RESEARCH ARTICLE

M. Chaisi · S. D. Maharaj

Compact anisotropic spheres with prescribed energy density

Abstract New exact interior solutions to the Einstein field equations for anisotropic spheres are found. We utilise a procedure that necessitates a choice for the energy density and the radial pressure. This class contains the constant density model of Maharaj and Maartens (Gen. Rel. Grav. **21**, 899–905 (1989)), and the variable density model of Gokhroo and Mehra (Gen. Rel. Grav. **26**, 75–84 (1994)), as special cases. These anisotropic spheres match smoothly to the Schwarzschild exterior and gravitational potentials are well behaved in the interior. A graphical analysis of the matter variables is performed which points to a physically reasonable matter distribution.

Keywords Anisotropic relativistic stars · Compact spheres

1 Introduction

In recent years a number of authors have studied solutions to the Einstein field equations corresponding to anisotropic matter where the radial component of the pressure differs from the angular component. The gravitational field is taken to be spherically symmetric and static since these solutions may be applied to relativistic stars. A number of researchers have examined how anisotropic matter affects the critical mass, critical surface redshift and stability of highly compact bodies. These investigations are contained in the papers [3, 6, 7, 10, 12, 15, 16, 20],

M. Chaisi
Astrophysics and Cosmology Research Unit, School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, South Africa; Department of Mathematics & Computer Science, National University of Lesotho, Roma 180, Lesotho
E-mail: m.chaisi@nul.ls

S. D. Maharaj (✉)
Astrophysics and Cosmology Research Unit, School of Mathematical Sciences, University of KwaZulu-Natal, Durban 4041, South Africa
E-mail: maharaj@ukzn.ac.za

among others. Some researchers have suggested that anisotropy may be important in understanding the gravitational behaviour of boson stars and the role of strange matter with densities higher than neutron stars. Mak and Harko [15] and Sharma and Mukherjee [22] suggest that anisotropy is a crucial ingredient in the description of dense stars with strange matter.

In this paper our objective is to generate a new class of exact solutions to the Einstein field equations corresponding to a physically reasonable form for the energy density. A particular motive is to find simple analytic forms for the gravitational and matter variables so that the physical interpretation of the model is simplified. Often the solutions are presented in terms of special functions or a numerical approach is required [6, 7]. We hope that our results in terms of elementary functions will assist in the analysis of gravitational behaviour of compact objects, and the study of anisotropy under strong gravitational fields. In Sect. 2 we develop the anisotropic stellar model and present the relevant field equations. A particular form for the energy density is chosen in Sect. 3, and the Einstein field equations are integrated. Special cases of physical interest are isolated from the general solution in Sect. 4. Some physical features of the anisotropic star are briefly considered in Sect. 5. In Sect. 6 we demonstrate that our model yields surface redshifts and masses that correspond to real sources, and make a few concluding remarks and suggestions for future research.

2 The anisotropic model

The line element for static spherically symmetric spacetimes is given by

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary functions. We study non-radiating relativistic spheres with anisotropic stress, and the energy-momentum tensor is of the form

$$T^{ab} = \mu u^a u^b + p h^{ab} + \pi^{ab} \tag{2}$$

where μ is the energy density, p the isotropic pressure, and the projection tensor $h^{ab} = u^a u^b + g^{ab}$ is measured relative to the four-velocity u^a. It is convenient to express the anisotropic stress in the form

$$\pi^{ab} = \sqrt{3}S(r)\left(c^a c^b - \frac{1}{3}h^{ab}\right)$$

where the unit spacelike vector **c** is orthogonal to the fluid four-velocity **u** and $|S(r)|$ is the magnitude of the stress tensor. This representation for π^{ab} is a consequence of the symmetries of the static spherically symmetric spacetimes [13]. The quantity S is a useful device to introduce

$$p_r = p + 2S/\sqrt{3}, \quad p_\perp = p - S/\sqrt{3}$$

which are the radial and tangential pressures respectively. Note that for isotropic matter $S = 0$ and $p_r = p_\perp = p$. The magnitude S provides a measure of anisotropy. We assume that the fluid four-velocity is comoving. This assumption implies that $u^a = e^{-\nu/2}\delta_0^a$, $c^a = e^{-\lambda/2}\delta_1^a$.

