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Viscous brane cosmology with a brane-bulk energy interchange term

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Abstract We assume a flat brane located at $y = 0$, surrounded by an AdS space, and consider the 5D Einstein equations when the energy flux component of the energy-momentum tensor is related to the Hubble parameter through a constant Q . We calculate the metric tensor, as well as the Hubble parameter on the brane, when Q is small. As a special case, if the brane is tensionless, the influence from Q on the Hubble parameter is absent. We also consider the emission of gravitons from the brane, by means of the Boltzmann equation. Comparing the energy conservation equation derived herefrom with the energy conservation equation for a viscous fluid on the brane, we find that the entropy change for the fluid in the emission process has to be negative. This peculiar effect is related to the fluid on the brane being a non-closed thermodynamic system. The negative entropy property for non-closed systems is encountered in other areas in physics also, in particular, in connection with the Casimir effect at finite temperature.

Keywords Brane cosmology · Viscous cosmology · Randall-Sundrum · Gravitons

1 Introduction

When considering brane world perturbative cosmology one is confronted with a plethora of phenomena, among which some are unknown even with respect to sign. For instance, as discussed recently by Durrer [1], during ordinary inflation gravitational waves are generated. For a given inflationary potential their amplitudes can be calculated. In a brane world context, a fraction of these waves will be radiated from the brane into the bulk and thereby reduce the gravitational wave amplitude. On the other hand there may also be gravitational waves generated in the bulk, and some of these may accumulate on the brane, increasing the amplitude

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of gravitational waves on the brane. Thus, depending on the circumstances, even the sign of the brane world effect on a gravitational wave background is unknown.

In view of this state of affairs it may seem desirable to allow for realistic, non-ideal, properties of the cosmic fluid on the brane. Therewith one may hope to get some guidance in restricting the number of possibilities in the description of physical processes. One natural option in this direction is to allow for a *bulk viscosity*. As is known from ordinary hydromechanics, one is easily led astray in many cases if one ignores the viscosity effects. A bulk viscosity, in contrast to a shear viscosity, is compatible with the assumption about complete isotropy of the cosmic fluid. We shall assume in the following that there is a constant bulk viscosity ζ present, but shall ignore the shear viscosity. We shall work in terms of Gaussian normal coordinates and assume that there is one single brane present at fixed position $y = 0$. That is, we adopt essentially the Randall-Sundrum type II model [2], although this model is strictly speaking non-cosmological. (It might in this context seem natural, as an alternative, to introduce a spherical model in which the brane is an expanding surface mimicking the cosmological expansion. This implies a 5D cosmological solution of Einstein's equations with a negative cosmological constant. Theories of this kind have been worked out in Ref. [3] and [4]; cf. also the recent review in Ref. [5].)

Dissipative cosmology theories were worked out some years ago – cf., for instance, the reviews [6, 7] – whereas the theory of viscous fluids in a brane context was recently investigated in Refs. [8–10]. The presence of a dissipative fluid on the brane gives us the possibility to compare with general thermodynamical principles; in particular, the behaviour of entropy in irreversible processes.

An important point in the present context is that we will relax the condition about zero energy flux from the brane, $T_{ty} = 0$, in the y direction. This means physically that we draw into consideration the production of gravitons. Since the emission of gravitons into the bulk can be described via the Boltzmann equation, this is the case that we will be henceforth interested in (we thus do not consider any further the absorption of gravitons on the brane). We shall describe this interchange effect in a simple way phenomenologically, by introducing a non-vanishing energy flux component T_{ty} . It is only this component of the 5D energy-momentum tensor T_{AB} that comes into play in the present context.

In Sect. 3 we will be concerned with the energy conservation equation for the viscous fluid on the brane. Comparing with the corresponding equation derived from the Boltzmann equation, it actually turns out that the emission process corresponds to a negative entropy change for the thermodynamic subsystem on the brane. This may be an unexpected result, but it does not simply run into conflict with basic thermodynamics all the time that the thermodynamic principles apply only to a closed system; in our case this means the brane fluid plus the bulk particles.

