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RESEARCH ARTICLE

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Post-newtonian parameters from alternative theories of gravity

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Abstract Alternative theories of gravity have been recently studied in connection with their cosmological applications, both in the Palatini and in the metric formalism. The aim of this paper is to propose a theoretical framework (in the Palatini formalism) to test these theories at the solar system level and possibly at the galactic scales. We exactly solve field equations in vacuum and find the corresponding corrections to the standard general relativistic gravitational field. On the other hand, approximate solutions are found in matter cases starting from a Lagrangian which depends on a phenomenological parameter. Both in the vacuum case and in the matter case the deviations from General Relativity are controlled by parameters that provide the Post-Newtonian corrections which prove to be in good agreement with solar system experiments.

Keywords Alternative theories of gravity · Post-Newtonian parameters

1 Introduction

The most striking and recent experimental discovery regarding Cosmology and the structure of the universe is related with the evidence of the acceleration of the universe, which is supported by experimental data deriving from different tests: i.e., from Type-Ia Supernovae, from CMWB and from the large scale structure of the universe [1]. Standard General Relativity is not able to provide a theoretical

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explanation to these experimental results unless some *exotic and invisible matter* is admitted to exist in the universe (*Dark Energy*). Proposals to explain the cosmic acceleration also arise from higher dimensional theories of Gravity [2]. Alternative theories to explain the acceleration of the universe have been recently proposed in the framework of higher order theories of Gravity [3], already introduced in the framework of cosmological models to explain the early time inflation [4]. Different models have been then studied both in the standard metric formalism [5] and in the first order Palatini formalism [6]. Higher order theories of Gravity have been studied also in a quantum framework and a quantization of $L(R)$ theories has been performed in [7].

To test the theoretical consistence of these theories with observational data is however necessary to examine and to fit the standard tests for General Relativity: in particular solar system experiments and the tests of gravity at galactic scales. General Relativity reproduces with an excellent precision the experimental results obtained at the solar system scale [8]. This naturally implies that each theory which pretends to be consistent with experimental results should surely reproduce General Relativity in this limit.

The aim of this paper is to provide a general theoretical framework to test the reliability of alternative theories of Gravity with solar system experiments. Such a problem was already studied from a different viewpoint in the standard metric formalism in [9] and in the Palatini formalism in [6, 10]. Some debate is still open on the accordance of experimental results with solar system experiments and some authors erroneously claim that only theories which do not differ too much from General Relativity do the job (see [6]); however, as we shall see, this is not true and, moreover, it is known that the Palatini formalism can naturally provide accordance with solar system experiments (see e.g. [10–12] and references quoted therein). Some interesting results are also present in literature regarding the accordance of alternative theories of Gravity with rotational curves of galaxies [13]. In this paper we shall study the problem of the reliability of alternative theories of Gravity with solar system experiments and give also some hints regarding the galactic scale tests of Gravity (which will be considered in a forthcoming paper [14]). We shall do this from a purely theoretical viewpoint, trying to understand which Newtonian or Post-Newtonian modifications to standard General Relativity arise from specific modifications of the Hilbert-Einstein Lagrangian. In particular we consider $L(R)$ theories where the Lagrangian depends on an arbitrary analytic function L of the scalar curvature R. Starting from the results already obtained in [15] and [16] we find an exact solution to field equations in vacuum. In that case field equations are controlled by a scalar-valued equation called the *structural equation*. It is relevant that modifications to the standard general relativistic gravitational field arise, and they turn out to be directly related to solutions of the structural equation and, consequently, to the particular form of the Lagrangian chosen (the choice of $L(R)$). We shall show how these modifications can be suitably interpreted as Post-newtonian parameters related to the non linearity of the theory.

We consider furthermore field equations in the case of matter universes (i.e. when the stress energy tensor is non vanishing). Considering a linear approximation of the metric, either with respect to a Minkowski flat space-time, or with respect to a de Sitter or an anti de Sitter space-times, the non-linear structure of the theory influences the gravitational field. We stress however that, in the first order approximation of the Palatini formalism, the presence of non-linear terms in the Lagrangian only influences the definition of R, while field equations remain unchanged. We finally derive the gravitational field for the particular Lagrangians $R + \alpha f(R)$ where α is an adimensional parameter. The corresponding gravitational potential contains then a term which is directly proportional to α, such that General Relativity is reproduced in the limit $\alpha = 0$, as it should be expected. This implies that the parameter α behaves as a sort of scale parameter which becomes relevant at large scales and it can be interpreted as a Post-Newtonian parameter ensuing from the non linearity of the Lagrangian.

Our approach, of course does not completely solve the problem of the generic reliability of alternative theories of gravity at solar system and galactic scale. However by introducing some Post-Newtonian parameters, it shows that General Relativity is certainly reproduced at small scales (as it is expected) for large families of Lagrangians. Further comparisons with other classical tests of General Relativity and applications to more general cases, as well as tests of Gravity at large (galactic) scale will be presented in the forthcoming paper [14].