Using (1) and (2), the Einstein field equations become

$$-\frac{e^{-\lambda}}{r^2}(1 - \lambda' r - e^\lambda) = \mu \tag{3a}$$

$$\frac{e^{-\lambda}}{r^2}(1 - e^\lambda + r\nu') = p_r \tag{3b}$$

$$\frac{e^{-\lambda}}{4}\left(2\nu'' - \nu'\lambda' + \nu'^2 + \frac{2\nu'}{r} - \frac{2\lambda'}{r}\right) = p_\perp \tag{3c}$$

for static spherically symmetric anisotropic matter. We are using units where the speed of light and the coupling constant are unity. The momentum conservation equation leads to

$$(\mu + p_r)\nu' + 2p_r' + \frac{4}{r}(p_r - p_\perp) = 0 \tag{4}$$

for the spacetime (1). This conservation equation is not independent and can be generated directly from the field Eq. (3). We define the mass function as

$$m(r) = \frac{1}{2}\int_0^r x^2\mu(x)\,\mathrm{d}x \tag{5}$$

following the treatment of Stephani [23]. With the help of (4) and (5) we can integrate (3a), and then get the equivalent system

$$e^{-\lambda} = 1 - \frac{2m}{r} \tag{6a}$$

$$r(r - 2m)\nu' = p_r r^3 + 2m \tag{6b}$$

$$(\mu + p_r)\nu' + 2p_r' = -\frac{4}{r}(p_r - p_\perp). \tag{6c}$$

The system (6) has the advantage of being a first order system of differential equations, and is linear in the gravitational potential ν which simplifies the integration process. For certain applications it is easier to use (6) rather than the original second order system (3), which is the approach that we follow in this paper. We seek explicit solutions to the Einstein field equations that describe realistic anisotropic relativistic stars by utilising an algorithm that was initially proposed by Maharaj and Maartens [14]. In their approach they expressed the field equations as the first order system of differential Eq. (6). The energy density μ (or equivalently m) and the radial pressure p_r are chosen on physical grounds. The remaining relevant quantities $(e^\nu, e^\lambda, p_\perp)$ then follow from the field equations. Note that $(e^\nu, e^\lambda, \mu$ (or m), $p_r, p_\perp)$ are not independent; there are five unknown functions and three field equations so that we have the freedom to choose any two of the quantities. In this paper we make explicit choices for μ and p_r.

3 General solution to the field equations

It is convenient to make the following choice for the energy density

$$\mu = \frac{j}{r^2} + k + \ell r^2 \tag{7}$$

where j, k and ℓ are constants. The roles of j, k and ℓ in the physics of the model are highlighted in examples considered later. An advantage of this form for μ is that it contains particular cases studied previously. Then (5) yields the following expression for the mass function

$$m = \frac{r}{2} \left(j + \frac{k}{3} r^2 + \frac{\ell}{5} r^4 \right) \tag{8}$$

with the particular energy density (7). Equation (6a) gives

$$e^{-\lambda} = 1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4 \tag{9}$$

and the gravitational potential λ has been determined.

