RESEARCH ARTICLE

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Smarr's formula for black holes with non-linear electrodynamics

Abstract It is known that for nonlinear electrodynamics the First Law of Black Hole Mechanics holds, however the Smarr's formula for the total mass does not. In this contribution we discuss the point and determine the corresponding expressions for the Bardeen black hole solution that represents a nonlinear magnetic monopole. The same is done for the regular black hole solution derived by Ayón–Beato and García [1], showing that in the case that variations of the electric charge are involved, the Smarr's formula is no longer valid.

Keywords Black hole · Magnetic monopole

The Reissner–Nordström (RN) solution is the unique static and asymptotically flat solution to the Einstein–Maxwell equations for spherical symmetry. It represents a black hole characterized by its mass and electric charge. The geometry of the RN black hole is singular at the origin of the radial coordinate, then it has been a subject of research the construction of its regular generalizations. A good candidate for the source term of the Einstein equations is the (classical) stress–energy tensor of nonlinear electrodynamics. The recent renewal of interest in nonlinear electrodynamics has to do also with the fact that such theories arose as limiting cases of certain formulations of string theory.

Nonlinear or logarithmic electromagnetic Lagrangians coupled to gravity have been studied in an attempt to remove some of the singularities associated with charged black holes. The exterior of such black holes is, at large distances, the same as the usual (RN) black holes of Einstein–Maxwell theory. Close to the black hole, however, things may be very different. Quantities defined at the horizon of such black holes have been useful in obtaining a more accurate description of

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the physics near the black hole. In this regard, the concept of *isolated horizon* has been used to provide a full Hamiltonian treatment of black holes [2]. This framework has been applied successfully not only to Einstein–Maxwell theory but also in more general cases like non-Abelian gauge theories [3]. At this point it is relevant to question the role of the Laws of Black Hole Mechanics (BHM) in this description, i.e. in situations more general than Einstein–Maxwell fields.

The zeroth and first laws of BHM refer to equilibrium situations and small departures therefrom. First Law of BHM is an identity relating the changes in mass, angular momentum and horizon area of a stationary black hole when it is perturbed. The variation applies for perturbations from one stationary axisymmetric solution of Einstein equations to another; moreover, it has been shown that the validity of this law depends only on very general properties of the field equations [4]. For the horizon mass M_Δ, the first law when static spherically symmetric solutions are considered [3], is

$$\delta M_\Delta = \frac{\kappa}{8\pi} \delta a_\Delta + \Phi_\Delta \delta Q_\Delta, \tag{1}$$

where κ is the surface gravity at the horizon, a is the area of the horizon, Q is the electric charge and Φ is the electric potential; the subindex Δ indicates that the quantity is evaluated at the horizon of the black hole.

On the other side, the total mass is given by the Smarr's formula

$$M_\Delta = \frac{\kappa a_\Delta}{4\pi} + \Phi_\Delta Q_\Delta. \tag{2}$$

In the case of Einstein–Maxwell theory, it is possible to deduce one, Eq. (1), directly from the other, Eq. (2), using the homogeneity of the mass as a function of $\sqrt{a}$ and Q. In the work by Ashtekar et al. [2] the first law of BHM, for quantities defined only at the horizon, arises naturally as part of the requirements for a consistent Hamiltonian formulation. In the case of non-linear electrodynamics, however, one no longer has homogeneity of the mass function and a priori one has no reason to expect that either of them holds.

Previous work on this line includes the derivation of the first law of black hole physics for some nonlinear matter models [5]. Rasheed [6] studied the Zeroth and First Laws of BHM in the context of non-linear electrodynamics coupled to gravity. In this case, the Zeroth Law, which states that the surface gravity of a stationary black hole is constant over the event horizon, is shown to hold even if the Dominant Energy Condition [7] is violated. In the same paper, it is found that the usual First Law (the general mass variation formula) holds true for the case of non-linear electrodynamics but the formula for the total mass, known as Smarr's formula, does not.