2 The theoretical framework of $L(R)$ gravity

We deal with a 4-dimensional gravitational theory on a Lorentzian manifold (M, g) with signature $(-, +, +, +)$.[1] The action is chosen to be:

$$A = A_{\text{grav}} + A_{\text{mat}} = \int [\sqrt{g} L(R) + 2\kappa L_{\text{mat}}(\psi, \nabla\psi)]\, d^4x \tag{1}$$

where $R \equiv R(g, \Gamma) = g^{\alpha\beta} R_{\alpha\beta}(\Gamma)$, $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of any torsionless connection Γ independent on a metric g, which is assumed to be the physical metric. The gravitational part of the Lagrangian is represented by any real analytic function $L(R)$ of one real variable, which is assumed to be the scalar curvature R. The total Lagrangian contains also a first order matter part L_{mat} functionally depending on yet unspecified matter fields Ψ together with their first derivatives, equipped with a gravitational coupling constant $\kappa = \frac{8\pi G}{c^4}$ (see e.g. [15]).

Equations of motion ensuing from the first order á la Palatini formalism are (see [6, 11, 16])

$$L'(R) R_{(\mu\nu)}(\Gamma) - \frac{1}{2} L(R) g_{\mu\nu} = \kappa T^{\text{mat}}_{\mu\nu} \tag{2}$$

$$\nabla^{\Gamma}_{\alpha} [\sqrt{g} L'(R) g^{\mu\nu}) = 0 \tag{3}$$

where $T^{\mu\nu}_{\text{mat}} = -\frac{2}{\sqrt{g}} \frac{\delta L_{\text{mat}}}{\delta g_{\mu\nu}}$ denotes the matter source stress-energy tensor and ∇^{Γ} means covariant derivative with respect to the connection Γ, which we recall to be independent on the metric g. In this paper the metric g and its inverse are used for lowering and raising indices.

We denote by $R_{(\mu\nu)}$ the symmetric part of $R_{\mu\nu}$, i.e. we set $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$. From (3) it follows that $\sqrt{g} L'(R) g^{\mu\nu}$ is a symmetric twice contravariant

[1] If not otherwise stated, we use units such that $G = c = 1$.

tensor density of weight 1, so that it can be used (if non degenerate) to define a new metric $h_{\mu\nu}$ by the prescription:

$$\sqrt{g}L'(R)g^{\mu\nu} = \sqrt{h}h^{\mu\nu} \tag{4}$$

which is generically invertible. This means that the two metrics h and g are conformally equivalent so that space-time M can be a posteriori endowed with a bi-metric structure (M, g, h) [16] equivalent to the original metric-affine structure (M, g, Γ). The corresponding conformal factor can be easily found to be $L'(R)$, since (4) gives:

$$h_{\mu\nu} = L'(R)\ g_{\mu\nu} \tag{5}$$

Therefore, as it is well known, Eq. (3) implies that $\Gamma = \Gamma_{LC}(h)$, i.e. the dynamical connection turns out a posteriori to be the Levi-Civita connection of the newly defined metric h, so that $R_{(\mu\nu)}(\Gamma_{LC}(h)) = R_{\mu\nu}(h) \equiv R_{\mu\nu}$ is now the metric Ricci tensor of the new metric h.

Equation (2) can be supplemented by the scalar-valued equation obtained by taking the g-trace of (2), where we set $\tau = \mathrm{tr}T = g^{\mu\nu}T^{\mathrm{mat}}_{\mu\nu}$:

$$L'(R)R - 2L(R) = \kappa\tau \tag{6}$$

Equation (6) is called the *structural equation* and it controls the solutions of Eq. (2). For any real solution $R = F(\tau)$ of (6) we have in fact that both $L(R) = L(F(\tau))$ and $L'(R) = L'(F(\tau))$ can be seen as functions of τ. For notational convenience we shall use the abuse of notation $L(\tau) = L(F(\tau))$ and $L'(\tau) = L'(F(\tau))$.

Now we are in position to introduce the generalized Einstein equations under the form[2]

$$R_{\mu\nu}(h) = \frac{L(\tau)}{2L'(\tau)}g_{\mu\nu} + \frac{\kappa}{L'(\tau)}T_{\mu\nu} \tag{7}$$

with $h_{\mu\nu}$ defined by (5) for a given $g_{\mu\nu}$ and $T^{\mathrm{mat}}_{\mu\nu}$ (see also [6, 11, 16]).