2 Einstein's equations, with an interchange term

As mentioned, we assume that there is one single brane located at $y = 0$. We take the spatial curvature k to be zero. The metric will be taken in the form

$$ds^2 = -n^2(t, y)dt^2 + a^2(t, y)\delta_{ij}dx^i dx^j + dy^2. \quad (1)$$

The quantities $n(t, y)$ and $a(t, y)$ are determined from Einstein's equations which are, with Λ the 5D cosmological constant,

$$R_{AB} - \frac{1}{2}g_{AB}R + g_{AB}\Lambda = \kappa^2 T_{AB}. \quad (2)$$

Here the coordinate indices are numbered as $x^A = (t, x^1, x^2, x^3, y)$, with $\kappa^2 = 8\pi G_5$ the 5D gravitational coupling. With the metric (1) Einstein's equations in a coordinate basis have been worked out before [10–14], but it is convenient to give them also here for reference purposes. When $k = 0$,

$$3\left\{\left(\frac{\dot{a}}{a}\right)^2 - n^2\left[\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right]\right\} - \Lambda n^2 = \kappa^2 T_{tt}, \quad (3)$$

$$a^2 \delta_{ij} \left\{ \frac{a'}{a} \left(\frac{a'}{a} + \frac{2n'}{n} \right) + \frac{2a''}{a} + \frac{n''}{n} + \frac{1}{n^2} \left[\dot{a} \left(-\frac{\dot{a}}{a} + \frac{2\dot{n}}{n} \right) - \frac{2\ddot{a}}{a} \right] + \Lambda \right\} = \kappa^2 T_{ij}, \quad (4)$$

$$3 \left(\frac{\dot{a} n'}{a n} - \frac{\dot{a}'}{a} \right) = \kappa^2 T_{ty}, \quad (5)$$

$$3 \left\{ \frac{a'}{a} \left(\frac{a'}{a} + \frac{n'}{n} \right) - \frac{1}{n^2} \left[\dot{a} \left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right] \right\} + \Lambda = \kappa^2 T_{yy}. \quad (6)$$

Here overdots and primes mean derivatives with respect to t and y , respectively. The energy-momentum tensor is taken in the form

$$T_{AB} = \delta(y) [-\sigma g_{\mu\nu} + \rho U_\mu U_\nu + \tilde{p} h_{\mu\nu}] \delta_A^\mu \delta_B^\nu, \quad (7)$$

where $h_{\mu\nu} = g_{\mu\nu} + U_\mu U_\nu$ is the projection tensor and $\tilde{p} = p - 3H_0\zeta$ is the effective pressure, $H_0 = \dot{a}_0/a_0$ being the Hubble parameter on the brane $y = 0$. As gauge condition we take $n_0(t) = 1$, which physically means that the proper time on the brane is equal to the cosmological time coordinate. As $n_0(t)$ is a constant, the scalar expansion $\theta = U^\mu{}_{;\mu} = 3H_0 + \dot{n}_0/n_0$ reduces to $3H_0$ [10]. The energy-momentum expression (7) is composed of two parts: one part which in an orthonormal frame means $T_{tt} = \delta(y)\sigma$, $T_{ij} = -\delta(y)\sigma\delta_{ij}$, which is in accordance with the equation of state $p = -\rho$ for a cosmic brane [15], and there is a second part describing the energy-momentum for a viscous fluid. We work henceforth in an orthonormal frame, where $U^\mu = (1, 0, 0, 0)$, and let generally the subscript zero be referring to the brane.

Consider next the junction conditions applied to Eqs. (3) and (4) at $y = 0$. For the distributional parts we have [10]

$$\frac{[a']}{a_0} = -\frac{1}{3}\kappa^2(\sigma + \rho), \quad (8)$$

$$[n'] = \frac{1}{3}\kappa^2(-\sigma + 2\rho + 3\tilde{p}), \quad (9)$$

where $[a'] = a'(0^+) - a'(0^-)$, and similarly for $[n']$.