However, we can propose the form that must have a Smarr-type formula for the horizon mass in order to be consistent with the variations expressed by the first law of BHM that indeed holds,

$$M_\Delta = \frac{\kappa a_\Delta}{4\pi} + \Phi_\Delta Q_\Delta + V(a_\Delta, Q_\Delta, P_\Delta), \tag{3}$$

where V is a so far undetermined potential that depends on the horizon parameters, $a_\Delta, Q_\Delta, P_\Delta$ (P_Δ is the magnetic charge), and also of the coupling constants

of the theory. In the variational principle this term plays no role, however in the Hamiltonian description it becomes essential.

Note that in the first law, Eq. (1), only variations of the electric charge are involved, and not variations of the magnetic charge. On the other hand, the horizon mass, Eq. (3) might depend on P_Δ through V.

The equations to determine the potential $V(a_\Delta, Q_\Delta, P_\Delta)$ arise from the condition that the first law holds and demanding consistency between Eq. (1) and the variations of Eq. (3),

$$
\begin{aligned}
a_\Delta \frac{\partial \beta}{\partial a_\Delta} + 8\pi r_\Delta Q_\Delta \frac{\partial \Phi}{\partial a_\Delta} + 8\pi r_\Delta \frac{\partial V}{\partial a_\Delta} &= 0, \\
\frac{r_\Delta}{2} \frac{\partial \beta}{\partial Q_\Delta} + Q_\Delta \frac{\partial \Phi}{\partial Q_\Delta} + \frac{\partial V}{\partial Q_\Delta} &= 0,
\end{aligned}
\tag{4}
$$

where $\beta = 1 - 2m'(r)$, $a_\Delta = 4\pi r_\Delta^2$ and r_Δ is the radius of the horizon (see Ref. [3] for details).

The condition of consistency determines the set of parameters that can vary independently, in this case, the magnetic charge becomes a function of the area and electric charge, $P_\Delta = P_\Delta(r_\Delta, Q_\Delta)$.

In what follows we shall determine the horizon mass in agreement with the first law of BHM for the Bardeen black hole and then for the regular black hole of ABG, both solutions of Einstein equations coupled with nonlinear electrodynamics.

We shall consider non-linear electrodynamics governed by an action of the form

$$
S = \int d^4 x \sqrt{-g} \{R(16\pi)^{-1} + \mathcal{L}\},
\tag{5}
$$

where R denotes the scalar curvature, $g := \det|g_{\mu\nu}|$ and $\mathcal{L}$, the electromagnetic part, is assumed to depend in nonlinear way on the invariants of the field strength tensor $F_{\mu\nu}$. As we mentioned above, this kind of fields have been studied with the aim of avoiding the singularities of black holes and other systems [8]. Successful advances on this line were the proposed nonlinear electrodynamics theory by Born and Infeld [9] which in fact succeeded in avoiding the electric field singularity at the charge position. Born–Infeld theory coupled to gravity were studied by Hoffmann and Infeld [10]. Also we must mention the pioneering work done by A. García as collaborator of Prof. J. Plebañski, in studying the problem of non-linear electrodynamics for the type D solutions of the Einstein–Born–Infeld coupled equations [11].

In the search for a regular black hole with nonlinear electrodynamics, there exists a no go theorem [12] which states that for Lagrangians depending on the invariant of the electromagnetic field, $\mathcal{L}(F)$, $F = F_{\mu\nu} F^{\mu\nu}$, with the Maxwell weak-field limit, there are no spherically symmetric static black hole solutions with a regular center. However, regular solutions with only *magnetic* charge may exist [13]. It is not excluded neither the possibility of regular solutions corresponding to Lagrangians depending on both invariants of the electromagnetic field, $\mathcal{L}(F, Q)$, $F = F_{\mu\nu} F^{\mu\nu}$, $Q = \tilde{F}_{\mu\nu} F^{\mu\nu}$, with $\tilde{F}_{\mu\nu}$ being the dual of $F_{\mu\nu}$. Recently, several solutions corresponding to regular black holes with nonlinear electrodynamics with Lagrangians of the form $\mathcal{L}(F)$, have been derived