3 Some exact solution of the field equations in vacuum

In this Section we look for a spherically symmetrical solution of the generalized Einstein equations in vacuum, starting from the results obtained in [15] and [16]. To this end first notice that Eqs. (2–3), in vacuum, can be written under the form

$$[L'(R)]R_{(\mu\nu)}(\Gamma) - \frac{1}{2}[L(R)]g_{\mu\nu} = 0 \tag{8}$$

$$\nabla^{\Gamma}_{\alpha}(\sqrt{g}\ [L'(R)]\ g^{\mu\nu}) = 0 \tag{9}$$

Furthermore, the structural Eq. (6) becomes

$$L'(R)R - 2L(R) = 0 \tag{10}$$

[2] Provided that $L'(\tau) \neq 0$: see below.

In order to solve (8)–(9), we follow the discussion outlined in [16]. Let us suppose that the structural Eq. (10) is not identically satisfied and has a countable set of (real) solutions ($i = 1, 2, \ldots$):

$$R = c_i \tag{11}$$

Then, we have two possibilities, depending on the value of the first derivative $L'(R)$ evaluated at the point $R = c_i$:

1. $L'(c_i) = 0$
2. $L'(c_i) \neq 0$

In the first case, Eq. (10) implies that also $L(c_i) = 0$, and, hence, the equations of motion (8)–(9) are identically satisfied. The only relation between g and Γ is the following

$$R(g, \Gamma) = c_i \tag{12}$$

Indeed, this equation is not sufficient in this case to determine an explicit relation between the metric and the connection. Hence, in what follows, we shall suppose that $L'(c_i) \neq 0$.

We remark that if the Lagrangian is in the form $L(R) = R^n$, with $n \geq 2$, $n \in \mathbb{N}$, $R = 0$ is solution of Eq. (10), and, moreover one has $L'(R = 0) = 0$. Consequently we exclude such Lagrangians.

If $L'(c_i) \neq 0$ then the solution of the equations of motion (8)–(9) is given by the Levi-Civita connection of the metric h, which turns out to be equivalent to Levi-Civita connection of the physical metric g (owing to the relation $h = L'(c_i)g$). Accordingly, the metric g is the solution of the generalized Einstein equations

$$R_{\mu\nu}(g) = \mu g_{\mu\nu} \tag{13}$$

where

$$\mu = c_i/4 \tag{14}$$

We look for a static solution of the field Eq. (13) describing the field outside a spherically symmetric mass distribution. Hence we may write the metric in the form

$$ds^2 = -e^{\Phi(r)}dt^2 + e^{\Lambda(r)}dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 \tag{15}$$

It is easy to check that the field Eq. (13) are satisfied if we set

$$-e^{\Phi(r)} = g_{tt} = -1 + \frac{C}{r} - \frac{\mu r^2}{3} \tag{16}$$

and

$$e^{\Lambda(r)} = g_{rr} = \left(1 - \frac{C}{r} + \frac{\mu r^2}{3}\right)^{-1} \tag{17}$$

where C is an arbitrary constant; in particular, the metric defined by (16)–(17) corresponds to the so called *Schwarzschild-de Sitter space-time* (see [17, 18]). The physical meaning of the constant C becomes clear when considering the limit of weak gravitational field. We know that in General Relativity in this limit we have

$$g_{tt} \simeq -(1 + 2\phi) \tag{18}$$

where

$$\phi = -\frac{M}{r} \tag{19}$$

is the Newtonian potential, M being the mass of the spherically symmetric source of the gravitational field. Consequently, in order to obtain the Newtonian limit we must set $C = 2M$. Moreover, from (16) it is evident that a further contribution to the standard Newtonian potential is present in higher order theories of gravity. In particular, this contribution is proportional to the values of the Ricci scalar, owing to the proportionality between μ and c_i (see (11) and (14)). This implies that the higher order contribution to the gravitational potential should be small enough not to contradict the known tests of gravity. In the case of small values of R (which surely occur at solar system scale) the Einsteinian limit (i.e. the Schwarzschild solution) and the Newtonian limit are recovered, as it is evident from (16). In this context, μ can be naturally thought of as a Post-Newtonian parameter, ensuing from the non linearity of the theory ($\mu = 0$ for the Hilbert-Einstein Lagrangian).

On the other hand this Post-Newtonian correction could play some role at larger scales and it could be interesting to test higher-order theories at galactic scales, as already done in the metric formalism in [13].