For the nondistributional parts we have

$$\left(\frac{\dot{a}}{na}\right)^2 - \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = \frac{1}{3}\Lambda, \quad (10)$$

$$\frac{a'}{a} \left(\frac{a'}{a} + \frac{2n'}{n}\right) + \frac{2a''}{a} + \frac{n''}{n} + \frac{1}{n^2} \left[\frac{\dot{a}}{a} \left(-\frac{\dot{a}}{a} + \frac{2\dot{n}}{n}\right) - \frac{2\ddot{a}}{a} \right] = -\Lambda. \quad (11)$$

We now turn attention to the energy flux component T_{ty} , attempting to model it in a simple way. It seems natural to assume that the energy flux from the brane was stronger in the early stages of the universe when the Hubble parameter $H = \dot{a}/a$ was large, than it is today. As ansatz we shall adopt a simple proportionality, i.e., $T_{ty} = -QH$, where Q is a constant. This ansatz actually leads to mathematical simplifications also. Namely, in (5) we can eliminate \dot{a}/a to get the equation

$$\frac{n'}{n} - \frac{\dot{a}'}{\dot{a}} = -\frac{1}{3}\kappa^2 Q \quad (12)$$

which, after integration with respect to y , yields

$$n(t, y) = \frac{\dot{a}(t, y)}{\dot{a}_0(t)} e^{-\kappa^2 Q|y|/3}, \quad (13)$$

where the Z_2 symmetry $y \rightarrow -y$ is taken into account. Note that the condition $n_0 = 1$ is obeyed, and that a positive value of Q leads to a decreasing value of the metric component $n(t, y)$ with increasing distances $|y|$ from the brane.

Consider next Eq. (10). After some algebra and use of (13) we can write it in the form

$$\frac{d}{dy} \left\{ \left(\frac{\dot{a}a}{n}\right)^2 - (aa')^2 - \frac{1}{6}\Lambda a^4 \right\} = \frac{2}{3}\kappa^2 Q a^2 \dot{a}_0^2 e^{2\kappa^2 Q|y|/3}. \quad (14)$$

A nonvanishing value of Q thus spoils the conservation of the expression between the curly parentheses and thereby changes the conventional 5D brane version of Friedmann's first equation.

We shall not solve (14) in general, but limit ourselves to the case when Q is small. Specifically, we shall consider the condition (assumed here that $Q > 0$)

$$\kappa^2 Q|y| \ll 1. \quad (15)$$

Stated in another way: we consider only distances $|y|$ from the brane for which (15) is satisfied. This region close to the brane is obviously also the one of main physical interest. Our calculation below will go only to the first order in Q .

When the exponential in (14) is replaced with unity, we need only the expression for $a(t, y)$ on the right hand side that pertains to the case $Q = 0$. For $y = 0$ we then have the equation

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{1}{6}\Lambda + \frac{\kappa^4}{36}(\sigma + \rho)^2 + \frac{C}{a_0^4}, \quad (16)$$

where $C = C(t)$ is an integration constant with respect to y . As $\rho = \rho(t)$, this equation can be solved for a_0 only in special cases. Let us give the explicit solution when $\rho = 0$ [14]:

$$a_0(t; \rho = 0) = \frac{1}{2\sqrt{\lambda}f(t)}[f^4(t) - 4\lambda C]^{1/2}, \quad (17)$$

where the constant

$$\lambda = \frac{1}{6}\Lambda + \frac{1}{36}\kappa^4\sigma^2 \quad (18)$$

can be interpreted as an effective four-dimensional cosmological constant in the five-dimensional theory, and

$$f(t) = e^{\sqrt{\lambda}(t+c_0)}, \quad (19)$$

c_0 being a new integration constant (recall that $k = 0$ is assumed).

Now considering the $Q = 0$ solution for $a(t, y)$ away from the brane, we shall assume only the AdS case, i.e., $\Lambda < 0$. We then have, for general y ,

$$a^2(t, y; Q = 0) = \frac{1}{2}a_0^2\left[1 + \frac{\kappa^4}{6\Lambda}(\sigma + \rho)^2\right] + \frac{3C}{\Lambda a_0^2} + \left\{\frac{1}{2}a_0^2\left[1 - \frac{\kappa^4}{6\Lambda}(\sigma + \rho)^2\right] - \frac{3C}{\Lambda a_0^2}\right\} \cosh(2\mu y) - \frac{\kappa^2}{6\mu}(\sigma + \rho)a_0^2 \sinh(2\mu |y|), \quad (20)$$

where $\mu = \sqrt{-\Lambda/6}$. (Note that the condition (15) does not necessarily imply that the argument $(2\mu y)$ is small.) The terms containing the quantity C are not of main interest here and will hereafter be omitted (the extra term C/a_0^4 in the Friedmann equation (16) is called the "radiation term").