[14]. Moreover, it has been of interest the interpretation given to the model of Bardeen for a regular black hole, as corresponding to a self-gravitating magnetic monopole. The Bardeen model was proposed some years ago as a regular black hole, however, only recently it has been shown [15] that it is an exact solution of the Einstein equations coupled to a kind of nonlinear electrodynamics characterized by the Lagrangian

$$\mathcal{L}(F) = \frac{2}{2sg^2} \left(\frac{2g^2F}{1 + \sqrt{2g^2F}} \right)^{5/2}. \tag{6}$$

The corresponding energy momentum tensor fulfills the weak energy condition and is regular everywhere. For a spherically symmetric space, the corresponding metric is given by

$$ds^2 = -\psi_B dt^2 + \psi_B^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7}$$

$$\psi_B = 1 - \frac{2m(r)}{r} = 1 - \frac{2mr^2}{(r^2 + g^2)^{3/2}}. \tag{8}$$

This solution is a self-gravitating magnetic monopole with charge g. The solution is regular everywhere, although the invariants of the electromagnetic field exhibit the usual singular behaviour of magnetic monopoles, $F = F^{\mu\nu}_{\mu\nu} = g^2/2r^4$. The asymptotic behaviour of the solution is

$$\psi_B = g_{tt} \approx 1 - \frac{2m}{r} + \frac{3mg^2}{r^3}, \tag{9}$$

it is this behavior at infinity, in which the constant g vanishes as $1/r^3$, and not as a Coulombian term ($1/r^2$), that allows to interpret the constant g as a magnetic charge.

The horizons are given by the roots of the equation $r = 2m(r)$. In this case these roots are not as easy to calculate as for RN. Hence the conditions which restrict the parameters in order that the solution corresponds to regions where $\psi_B \geq 0$ are more difficult to find. The Bardeen solution does not involve electric charge, then the horizon mass depends only on the area of the horizon,

$$M_\Delta = \frac{1}{8\pi} \int \kappa \, da = \int (1 - m') dr, \tag{10}$$

the condition that the horizon mass be positive, from Eq. (10), amounts to $m(r) \leq r$, this condition also guarantees that $g_{tt} \geq 0$. Using the expression for g_{tt} it amounts to $(r^2 + g^2)^3 \geq 4m^2r^4$. In this case when $g^2 = (16/27)m^2$ the two horizons that could be present shrink into a single one, being this value of g the corresponding to the extreme black hole; for $g^2 < (16/27)m^2$ there exist both inner and event horizon.

The potential V for the Smarr-type formula, Eq. (3), for the Bardeen black hole turns out to be, undetermined until an integration constant which we have put zero,

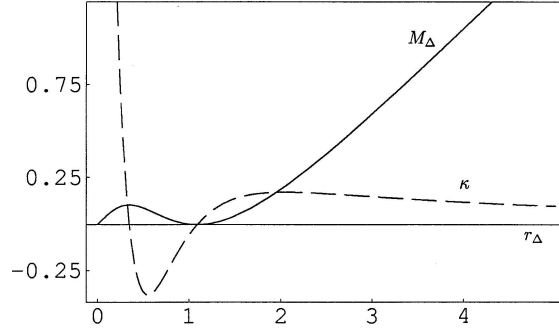$$V = mr^3 \frac{2g^2 - r^2}{(g^2 + r^2)^{3/2}}, \tag{11}$$

Fig. 1 Horizon mass M_Δ and surface gravity κ, as functions of the horizon radius, for the extreme Bardeen black hole, in this case the magnetic charge has the value $g^2 = 16m^2/27$

Substituting V in the Smarr-type formula one obtains the horizon mass

$$M_\Delta = \frac{r}{2} - \frac{mr^3}{(r^2 + g^2)^{3/2}}. \tag{12}$$

This value for the horizon mass coincides with the one determined by integrating the first law, Eq. (10).