4 Field equations in linear approximation

We aim at writing the field equations for Lagrangians $L(R) = R + \alpha f(R)$ in linear approximation: that is, we are going to solve the field equation at first order approximation with respect to a given background. In other words, we suppose to know a background solution of field Eqs. (2) and (3) determined by the affine connection ${}^{(0)}\Gamma$ and the metric ${}^{(0)}g$.[3] We now perturb this solution by writing

$$\Gamma^{\alpha}_{\mu\nu} = {}^{(0)}\Gamma^{\alpha}_{\mu\nu} + {}^{(1)}\Gamma^{\alpha}_{\mu\nu} \tag{20}$$

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + {}^{(1)}g_{\mu\nu} \tag{21}$$

Furthermore, the matter source stress-energy tensor is written with respect to this perturbation in the form:

$$T^{\text{mat}}_{\mu\nu} = {}^{(0)}T^{\text{mat}}_{\mu\nu} + {}^{(1)}T^{\text{mat}}_{\mu\nu} \tag{22}$$

As a consequence, the equation

$$L'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}L(R)g_{\mu\nu} = \kappa T^{\text{mat}}_{\mu\nu} \tag{23}$$

[3] Here and henceforth, the superscripts ${}^{(0)}$ and ${}^{(1)}$ refer to the background and perturbed quantities, respectively.

can be written under the form[4]

$$L'\left({}^{(0)}R\right)\,{}^{(1)}R_{\mu\nu} + L'\left({}^{(1)}R\right)\,{}^{(0)}R_{\mu\nu} - \frac{1}{2}\,{}^{(1)}g_{\mu\nu}L\left({}^{(0)}R\right) - \frac{1}{2}\,{}^{(0)}g_{\mu\nu}L\left({}^{(1)}R\right) = \kappa\,{}^{(1)}T^{\mathrm{mat}}_{\mu\nu} \tag{25}$$

The Ricci curvature ${}^{(0)}R_{\mu\nu}$ (and the corresponding Ricci scalar ${}^{(0)}R$) refer to the background solution; in terms of the perturbation of this solution we may write

$$R_{\mu\nu} = {}^{(0)}R_{\mu\nu} + {}^{(1)}R_{\mu\nu} \tag{26}$$

and

$$R = {}^{(0)}R + {}^{(1)}R + {}^{(1)}g^{\mu\nu}\,{}^{(0)}R_{\mu\nu} \tag{27}$$

So, in order to explicitly write the perturbed field Eq. (25) we have to evaluate the perturbed Ricci curvature and scalar in terms of the fields g and Γ. In general, we have [11]:

$$R_{\mu\nu}(\Gamma) - R_{\mu\nu}(g) = \nabla_{(\mu}Q^{\alpha}_{\nu)\alpha} - \nabla_{\alpha}Q^{\alpha}_{\mu\nu} + Q^{\alpha}_{\beta(\mu}Q^{\beta}_{\nu)\alpha} - Q^{\alpha}_{\mu\nu}Q^{\beta}_{\alpha\beta} \tag{28}$$

where

$$Q^{\alpha}_{\mu\nu} \doteq \left\{{\alpha \atop \mu\nu}\right\} - \Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\nabla_{\mu}g_{\nu\beta} + \nabla_{\nu}g_{\mu\beta} - \nabla_{\beta}g_{\mu\nu}) \tag{29}$$

in terms of the Christoffel symbols

$$\left\{{\alpha \atop \mu\nu}\right\} = \frac{1}{2}g^{\alpha\beta}(g_{\nu\beta,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}) \tag{30}$$

Notice that $\nabla_{\mu} \doteq \nabla^{\Gamma}_{\mu}$ here and henceforth and we denote moreover with $g_{\mu\beta,\nu}$ the partial derivative $\partial_{\nu}g_{\mu\beta}$. The second set of field equations

$$\nabla_{\alpha}(\sqrt{g}\;[L'(R)]\;g^{\mu\nu}) = 0 \tag{31}$$

can be now written in the form

$$\nabla_{\alpha}g_{\mu\nu} = b_{\alpha}g_{\mu\nu} \tag{32}$$

We have here defined:

$$b_{\alpha} \doteq -\nabla_{\alpha}[\ln L'(R)] \tag{33}$$

From the structural Eq. (6) and from (33), we obtain then

$$b_{\alpha} \doteq -\kappa\frac{L''(R)}{L'(R)}\frac{\tau_{,\alpha}}{L''(R)R - L'(R)} \tag{34}$$

As a consequence, from Eq. (29) we may write

$$Q^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(b_{\mu}g_{\nu\beta} + b_{\nu}g_{\mu\beta} - b_{\beta}g_{\mu\nu}) \tag{35}$$

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[4] We have taken into account the fact that, on the background,

$$L'({}^{(0)}R)\,{}^{(0)}R_{(\mu\nu)}({}^{(0)}\Gamma) - \frac{1}{2}L({}^{(0)}R)\,{}^{(0)}g_{\mu\nu} = \kappa\,{}^{(0)}T^{\mathrm{mat}}_{\mu\nu} \tag{24}$$

The expression of the Ricci tensor of the affine connection reads then

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(g) + \nabla_{(\mu}b_{\nu)} - \frac{1}{2}b_\mu b_\nu + g_{\mu\nu}b_\alpha b^\alpha + \frac{1}{2}\nabla_\alpha b^\alpha g_{\mu\nu} = R_{\mu\nu}(g) + B_{\mu\nu} \tag{36}$$

by introducing the tensor

$$B_{\mu\nu} \doteq \nabla_{(\mu}b_{\nu)} - \frac{1}{2}b_\mu b_\nu + g_{\mu\nu}b_\alpha b^\alpha + \frac{1}{2}\nabla_\alpha b^\alpha g_{\mu\nu} \tag{37}$$

This expression (36) holds in the exact theory, so the task of writing its linear approximation is fulfilled by separately approximating the metric Ricci tensor $R_{\mu\nu}(g)$ and the $B_{\mu\nu}$ tensor; the latter, in particular, depends on the analytic expression of $L(R)$.