With the help of (8), we can write (6b) as

$$
\begin{aligned}
\nu' &= \frac{r p_r}{1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4} + \frac{j + \frac{k}{3} r^2 + \frac{\ell}{5} r^4}{r \left(1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4 \right)} \\
&= \frac{r p_r}{1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4} + \frac{j}{r (1 - j)} \\
&\quad + \frac{\frac{k}{3} r + \frac{\ell}{5} r^3}{(1 - j) \left(1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4 \right)}
\end{aligned}
\tag{10}
$$

where we have used partial fractions. On integration, (10) can be expressed as

$$\nu = I_1 + \frac{j}{1 - j} \ln r + \frac{1}{1 - j} I_2 + \ln B \tag{11}$$

where $\ln B$ is a constant of integration and we have set

$$I_1 = \int \frac{r p_r}{1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4} \mathrm{d}r$$

$$I_2 = \int \frac{\frac{k}{3} r + \frac{\ell}{5} r^3}{1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4} \mathrm{d}r$$

At this point we could choose a barotropic equation of state $p_r = p_r(\mu)$. However this is an approach that we intend to follow in future work. In this treatment we make a choice for the radial pressure p_r which is physically reasonable and is a generalisation of earlier studies. We make the choice

$$p_r = \frac{C}{1 - j} \left(1 - j - \frac{k}{3} r^2 - \frac{\ell}{5} r^4 \right) \left(1 - \frac{r^2}{R^2} \right)^n. \tag{12}$$

When $j = \ell = 0$, we obtain the radial pressure postulated by Maharaj and Maartens [14]. For $j = 0$, we regain the radial pressure of Gokhroo and Mehra [8]. The form (12) for p_r is physically reasonable because $p_r > 0$ in the interval $(0, R)$ for relevant choices of the constants, $p_r = C$ at the centre $r = 0$, $p_r = 0$ at the boundary $r = R$, and p_r is continuous and well behaved in the interval $[0, R]$.

The first integral I_1 simplifies to

$$I_1 = \frac{C}{1-j} \int \left(1 - \frac{r^2}{R^2}\right)^n r \, dr$$

$$= -\frac{CR^2}{2(1-j)(n+1)} \left(1 - \frac{r^2}{R^2}\right)^{n+1}$$

for the choice of (12). To evaluate the second integral I_2 we need to consider two cases: $\ell = 0$ and $\ell \neq 0$.

Case I: $\ell = 0$.

In this case the integration is straightforward and we obtain

$$I_2 = -\frac{1}{2} \ln \left\{1 - j - \frac{k}{3} r^2\right\}$$

Case II: $\ell \neq 0$

For this case we let

$$u = r^2 + \frac{5k}{6\ell}, \quad q^2 = 1 - j + \frac{5k^2}{36\ell}$$

and obtain

$$I_2 = \int \frac{\frac{\ell r}{5}\left(r^2 + \frac{5k}{3\ell}\right)}{1 - j + \frac{5k^2}{36\ell} - \frac{\ell}{5}\left(r^2 + \frac{5k}{6\ell}\right)^2} \, dr$$

$$= \frac{\ell}{10} \int \frac{u + \frac{5k}{6\ell}}{q^2 - \frac{\ell}{5} u^2} \, du$$

$$= \frac{\ell}{10} \left(-\frac{5}{2\ell} \ln\left\{q^2 - \frac{\ell}{5} u^2\right\} + \frac{5k}{6\ell}\left(\frac{\sqrt{5}}{q\sqrt{\ell}}\right) \tanh^{-1}\left\{\frac{u\sqrt{\ell}}{q\sqrt{5}}\right\}\right).$$

Hence we can collectively write for both *Case I* and *Case II* that

$$
I_2 =
\begin{cases}
-\dfrac{1}{2}\ln\left\{1 - j - \dfrac{k}{3}r^2\right\}, & \text{for } \ell = 0 \\[4mm]
-\dfrac{1}{4}\ln\left\{1 - j + \dfrac{5k^2}{36\ell} - \dfrac{\ell}{5}\left(r^2 + \dfrac{5k}{6\ell}\right)^2\right\} \\[4mm]
+\left(\dfrac{5}{\ell}\right)^{\frac{1}{2}}\left(\dfrac{k}{12\sqrt{1 - j + \dfrac{5k^2}{36\ell}}}\right)\tanh^{-1}\left\{\left(\dfrac{\ell}{5}\right)^{\frac{1}{2}}\dfrac{r^2 + \dfrac{5k}{6\ell}}{\sqrt{1 - j + \dfrac{5k^2}{36\ell}}}\right\}, \\[4mm]
\quad \text{for } \ell \neq 0
\end{cases}
\tag{13}
$$

The integrals I_1 and I_2 are given in terms of elementary functions which helps in the physical analysis of the model.