The expression (20) is easily integrated with respect to y , and so the whole Eq. (14) can be integrated. Again omitting a "radiation" type term we obtain

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{1}{6}\Lambda + \left(\frac{a'}{a}\right)^2 + \frac{2}{3}\kappa^2 Q \frac{\dot{a}_0^2}{a^4} \left\{\frac{1}{2}a_0^2\left(1 + \frac{\kappa^4\sigma^2}{6\Lambda}\right)y + \frac{1}{4\mu}a_0^2\left(1 - \frac{\kappa^4\sigma^2}{6\Lambda}\right) \sinh 2\mu y - \frac{\kappa^2\sigma}{12\mu^2}a_0^2 \cosh 2\mu y\right\}. \quad (21)$$

Note that it is equivalent here whether we insert $a_0^2(t; Q = 0)$ or the more general expression $a_0^2(t; Q) \equiv a_0^2(t)$ on the right hand side, all the time that we work to the first order in Q .

The expression (21) can be evaluated on the brane. Let us take $y = 0^+$; then $a'_0/a_0 = -\frac{1}{6}\kappa^2(\rho + \sigma)$. Recalling that $n_0(t) = 1$ we get

$$H_0^2 = \frac{\lambda + \frac{1}{18}\kappa^4\sigma\rho + \frac{1}{36}\kappa^4\rho^2}{1 + \frac{\kappa^4}{18\mu^2}\sigma Q}. \quad (22)$$

This is our main result. It shows how Friedmann's first equation becomes modified in the presence of a nonvanishing Q . When $Q = 0$, (22) reduces to (28) in Ref. [10] (when the radiation term is omitted).

We note the following points:

- (1) There is no influence from the viscosity in (22). This arises from the fact that Friedmann's first equation refers to the energy in the fluid, not to the pressure, and it is only in the latter context that viscosity plays a role (cf. $p \rightarrow \tilde{p}$ above).
- (2) If $\sigma > 0$ (i.e., a positive tensile stress on the brane), and if $Q > 0$, then the magnitude of the Hubble parameter becomes *diminished* by the presence of Q .
- (3) If the brane is tensionless, $\sigma = 0$, there is no influence from Q on the Hubble parameter at all, irrespective of the value of ρ .

What is the *sign* of the constant Q ? Although we have not discussed this issue in detail, it is fairly obvious from the expression (13) that in order for the present model to be physically reasonable one should have $Q > 0$. The influence from T_{ly} on the brane has to decay with increasing distances $|y|$ from the brane.

In the next section we shall consider the energy conservation equation for the fluid on the brane. This equation can be derived from the Boltzmann equation describing the emission process, or alternatively from the standard conservation equation for a viscous fluid. Comparison of the equations will enable us to discuss the behaviour of entropy.

3 Radiating brane: energy conservation equation in the presence of viscosity

A reasonable physical model for the brane-bulk interaction is to assume that bulk gravitons are produced by fluctuations of brane matter. Assuming as before an AdS bulk, we can take the gravitons to be created by the collision of pairs of particles on the brane. The process can be described in various ways. Let us briefly review here the kind of approach advocated by Langlois et al. [16–18]. The process can be described as

$$\psi \bar{\psi} \rightarrow g, \quad (23)$$

where ψ is a standard model particle and g is a graviton. The equation of state is written in the conventional form

$$p = w\rho, \quad (24)$$

where $w = 1/3$ for highly relativistic particles. One can make use of the Boltzmann equation on the brane,

$$\dot{\rho} + 3H_0(\rho + p) = - \int \frac{d^3 p}{(2\pi)^3} C[f], \quad (25)$$

where the collision term is

$$C[f] = \frac{1}{2} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \frac{d^3 p_2}{(2\pi)^3 2E_2} \sum |M|^2 f_1 f_2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p). \quad (26)$$

Here f is the distribution function for gravitons, and M is the scattering amplitude. The calculation leads to the following result, with $w = 1/3$,

$$\dot{\rho} + 4H_0 \rho = -\frac{315}{512\pi^3} \hat{g} \kappa^2 T^8, \quad (27)$$

when the matter is in thermal equilibrium at a temperature T . Here $\hat{g} = (2/3)g_s + g_f + 4g_v = 188.7$, where $g_s = 4$, $g_f = 90$, $g_v = 24$ refer to the degrees of freedom for scalars, fermions, and vectors, respectively, in the standard model.