The perturbation of the metric Ricci tensor is given by (see [11, 19]):

$$^{(1)}R_{\mu\nu}(g) = \frac{1}{2}\,{}^{(0)}g^{\alpha\beta}\left({}^{(1)}g_{\beta\mu|\nu\alpha} - {}^{(1)}g_{\alpha\beta|\mu\nu} + {}^{(1)}g_{\beta\nu|\mu\alpha} - {}^{(1)}g_{\mu\nu|\beta\alpha}\right) \tag{38}$$

where $_|$ stands for the (metric) covariant derivative with respect to the background.[5] By perturbing the $B_{\mu\nu}$ tensor, we obtain

$$B_{\mu\nu} = {}^{(0)}B_{\mu\nu} + {}^{(1)}B_{\mu\nu} \tag{39}$$

where

$$^{(0)}B_{\mu\nu} = {}^{(0)}b_{(\mu;\nu)} - \frac{1}{2}\,{}^{(0)}b_\mu\,{}^{(0)}b_\nu + {}^{(0)}g_{\mu\nu}\,{}^{(0)}b_\alpha\,{}^{(0)}b^\alpha + \frac{1}{2}\,{}^{(0)}b^\alpha_{;\alpha}g_{\mu\nu} \tag{40}$$

and

$$\begin{aligned} ^{(1)}B_{\mu\nu} = {} & {}^{(1)}b_{\mu,\nu} - {}^{(0)}\Gamma^\alpha_{\nu\mu}\,{}^{(1)}b_\alpha - {}^{(1)}\Gamma^\alpha_{\nu\mu}\,{}^{(0)}b_\alpha - \frac{1}{2}\,{}^{(0)}b_\mu\,{}^{(1)}b_\nu \\ & - \frac{1}{2}\,{}^{(1)}b_\mu\,{}^{(0)}b_\nu + h_{\mu\nu}\,{}^{(0)}b_\alpha\,{}^{(0)}b^\alpha + {}^{(0)}g_{\mu\nu}\,{}^{(0)}b_\alpha\,{}^{(1)}b^\alpha \\ & + {}^{(0)}g_{\mu\nu}\,{}^{(1)}b_\alpha\,{}^{(0)}b^\alpha + \frac{1}{2}\,{}^{(0)}g_{\mu\nu}\,{}^{(1)}b^\alpha_{,\alpha} + \frac{1}{2}\,{}^{(0)}g_{\mu\nu}\,{}^{(0)}\Gamma^\alpha_{\alpha\gamma}\,{}^{(1)}b^\gamma \\ & + \frac{1}{2}\,{}^{(0)}g_{\mu\nu}\,{}^{(1)}\Gamma^\alpha_{\alpha\gamma}\,{}^{(0)}b^\gamma + \frac{1}{2}h_{\mu\nu}\,{}^{(0)}b^\alpha_{;\alpha} \end{aligned} \tag{41}$$

Notice that $_;$ stands here for the covariant derivative with respect to the unperturbed connection $^{(0)}\Gamma$.

[5] The covariant derivative defined by$_|$ is such that $^{(0)}g_{\mu\nu|\alpha} = 0$.

4.1 Perturbation of flat space-time

We recall here that field equations are

$$L'(R)R_{(\mu\nu)}(\Gamma) - \frac{1}{2}L(R)g_{\mu\nu} = \kappa T^{\text{mat}}_{\mu\nu} \tag{42}$$

$$\nabla_\alpha g_{\mu\nu} = b_\alpha g_{\mu\nu} \tag{43}$$

where b_α is defined by formula (34). It is easy to check that the pair $g_{\mu\nu} = \eta_{\mu\nu}$, $\Gamma = 0$, i.e. the Minkowski flat space-time is a solution of the field Eqs. (42) and (43) iff

$$L(R = 0) = 0 \tag{44}$$

In fact, in vacuum $T \equiv 0$, hence $b_\alpha = 0$.[6]

Now, we have to solve the field equations in terms of a perturbation of the Minkowski flat solution. In particular, we look for solutions in the form

$$\Gamma^\alpha_{\mu\nu} = {}^{(0)}\Gamma^\alpha_{\mu\nu} +^{(1)} \Gamma^\alpha_{\mu\nu} =^{(1)} \Gamma^\alpha_{\mu\nu} \tag{45}$$