On substituting I_1 in (11) we obtain

$$
e^\nu = B r^{\frac{j}{1-j}} \exp\left\{\frac{I_2}{1-j} - \frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\}
\tag{14}
$$

for the gravitational potential e^ν where $I_2 = I_2(r)$ has the functional representation given above in (13) for $\ell = 0$ and $\ell \neq 0$. To match the interior solution to the Schwarzschild interior we require that $e^{\nu(R)} = 1 - 2M/R$ which implies that $B = R^{\frac{-j}{1-j}}(1 - 2M/R)\exp\{-I_2(R)/(j-1)\}$. Finally the last field Eq. (6c) gives the tangential pressure $p_\perp$:

$$
\begin{aligned}
p_\perp = p_r &+ \frac{C}{2(1-j)}\left(j - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n \\
&+ \frac{r^2}{2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1} \\
&\times \left\{\frac{C^2}{2(1-j)^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n}\right. \\
&\quad - \frac{2nC}{(1-j)R^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1} \\
&\quad \left. + \frac{1}{2r^2}\left(\frac{j}{r^2} + k + \ell r^2\right)\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right)\right\}
\end{aligned}
\tag{15}
$$

where we have used (7), (12) and (14). The anisotropic factor $S(r)$ is given by

$$
S = -\frac{C}{2\sqrt{3}\,(1-j)}\left(j - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n
$$
$$
-\frac{r^2}{2\sqrt{3}}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1}
$$
$$
\times\left\{\frac{C^2}{2(1-j)^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n}\right.
$$
$$
-\frac{2nC}{(1-j)R^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1}
$$
$$
\left.+\frac{1}{2r^2}\left(\frac{j}{r^2} + k + \ell r^2\right)\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right)\right\}
\tag{16}
$$

which follows from (12) and (15).

Thus we have generated a new class of solutions to the Einstein field Eq. (6). Collecting the various results given above we can express the exact solution as

$$
\mu = \frac{j}{r^2} + k + \ell r^2
\tag{17a}
$$

$$
p_r = \frac{C}{1-j}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n
\tag{17b}
$$

$$
p_\perp = p_r + \frac{C}{2\,(1-j)}\left(j - \frac{\ell}{5}r^4\right)\left(1 - \frac{r^2}{R^2}\right)^n + \frac{r^2}{2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^{-1}
$$
$$
\times\left\{\frac{C^2}{2(1-j)^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{2n}\right.
$$
$$
-\frac{2nC}{(1-j)R^2}\left(1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4\right)^2\left(1 - \frac{r^2}{R^2}\right)^{n-1}
$$
$$
\left.+\frac{1}{2r^2}\left(\frac{j}{r^2} + k + \ell r^2\right)\left(j + \frac{k}{3}r^2 + \frac{\ell}{5}r^4\right)\right\}
\tag{17c}
$$

$$
e^\nu = B r^{\frac{j}{1-j}} \exp\left\{\frac{I_2}{1-j} - \frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\}
\tag{17d}
$$

$$
e^\lambda = \frac{1}{1 - j - \frac{k}{3}r^2 - \frac{\ell}{5}r^4}
\tag{17e}
$$

where I_2 (given in (13)) contains the two cases, $\ell = 0$ and $\ell \neq 0$. The exact solution (17) represents the interior of an anisotropic star corresponding to the energy density $\mu = j/r^2 + k + \ell r^2$. Clearly other choices for μ and p_r will yield

new solutions to the field equations; however these choices may not correspond to realistic matter or the integrals I_1 and I_2 may not be expressible in closed form. This solution does not have a barotropic equation of state $p_r = p_r(\mu)$. To obtain a model with an equation of state we need to specify this explicitly when evaluating the integral I_1. An equation of state is a desirable physical feature which we hope to incorporate in future models. Note that the cosmological constant is absent from our model. This quantity can be easily included by adding a constant to the energy density and the pressure function.