In Fig. 1 we have depicted both the horizon mass M_Δ and the surface gravity $\kappa = (1 - 6mr_\Delta^2 g^2 (r_\Delta^2 + g^2)^{5/2})/2r_\Delta$, for the extreme case. In Fig. 2 are shown the horizon mass and the surface gravity in the case when inner and event horizon appear. In both figures there are regions where the horizon mass and the surface gravity are negative, these are regions inner to the event horizon, so they have no meaning as the formalism is valid for the regions $M_\Delta \geq 0$. As we pointed out above, the condition for the positiveness of M_Δ is the same for g_{tt}. Furthermore, we remind that for the regions where $g_{tt} \leq 0$ the signature of the metric changes, as does the character of the Killing vectors, in such a manner that it is a spatially homogeneous region that is not static; then the situation does not correspond to an always increasing area of the horizon, but on the contrary, as we penetrate the
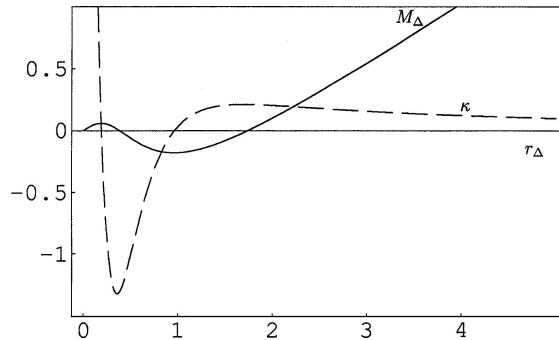

Fig. 2 Horizon mass M_Δ and surface gravity κ, for a Bardeen black hole with magnetic charge $g^2 = 18m^2/27$, in this case the black hole present an inner and an event horizon. M_Δ has meaning only in the range for which $M_\Delta \geq 0$

spatially homogeneous region, the area decreases reaching a minimum and then increasing again. The Bardeen black hole turns out to be stable with respects to arbitrary linear fluctuations of the metric and electromagnetic field [16].

Note that the horizon mass of the Bardeen black hole involved dependence only on the horizon area, since the magnetic charge is not considered as a varying parameter of the horizon. So far we have shown the agreement between the horizon mass calculated with the first law of BHM and when it is determined by adding the appropriate potential to a Smarr-type formula.

However, things are not so easy when we consider situations involving electric charge in nonlinear electrodynamics. This is the case of the regular black hole derived by Ayón–Beato and García (ABG) [1], that we have to take into account in the corresponding potential V, the term of the variation of the electric charge. The metric of the ABG black hole is spherically symmetric, given by

$$ds^2 = -\psi_{\text{ABG}}dt^2 + \psi_{\text{ABG}}^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (13)$$

$$\psi_{\text{ABG}} = 1 - \frac{2mr^2}{(r^2 + q^2)^{3/2}} + \frac{q^2 r^4}{(r^2 + q^2)^2}. \qquad (14)$$

Asymptotically this solution behaves as a RN. The ABG line element is not a solution of the standard nonlinear electrodynamics and the effective geometry (i.e. the geometry affecting the photons of the nonlinear theory) is singular. This regular black hole is an exact solution of Einstein equations coupled with a Lagrangian matter of the form

$$\mathcal{L} = P \frac{1 - 8\sqrt{-2q^2 P} - 6q^2 P}{(1 + \sqrt{-2q^2 P})^4} - \frac{3}{4q^2 s} \frac{(-2q^2 P)^{5/4}(3 - 2\sqrt{-2q^2 P})}{(1 + \sqrt{-2q^2 P})^{7/2}}, \quad (15)$$

where P is the invariant of the electromagnetic field tensor $P_{\mu\nu}$ and $s = |q|/2m$. The no go theorem about a regular static spherically solution with electric charge can be eluded reinterpreting the ABG solution as describing a magnetically charged regular solution of the coupled equations of nonlinear electrodynamics of Eq. (15) and gravitation with much more regular behaviour of the effective geometry.