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} +^{(1)} g_{\mu\nu} = \eta_{\mu\nu} +^{(1)} g_{\mu\nu} \tag{46}$$

In what follows we use Cartesian coordinates adapted to the background metric ${}^{(0)}g_{\mu\nu} = \eta_{\mu\nu}$; furthermore, the latter is used to raise and lower indices. The matter source stress-energy tensor is written in the form:

$$T^{\text{mat}}_{\mu\nu} = {}^{(0)}T^{\text{mat}}_{\mu\nu} + {}^{(1)}T^{\text{mat}}_{\mu\nu} = {}^{(1)}T^{\text{mat}}_{\mu\nu} \tag{47}$$

The Ricci curvature is written in the form

$$R_{\mu\nu} = {}^{(0)}R_{\mu\nu} + {}^{(1)}R_{\mu\nu} = {}^{(1)}R_{\mu\nu} \tag{48}$$

and the corresponding Ricci Scalar (owing to ${}^{(0)}R_{\mu\nu}(\eta) = 0$):

$$R = {}^{(0)}R + {}^{(1)}R + {}^{(1)}g^{\mu\nu}\, {}^{(0)}R_{\mu\nu} = {}^{(1)}R \tag{49}$$

Notice that both the Ricci curvature and the Ricci scalar, when it is not explicitly stated (like in the above equations), refer to the connection Γ. As a consequence, Eq. (24) can be written in the form

$$L'(0)\, {}^{(1)}R_{\mu\nu}(\Gamma) - \frac{1}{2}\eta_{\mu\nu}L({}^{(1)}R) = \kappa\, {}^{(1)}T^{\text{mat}}_{\mu\nu} \tag{50}$$

From Eq. (36), the perturbed Ricci tensor is made of two contributions:

$${}^{(1)}R_{\mu\nu}(\Gamma) = {}^{(1)}R_{\mu\nu}(g) + {}^{(1)}B_{\mu\nu} \tag{51}$$

The perturbation of the metric part of the Ricci tensor is obtained by replacing the covariant derivative $_|$ with the ordinary derivative in (38), since our background is Minkowski flat space-time:

$${}^{(1)}R_{\mu\nu}(g) = \frac{1}{2}\, {}^{(0)}g^{\alpha\beta}\left({}^{(1)}g_{\beta\mu,\nu\alpha} - {}^{(1)}g_{\alpha\beta,\mu\nu} + {}^{(1)}g_{\beta\nu,\mu\alpha} - {}^{(1)}g_{\mu\nu,\beta\alpha}\right) \tag{52}$$

[6] Notice that in order to have a well posed definition of b_α, we must have $L'(R) \neq 0$, and $L''(R)R - L'(R) \neq 0$.

On the other hand, since on the background one has ${}^{(0)}b_\alpha \equiv 0$, the perturbation of the $B_{\mu\nu}$ tensor reads now as:

$$ {}^{(1)}B_{\mu\nu} = {}^{(1)}b_{(\mu,\nu)} + \frac{1}{2}\eta_{\mu\nu}\,{}^{(1)}b^{\alpha}_{,\alpha} \tag{53} $$

Hence, the perturbed Ricci tensor turns out to be

$$ \begin{aligned} {}^{(1)}R_{\mu\nu}(\Gamma) = \frac{1}{2}\eta^{\alpha\beta}\big({}^{(1)}g_{\beta\mu,\nu\alpha} - {}^{(1)}g_{\alpha\beta,\mu\nu} + {}^{(1)}g_{\beta\nu,\mu\alpha} - {}^{(1)}g_{\mu\nu,\beta\alpha}\big) \\ + {}^{(1)}b_{(\mu,\nu)} + \frac{1}{2}\eta_{\mu\nu}\,{}^{(1)}b^{\alpha}_{,\alpha} \end{aligned} \tag{54} $$

By exploiting gauge freedom, we may arbitrarily impose the following gauge condition

$$ g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0 \tag{55} $$

which, in linear approximation and by forgetting vanishing terms, becomes:

$$ \eta^{\mu\nu}\,{}^{(1)}\Gamma^{\alpha}_{\mu\nu} = 0 \tag{56} $$

By taking the linear approximations of Eqs. (29), (30) and (35), condition (55) simply becomes

$$ {}^{(1)}g_{\mu\alpha,}{}^{\alpha} - \frac{1}{2}\,{}^{(1)}g^{\alpha}_{\alpha,\mu} + {}^{(1)}b_\mu = 0 \tag{57} $$