4 Special cases

We consider some special cases contained in the new class of solution presented in Sect. 3; two of these cases lead to particular models that have been studied previously.

Solution I: $j = 0$

In this case the energy density is given by

$$\mu = k + \ell r^2$$

and the line element has the form

$$\mathrm{d}s^2 = - \left(B \exp\left\{ I_2 - \frac{CR^2}{2(n+1)} \left(1 - \frac{r^2}{R^2} \right)^{n+1} \right\} \right) \mathrm{d}t^2$$
$$+ \left(1 - \frac{k}{3}r^2 - \frac{\ell}{5}r^4 \right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \qquad (18)$$

The particular solution (18) was found by Gokhroo and Mehra [8]. Their solution is regained when we set $k = \rho_0$, $\ell = -\rho_0 K/a^2$. Note that if we require $\mu' < 0$ then the constant $\ell < 0$ for a monotonically decreasing energy density as we approach the boundary $r = R$ from the centre.

Solution II: $j = \ell = 0$

For this case the energy density

$$\mu = k$$

is a constant. The line element has the representation

$$\mathrm{d}s^2 = - \left(B \exp\left\{ -\frac{1}{2} \ln\left(1 - \frac{k}{3}r^2 \right) - \frac{CR^2}{2(n+1)} \left(1 - \frac{r^2}{R^2} \right)^{n+1} \right\} \right) \mathrm{d}t^2$$
$$+ \left(1 - \frac{k}{3}r^2 \right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \qquad (19)$$

The particular solution (19) was found by Maharaj and Maartens [14]. Their solution is regained when we let $k = 6M/R^3$. Since μ is constant we may interpret

this solution as an anisotropic generalisation of the incompressible Schwarzschild interior sphere; however note that the anisotropy factor $S(r) \neq 0$ everywhere except at the centre $r = 0$.

Solution III: $k = \ell = 0$

In this case the energy density has the form

$$\mu = \frac{j}{r^2}$$

The line element is given by

$$
\begin{aligned}
\mathrm{d}s^2 = -Br^{\frac{j}{1-j}} \\
\times \exp\left\{-\frac{1}{2}\ln(1-j) - \frac{CR^2}{2(1-j)(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right\}\mathrm{d}t^2 \\
+ (1-j)^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)
\end{aligned}
\tag{20}
$$

Even though (20) has a very simple form, we believe that it is a new anisotropic solution to the Einstein field equations and has not been published before. Since $\mu \propto r^{-2}$ we may relate (20) to the results of other treatments. Dev and Gleiser [6], Herrera and Santos [11] and Petri [19] found solutions to the anisotropic Einstein field equations involving $\mu \propto r^{-2}$. In each of these papers a different set of assumptions to that utilised in this paper was used; in our treatment we have chosen a form for the radial pressure p_r. Therefore their solutions are necessarily different from (20) for the corresponding energy density choice $\mu \propto r^{-2}$.