We now go back to the formalism of the previous section. Evaluating (12) on both sides of the brane, and inserting the expressions (8) and (9) for the jumps $[a']$ and $[n']$, we obtain the energy conservation equation in the following form:

$$\dot{\rho} + 3H_0(\rho + p) = 9\zeta H_0^2. \quad (28)$$

In this equation Q does not appear explicitly, but its influence is hidden in the Q -dependent expression for H_0 ; cf. (22). It is notable that (28) has precisely the same form as in ordinary viscous four-dimensional cosmology [6, 19, 20], although there is seemingly no simple physical reason why this should be so.

Let us compare Eqs. (27) and (28). When $w = 1/3$ their left hand sides are the same, but on the right hand sides there is a striking difference in that the *signs* are opposite. As we know from ordinary thermodynamics the bulk viscosity ζ (as well as the shear viscosity η) are taken to be positive quantities; this arising from the condition that the entropy change in an irreversible process for a closed system is positive [21].

So, the following conclusion naturally emerges: The emission of gravitons into the bulk, as described by (27), is accompanied by a *negative* entropy change in the cosmic fluid on the brane. It corresponds to a negative ζ . The negativity of the entropy change is counterintuitive, but does not violate thermodynamics as one would be inclined to conclude at first. The reason is that the fluid on the brane forms a *non-closed* thermodynamic system; in order to close the system one has to include the particles in the bulk also. General relationships, such as the law about entropy increase in an irreversible process, applies to a closed thermodynamic system only.

This particular effect is not so uncommon after all. It becomes natural here to compare with the theory of the Casimir effect. Imagine that there are two parallel metal plates, separated by a fixed gap a of the order of $1 \mu\text{m}$, at a temperature T . At low T , there exists a finite temperature interval in which the Casimir free energy F is *increasing* with increasing values of T , keeping a constant. This corresponds to a negative entropy $S = -\partial F/\partial T$ in the actual temperature interval. The reason is evidently that these Casimir quantities are not concerned with the thermodynamic quantities of the closed system, but only with the interaction part of it. And the Casimir force is derived just from the interaction part of the free energy. We have considered this effect repeatedly earlier [22–25], and the theory has been corroborated by other works considering the Casimir effect from different viewpoints - cf. for instance Sernelius et al. [26–28] – although it is only fair to say that a full consensus on this point has not so far been achieved in the literature.

4 Summary

Our starting point was the metric (1), corresponding to zero spatial curvature, whereby the 5D Einstein equations (2) took the form (3)-(6). On the $y = 0$ brane, endowed with a constant tension σ , a fluid with density ρ and constant bulk viscosity ζ was assumed, corresponding to the energy-momentum tensor in the form of (7).

The main results of the present paper are:

- Assuming the energy flux component to satisfy the proportionality $T_{ty} = -QH$ with Q a constant, we found the metric component $n(t, y)$ to be given by (13). This expression shows that $Q > 0$ in order for the present kind of theory to be meaningful; the influence from the brane-bulk interaction is expected to decay for increasing values of $|y|$.
- In the limit of small Q , the metric component $a(t, y)$ is determined from (21), which in turn leads to the expression (22) for the Hubble parameter H_0 on the brane. This expression shows, in particular, that the influence from Q on H_0 is absent if the brane is tensionless, $\sigma = 0$.
- The emission of gravitons can be described through the Boltzmann equation on the brane [16–18]. The corresponding energy conservation Eq. (27), when compared with the energy conservation Eq. (28) for a viscous fluid on the brane, shows that ζ has to be negative, corresponding to a negative entropy change. This counterintuitive effect relates to the fact that the fluid on the brane is a non-closed thermodynamic system. The negative entropy effect has parallels in other areas of physics also, notably in connection with the Casimir effect at a finite temperature [22–28].

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Note added in proof

We have recently become aware of the following two relevant papers, which ought to have been referred to:

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