The gauge condition (57) allows us to write the perturbed Ricci tensor and the corresponding scalar curvature under the form

$$ {}^{(1)}R_{\mu\nu}(\Gamma) = -\frac{1}{2}\,{}^{(1)}g_{\mu\nu,\alpha}{}^{\alpha} + \frac{1}{2}\eta_{\mu\nu}\,{}^{(1)}b^{\alpha}_{,\alpha} \tag{58} $$

$$ {}^{(1)}R(\Gamma) = \eta^{\mu\nu}\,{}^{(1)}R_{\mu\nu}(\Gamma) = -\frac{1}{2}\,{}^{(1)}g^{\mu}_{\mu,\alpha}{}^{\alpha} + 2\,{}^{(1)}b^{\alpha}_{,\alpha} \tag{59} $$

Now we are in position to explicitly write the field Eq. (50). By taking into account (44), we may now suppose that the function $L(R)$ has the explicit form

$$ L(R) = R + \alpha f(R) \tag{60} $$

where α is a constant parameter, and $f(R)$ is some function that for simplicity we may think to be as a polynomial of degree higher than one. A similar analysis holds for the non-polynomial but still real analytic functions $f(R)$. Consequently, up to linear order we have

$$ L'(0) = 1 \quad L\big({}^{(1)}R\big) \simeq {}^{(1)}R \tag{61} $$

and field equations become:

$$ {}^{(1)}R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,{}^{(1)}R = \kappa\,{}^{(1)}T^{\mathrm{mat}}_{\mu\nu} \tag{62} $$

By substituting the expression of the Ricci tensor and the scalar curvature (58), (59), we obtain

$$-\frac{1}{2}\,{}^{(1)}g_{\mu\nu,\alpha}{}^{\alpha}+\frac{1}{4}\eta_{\mu\nu}\,{}^{(1)}g^{\mu}{}_{\mu,\alpha}{}^{\alpha}=\kappa\,{}^{(1)}T^{\rm mat}_{\mu\nu}+\frac{1}{2}\eta_{\mu\nu}\,{}^{(1)}b^{\alpha}{}_{,\alpha} \tag{63}$$

Now, from (34), up to first order we may write

$${}^{(1)}b^{\alpha}{}_{,\alpha}\simeq-\kappa\frac{L''(0)}{(L'(0))^2}\,{}^{(1)}T^{\rm mat}{}_{,\alpha}{}^{\alpha} \tag{64}$$

Furthermore we may introduce the tensor $\overline{h}_{\mu\nu}$ defined by

$$\overline{h}_{\mu\nu}\doteq{}^{(1)}g_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\,{}^{(1)}g^{\alpha}{}_{\alpha} \tag{65}$$

Then by means of (64) and (65) the field Eq. (63) simplify to

$$\Box\left[\overline{h}_{\mu\nu}-\eta_{\mu\nu}\kappa L''(0)\,{}^{(1)}\tau\right]=-2\kappa\,{}^{(1)}T^{\rm mat}_{\mu\nu} \tag{66}$$

If we set

$$L(R)=R+\alpha f(R)=R+\alpha R^2+P(R) \tag{67}$$

where $P(R)$ is a polynomial of degree higher than 2, the field Eq. (66) can consequently be written in the form

$$\Box\left[\overline{h}_{\mu\nu}-2\eta_{\mu\nu}\kappa\alpha\,{}^{(1)}T\right]=-2\kappa\,{}^{(1)}T^{\rm mat}_{\mu\nu} \tag{68}$$

By setting

$$\mathcal{H}_{\mu\nu}\doteq\overline{h}_{\mu\nu}-2\eta_{\mu\nu}\kappa\alpha\tau^{\rm mat} \tag{69}$$

the field equations take the simple expression

$$\Box\mathcal{H}_{\mu\nu}=-2\kappa\,{}^{(1)}T^{\rm mat}_{\mu\nu} \tag{70}$$

The solution of (70) can now be written in terms of retarded potentials:

$$\mathcal{H}_{\mu\nu}=4\frac{G}{c^4}\int\frac{T^{\rm mat}_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' \tag{71}$$

where we have explicitly written $\kappa=8\pi G/c^4$. Hence

$$\overline{h}_{\mu\nu}=4\frac{G}{c^4}\int\frac{T^{\rm mat}_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'+\frac{16\pi G}{c^4}\eta_{\mu\nu}\,\alpha\tau^{\rm mat} \tag{72}$$

From the above calculations, which have been performed step by step to exactly clarify what happens, it turns out that the first order perturbation does not influence the form of field Eq. (50), while it enters into the definition of the perturbed Ricci tensor (51) and consequently of the scalar curvature.

Specifying to the case of the Lagrangian (67) we see that the solution of the perturbed field equations, written in terms of the retarded potentials, contains two terms: (i) the first one, in the weak field approximation, reduces to the standard Newtonian potential, (ii) the second one is related to the Lagrangian chosen for the alternative theory of gravity. In particular it vanishes in the limit $\alpha \to 0$, i.e. exactly reproducing the weak field limit of standard General Relativity. It is thus clear that α can be identified with a scale parameter, vanishing at small (solar system) scales and consequently reproducing General Relativity. The same reasoning can be done by supposing that all the term $\alpha f(R)$ becomes in fact irrelevant at solar system scales.