5 Physical conditions and analysis

One of the original reasons for studying anisotropic matter was to generate models that permit redshifts higher than the critical redshift z_c of isotropic matter [3]. Observational results indicate that certain isolated objects have redshifts higher than z_c. The surface redshift is given by

$$z = \left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} - 1$$

The critical redshift $z_c = 2$ is the limiting value for the perfect fluid spheres, and is attained when $2M/R = 8/9$ [4]. For the range of values falling in the interval $8/9 < 2M/R < 1$ the redshift is greater than z_c; this phenomenon may be explained by allowing for anisotropy. For values of $2M/R$ close to unity, the surface redshift becomes infinitely large. The feasibility of higher redshifts for anisotropic matter, in both Newtonian and relativistic models, was firmly established by Bondi [2]. It is interesting to note that Bondi, Binney and Tremaine [1], Cuddeford [5] and Michie [17] emphasise the significance of anisotropies in stellar clusters and galaxies, in addition to individual stars.

The gravitational potential e^λ is finite at the centre $r = 0$ and at the boundary $r = R$. The function e^λ is well behaved in the interior of the relativistic star. The

gravitational potential e^ν is continuous and well behaved in the interior and finite at the boundary of the star $r = R$. There is a singularity at the centre $r = 0$ in the potential e^ν. The singularity in e^ν is removable for a specific choice of parameter values. This singularity is eliminated by setting $j = 0$ which corresponds to the solution of Gokhroo and Mehra [8].

The energy density (7) chosen describes relativistic stars as we demonstrate later. The form of $\mu \propto r^{-2}$ ($k = \ell = 0$) is usually used in domains where it is not possible to use a single equation of state; particularly where the origin is excluded, like a body with a constant density core and matter density distribution around the core going like r^{-2} [6, 22]. It is interesting to observe that the r^{-2} profile in the energy density also arises in isothermal spheres in Newtonian configurations that correspond to a Maxwell-Boltzmann gas in galactic systems [21]. Densities with $j \neq 0$ and $k \neq 0$ are also physically reasonable. For example, Misner and Zapolsky [18] propose that the term jr^{-2} models the physical configuration of a relativistic Fermi gas for some particular value of the parameter j. Another example is due to Dev and Gleiser [6] who suggest that for some particular value of j and $k \neq 0$ the energy density function $jr^{-2} + k$ describes a relativistic Fermi gas core immersed in a constant density background.

The radial pressure p_r is continuous and well behaved in the interior of the star. Also $p_r > 0$ in the interval $(0, R)$, regular at the centre $(p_r(r = 0) = C)$, and vanishes at the boundary $(p_r(r = R) = 0)$. The tangential pressure $p_\perp$ has a singularity at the centre, but is otherwise well behaved throughout the interior of the star and finite at the boundary. The singularity in $p_\perp$ may be eliminated by suitable particular choice of parameter values. In general the tangential pressure is not zero at the boundary of the star $(p_\perp(r = R) \neq 0)$ which is different from the radial pressure $(p_r(r = R) = 0)$. It is also important to observe that the magnitude of the stress tensor

$$S = \frac{1}{\sqrt{3}} (p_r - p_\perp)$$

is a nonzero function in general. Hence this class of solutions is generally anisotropic and does not have an isotropic limit (the isotropic limit results when we set particular values for the constants in our ansatz). It is not possible to eliminate S and obtain an isotropic counterpart. This means that the model remains anisotropic. An analogous situation rises in Einstein-Maxwell solutions modelling charged relativistic stars in which the electric field is always present. An example of such a charged star is given by Hansraj [9].

