatial homogeneity.

This work is theoretically interesting but of little practical consequence because the observed universe seems to be so isotropic on large scales. All one obtains from this work are upper limits on the anisotropy parameters (the shear, vorticity, and Weyl tensor components).

4.2 Element formation

Because anisotropies in the early universe will affect the expansion rate, nucleosynthesis will be different in Bianchi models than FLRW models. Thorne investigated this effect in Bianchi I models [57], and since then for example helium abundances have been calculated in many orthogonal Bianchi models. Comparison with element abundance observations give strong limits on anisotropy of the models at the time of nucleosynthesis, and hence (dynamically projecting forward form this data) at the present day. The resulting limits on σ_0/H_0 are of the order of 10^{-9} to 10^{-13}, showing how isotropic the universe must have been at that time. A summary with many references is given in [61], Sect. 3.3.3. However this does not imply it was so isotropic at earlier times; decaying modes that were small then could have been much larger earlier.

4.3 CBR anisotropy

Cosmic background radiation anisotropies will result from Bianchi universe anisotropies, depending on the model type. Some models give localised hot spots in the CBR temperature and associated spiral patterns around these hot spots; others give simple anisotropy patterns, in the Class A case often invariant under discrete isotropies. Comparing these theoretical predictions with observations puts quite strong limits on the anisotropy of the universe at the time of decoupling, and hence at the present day; however these limits are quite model dependent. The resulting limits on σ_0/H_0 and ω_0/H_0 are of the order of 10^{-3} to 10^{-5}. A summary with many references is given in [61], Sect. 3.2.4. This is quite a bit weaker than the limits from nucleosynthesis, essentially because those observations probe much earlier times.

The CBR spectrum can also give limits on early universe anisotropy in those cases where late reheating takes place, and hence scattering of radiation that mixes the spectrum received from different directions, hence limits on these homogeneous modes from the CBR spectrum.

4.4 Overall

We get strong limits on anisotropic modes from these observations, typically $[\sigma/H]_0 < 10^{-9}$, $[\omega/H]_0 < 10^{-6}$. Nevertheless anisotropic spatially homogeneous modes can still dominate the universe at early times (before nucleosynthesis) and at late times (long after the present). They remain possibilities in the real universe we see around us.

5 Application?

Are these models significant, given that the observable regime of the universe seems to be very close to FLRW. Why do we need to study anisotropic universe models?

5.1 Non-linear dynamics

First, they give us an unparalleled study of the exact dynamics of the Einstein Field Equations in a spatially homogeneous context. Particularly through the dynamical system studies, one can see the relation between families of solutions with generic and special behaviour. In particular, one sees how the higher symmetry solutions, and particularly self-similar models, provide a skeleton guiding the evolution of the lower symmetry models, and one sees basins of attraction, sources, sinks, saddle points, etc. It is through looking at these generic behaviours that we begin to fully understand the nature of the high symmetry models, particularly the FLRW models, which in most cases are saddle points in the phase space for solutions with less symmetry. The overall point is that we can't understand isotropic behaviour fully unless we understand the alternatives as well.

5.2 Early and late universe

Second, while the universe is apparently very close to being isotropic and spatially homogeneous in the observable past, it could be very different in the very early or very late universe [38]. Indeed if the cosmological constant is zero, there are anisotropic modes which will eventually dominate at late times if non-zero, and these modes will indeed be present if we believe in genericity of initial conditions – the philosophy underlying the inflationary universe paradigm. In generic models these modes will occur with anisotropy dominant at both late and early times. If the cosmological constant is non-zero at all late times this will no longer apply in the far future [63]; hence the importance of the question of whether the present effective cosmological constant will die away or not in the far future. Thus the cosmological constant could mean these modes are not important at late times. But to show that this is the case, we have to study these models and their dynamical behaviour! Tilted models may behave quite differently from orthogonal models [28].

As to the universe in the far past, it could have been anisotropic there too, but with the anisotropy dying away in an inflationary era; or even with anisotropy preventing an inflationary era, but dying away later. The further interesting point is that intermediate isotropisation is possible: that is, models can start off quite unlike the RW geometries, then become arbitrarily similar to them for an arbitrarily long time, and then become quite unlike them again at later times [20, 60].Observation of a very isotropic present state of the universe does not preclude this from occurring; and genericity of initial conditions will mean all possible Bianchi modes are present in some combination in the initial data. Thus Bianchi models could conceivably be important at very early or very late times as models of the universe domain in which we live.

5.3 Basis for more general geometric studies

The dynamical systems approach used to study Bianchi models gives a sound platform from which to generalise and look at the dynamics of inhomogeneous models. Indeed a research programme under way does just that, generalising to G_2 and the G_0 models. This programme seems to indicate that, as suggested by Belinskii, Lifshitz and Khalatnikov [1], oscillatory solutions are indeed the generic case (see [38, 59] and references there). The question of how generic inflation is in this context is not yet fully answered.

5.4 Basis for more general dynamic studies

The Bianchi models also provide a platform to study evolution of cosmologies with other dynamics, e.g. brane models (e.g. [7]), string cosmology models (e.g. [8, 14]), and loop quantum gravity models (see for example [5]). Higher dimensional versions of these universes have been studied in these various contexts, benefiting from the knowledge obtained in the context of standard general relativity. Thus we may expect them to continue being of interest in the future.

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