4.2 Perturbation of the de Sitter space-time

The calculation performed in the previous sub-section can be generalized to the case of space-times which do not admit a Minkowski background solution. However, as we have seen in the previous sub-section, having a Minkowski solution heavily constrains the available Lagrangians, since it implies that $L(R=0)=0$: in particular, these Lagrangians are not interesting for cosmological applications (see [3] and [12]). This case was already studied in [12], where it is shown that theories with singular $L(R)$ and $\frac{d^2L}{dR^2}({}^{(0)}R) = 0$ provide the correct Newtonian limit and they are good candidates to explain the cosmic acceleration.

We want hereafter to comment this case and to apply it to the particular Lagrangian $L(R) = R+\alpha f(R)$. We skip calculations as they can be reproduced step by step following the headlines of the previous chapter and moreover they have been already performed in [12]. We consider as a background metric the (anti) de Sitter metric:

$$ {}^{(0)}g = -dt^2 + e^{2t\sqrt{\frac{\Lambda}{3}}}(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2) \tag{73} $$

which satisfies the field equations:

$$ {}^{(0)}R_{\mu\nu} = -\Lambda\ {}^{(0)}g_{\mu\nu} \tag{74} $$

Considering the Lagrangian $L(R) = R + \alpha f(R)$ we obtain that the resulting Newtonian potential is (see [12] for details on calculations):

$$ V(\mathbf{x}) = e^{2t\sqrt{\frac{\Lambda}{3}}} C \int \frac{\rho(\mathbf{x}') \exp(-|\mathbf{x}-\mathbf{x}'| e^{2t\sqrt{\frac{\Lambda}{3}}})}{\mathbf{x}-\mathbf{x}'} d^3x' + A\rho(\mathbf{x}') \tag{75} $$

where $\rho(\mathbf{x})$ represents the energy density while:

$$ \begin{cases} A = \alpha \dfrac{\kappa\ {}^{(0)}f''}{2\ {}^{(0)}L'(4\Lambda\alpha\ {}^{(0)}f'' + {}^{(0)}L')} \\[2ex] C = \dfrac{8\Lambda\alpha\ {}^{(0)}f''\ {}^{(0)}L' + ({}^{(0)}L')^2}{({}^{(0)}L')^2(4\Lambda\alpha\ {}^{(0)}f'' + {}^{(0)}L')} \end{cases} $$

Also in this case it is evident that in the limit $\alpha \to 0$, for any current experiment and observation, the first term in (75) reduces to the standard Newtonian potential [12], and once again we obtain the weak field limit of standard General Relativity. Moreover α naturally behaves as a Post-Newtonian parameter in the potential, which is supposed to vanish at small scales. Once more α behaves like a scale parameter and the accordance with experimental results is supported.

5 Conclusions

In this paper we have shown that, both in the case of vacuum universes and in the case of matter universes, solar system experiments can be theoretically explained and reproduced in the framework of alternative theories of Gravity for specific classes of Lagrangians. The gravitational potential of alternative theories of Gravity reduces, under suitable hypotheses, to the standard Newtonian potential at the solar system scale. This has been proven both in the case of vacuum and matter universes (with a flat or an (anti) de-Sitter background). Gravitational effects due to the (alternative) form of the Lagrangian generate Post-Newtonian parameters appearing in the gravitational potential, which vanish when the corrections to the standard Hilbert Lagrangian are cancelled. Moreover we stress that these corrections are negligible when we consider values of the scale parameter α which is necessary to explain cosmic acceleration (see e.g. [3]). These contributions become however relevant when considering larger scales (cosmology [6] and, hopefully, galactic scales). This implies that higher order corrections to the standard Hilbert-Einstein theory could behave as a scale effect, ruled by a scale parameter which vanishes at solar system scales. General Relativity is consequently reproduced at the solar system scale, as it has to be surely expected.

The results obtained here slightly differ from some results already presented in literature [6] and in particular from recent results obtained by G.J. Olmo, (see again [6]). It was there argued that only small corrections to the Hilbert-Einstein Lagrangian can pass the solar system experiments. However, calculations were there performed by means of a conformal transformation on a flat Minkowski background spacetime. We have here shown that in the particular case of a flat spacetime, the theory $\frac{1}{R}$ is not viable, owing to the conditions (44). This implies that the results obtained by G.J. Olmo does not exclude the reliability of $\frac{1}{R}$-like theories, which should however be examined in the (anti)de-Sitter background framework. In fact $R = 0$ is singular for all Lagrangians which contain inverse powers or logarithms. Moreover, it is not at all evident that our universe should be asymptotically flat. We have here proven the accordance of such theories with solar system experiments, at least when the scale (post-Newtonian) parameter becomes small enough.

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