Figures 1–3 are illustrations of the behaviour of the energy density μ, the radial pressure p_r, and the anisotropy factor S respectively, for particular chosen values of the constants in the exact solution (17). The radial distance is over the interval $0 \leq r \leq 1$ and the boundary of the star has been normalised to be $r = R = 1$. Note that Plot A corresponds to $j = 0$ case (*Solution I*), Plot B corresponds to $j = \ell = 0$ (*Solution II*) and Plot C corresponds to the general solution (17) where $j \neq 0$, $k \neq 0$, and $\ell \neq 0$. In Fig. 1 for μ, Plots A and B are continuous throughout the interval $0 \leq r \leq 1$; however Plot C indicates unphysical behaviour as we approach the centre. This undesirable feature in Plot C arises because $j \neq 0$ and indicates that another solution has to be utilised around the centre in a core-envelope model [6, 11, 19]. In Fig. 2 for p_r, the radial pressure is monotonically
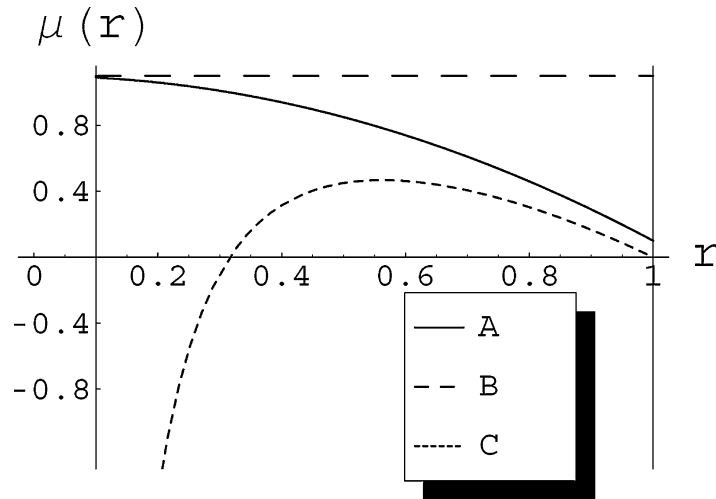
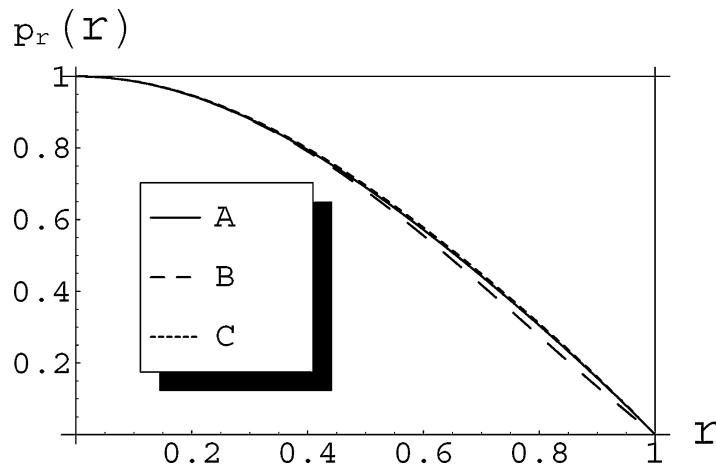**Fig. 1** Energy density $\mu(r)$ plots

Fig. 2 Radial pressure $p_r(r)$ plots

decreasing from the centre to the boundary for all Plots A, B, and C. In Fig. 3 for S, Plots A and B are continuous throughout the interval $0 \leq r \leq 1$; however Plot C indicates a singularity as we approach $r = 0$. We suspect that this singularity in Plot C is related to the fact that $j \neq 0$. We observe that the gradient of S is greatest for Plot C, corresponding the case for $j \neq 0$, $k \neq 0$, and $\ell \neq 0$, as the boundary is approached. Hence $S(R)$ has the largest value at the boundary for the general solution (17) in this case. The behaviour of S outside the centre is likely to correspond to physically reasonable anisotropic matter: Plot B has a profile similar to the behaviour of the anisotropic boson stars studied by Dev and Gleiser [6]. However the general solution obtained by Dev and Gleiser [6] for the choice $\mu = jr^{-2} + k$ is given in terms of hypergeometric functions. Our corresponding solution has the advantage of being expressed in terms of elementary functions.
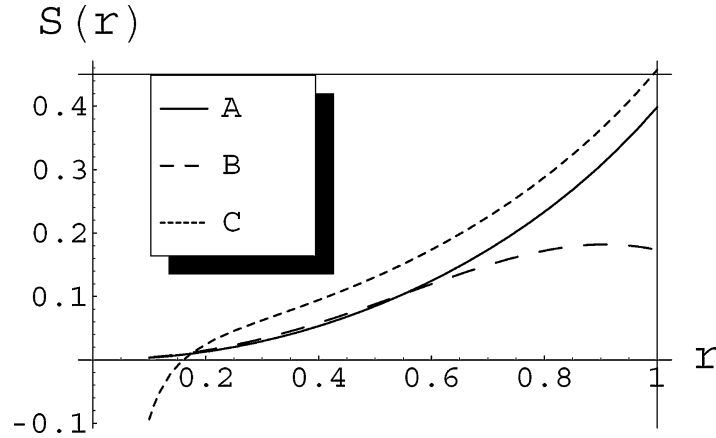
Fig. 3 Anisotropy factor $S(r)$ plots