The Smarr-type formula Eq. (3), with the potential V determined from Eq. (5) amounts to

$$M_\Delta = \frac{r}{2} - \frac{3mr}{2\sqrt{r^2 + q^2}} - \frac{r^5}{4(r^2 + q^2)^2} + \frac{3m}{2} \ln\left[\frac{r + \sqrt{r^2 + q^2}}{q}\right]$$

$$+ \frac{3mr^3 q^2}{2(r^2 + q^2)^{\frac{5}{2}}} - \frac{r^3 q^4}{(r^2 + q^2)^3}. \qquad (16)$$

While the expression for the horizon mass determined from the first law of BHM, Eq. (1), taking into account the presence of the electric charge, is

$$M = \frac{1}{8\pi} \int \kappa da + \int \phi dq,$$

$$= \frac{r}{2} - \frac{3mr}{2\sqrt{r^2 + q^2}} - \frac{r^5}{4(r^2 + q^2)^2} + \frac{3m}{2} \ln\left[\frac{r + \sqrt{r^2 + q^2}}{q}\right]$$

$$- \frac{3mr^3}{2(r^2 + q^2)^{3/2}} + \frac{r^3 q^2}{2(r^2 + q^2)^2}. \qquad (17)$$
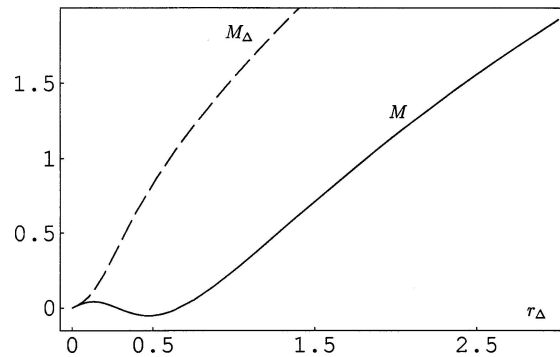
Fig. 3 The horizon mass for the Ayón–Beato–García black hole is displayed. The discontinuous curve corresponds to M_Δ calculated using a Smarr-type formula while the continuous one, also for the horizon mass M, was determined by using the first law of BHM

The two expressions differ in the last two terms, the mismatch is showed in Fig. 3 for the horizon mass. Remains as an open problem the reason why the potential V determined in agreement with the first law of BHM can not give the appropriate dependence for the terms corresponding to the electric charge. Equation (5) do not describe in a feasible form the potential V in situations where nonlinear electromagnetic fields are present. It might be that the dependence of V on the charge is of a nonlinear nature that can not be approached with Eq . (5).

If in contrast, we reinterpret the ABG solution as describing a magnetically charged regular solution, then the potential does not depend on the magnetic charge, but solely on the horizon area. In this case the agreement between the two procedures to calculate the horizon mass is hold and the corresponding expression is

$$M_\Delta = \frac{r}{2} - \frac{mr^3}{(r^2 + q^2)^{3/2}} + \frac{r^3 q^2}{2(r^2 + q^2)^2}. \tag{18}$$

In this contribution we have illustrated that in cases involving nonlinear electromagnetic fields, the horizon mass calculated with a Smarr type formula that is consistent with the fist law of BHM, is feasible only for the magnetic sector of the solutions. If the variation of electric charge is taken into account in the potential V of the Smarr formula, the mentioned consistency does not longer hold. The cases we addressed in this regard were the Bardeen magnetic monopole and the regular black hole derived by Ayón–Beato and García.

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