6 Discussion

It is possible to demonstrate that the solutions found can be utilised to discuss the structure of neutron stars and quasi-stellar objects. We write the surface density in the particular form $\mu_s = \mu_0(1 - \tilde{j} - \tilde{\ell})$ where μ_0 is the central density and the constants j, k and ℓ have been chosen so that the energy density can be easily expressed in c.g.s. units. Now consider a neutron star of radius 10km and surface density of 2×10^{14} gcm^{-3}. Then the parameters $\mu_0, \mu_0 R^2, 2M/R$, surface redshift $z = (1 - 2M/R)^{-1/2} - 1$, and mass M in terms of the solar masses $M_\odot$ can be calculated. We choose values of $\tilde{j}$ and $\tilde{\ell}$ so that comparison with Gokhroo and Mehra [8] is facilitated. The results are given in Table 1. In this category of results the surface redshifts range up to 0.566, and masses extend to $2.00M_\odot$. This range of values is consistent with the results of Gokhroo and Mehra [8]. Hence our solutions yield values for surface redshifts and masses that correspond to realistic stellar sources such as Her X-1 and Vela X-1. Clearly higher values for z and M can be generated by adjusting $\tilde{j}$ and $\tilde{\ell}$.

Table 1 Densities and redshifts for neutron stars

$\tilde{j}$	$\tilde{\ell}$	$\mu_0 \times 10^{14}$	$\mu_0 R^2$	$2M/R$	z	$M(M_\odot)$
0	0	1.48	.015	.124	.068	.42
.001	.1	1.65	.017	.130	.072	.44
.002	.2	1.86	.019	.136	.076	.46
.003	.3	2.13	.021	.145	.081	.49
.004	.4	2.49	.025	.156	.089	.53
.005	.5	3.00	.030	.172	.099	.58
.006	.6	3.77	.038	.196	.116	.66
.007	.7	5.07	.051	.237	.145	.80
.008	.8	7.73	.077	.321	.214	1.09
.009	.9	16.32	.163	.592	.566	2.00

In this paper we have found a new class of solutions to the Einstein field equations for an anisotropic matter distribution utilising the algorithm of Maharaj and Maartens [14]. These solutions correspond to the energy density $\mu = jr^{-2} + k + \ell r^2$ and contain particular solutions found previously. We note that the term containing jr^{-2} is physically important and arises in a number of applications [6, 7, 18, 22]. Our results indicate that anisotropic solutions for the physically reasonable energy density $\mu \propto r^{-2}$ can be generated with the simple solution generating mechanism of Maharaj and Maartens [14]. Our ongoing investigations indicate that a general class of anisotropic models are possible, for different choices of μ, such that the desired limit $\mu \propto r^{-2}$ is regained as we approach the boundary. This work is in preparation. Observe from (16) that the anisotropy factor S is nonzero in general in the interior of the star. This means that the exact solution (17) remains anisotropic and does not have an isotropic limit. We would need to use another approach of integrating the anisotropic Einstein field equations than the algorithm used in this paper, if an isotropic limit is to be contained in the stellar model.

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