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LETTER

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Noether symmetry in the higher order gravity theory

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Abstract We explore the conditions for the existence of Noether symmetries for higher order gravity theory, after introducing an auxiliary variable, which gives the correct quantum description of the theory. It turns out that the application of Noether theorem in higher order theory of gravity is a powerful tool to find the solution of the field equations. A few such physically reasonable solutions like power law inflation are presented.

Keywords Fourth-order gravity · Cosmology · Inflation

1 Introduction

The higher order gravity theory plays an important role in the physics of the early universe. Actually, the relevance of fourth order gravity in the gravitational action was explored by several authors. Starobinsky [1] first presented a solution of the inflationary scenario without invoking phase transition in the early universe, but considering only a geometric term in the field equations. In the same direction, Hawking and Luttrel [2] have shown that the curvature squared term in the action

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mimics the role of a massive scalar field. Further, Starobinsky and Schmidt [3] have shown that the inflationary phase is an attractor in the neighbourhood of the solution of the fourth order gravity theory. Low energy effective action, corresponding to Brane world cosmology, also contains higher order curvature invariant terms.

However, in order to elucidate the effect of the fourth order theory of gravity in the early universe, it would be required to find exact solutions of the classical field equations, which are, at contrary, very few in the literature, due to the fourth degree of nonlinearity. Moreover, almost all the known solutions are obtained under certain oversimplified assumptions. The situation becomes even more complicated if the curvature squared term in the action is studied in the arena of a scalar tensor theory of gravity. Therefore it seems likely to find an alternative technique to solve such fourth order field equations. The first step in this direction is achieved by casting the equations in a simplified form by introducing an auxiliary variable, following the prescription of Boulware et al. [4]. The introduction of such an auxiliary variable in the action, effectively transforms the fourth order field equations to second order.

In some recent publications [5–7] the minisuperspace quantization of fourth order gravity has been presented, after introducing auxiliary variable, generalizing the prescription given by Boulware et al. [4] . Actually, it turned out that, to obtain a correct quantum description of the theory, the auxiliary variables should be introduced only after elising all the removable total derivatives from the action. In such a way, the theory yields Schrödinger-like equation, with a meaningful definition of quantum mechanical probability, and the extremization of the effective potential, appearing in the quantum dynamical equation, lead to the vacuum Einstein equation. Thus, a correct choice of auxiliary variable becomes the turning point in yielding a transparent and simple quantum mechanical equation, and might also play an important role in extracting the solutions of classical field equations.

However, as already mentioned, the introduction of an auxiliary variable only partially simplifies the form of the classical field equations, and does not help to obtain a solution; in the presence of matter, for example, the situation is also underdetermined, i.e., the number of field equations is less than that of the field variables. Moreover, in the frame of the scalar tensor theory of gravity the situation is also worse, since not only the scalar field potential, but also the form of the coupling parameter are unknown.

In this paper we show that the only request that the action admits some Noether symmetry furnishes the forms of the coupling parameter and the potential. Further, the Noether symmetry is associated with a conserved current, and with a cyclic variables, which allow to find exact solutions. Despite the constant of motion, which is an outcome of Noether's theorem, does not admit generally any simple physical meaning, we can state that, in demanding such symmetry, we are looking for a relationship among the scale factor, the scalar field and their derivatives such that it yields a constant of motion. Earlier, Capozziello et al. [8] attempted to find Noether symmetry of higher order theory of gravity by Lagrange multiplier method, without invoking auxiliary variable. From this point of view our results are then completely different, and complementary.

In the following section we consider an action which incorporates nonminimally coupled scalar tensor theory with a curvature squared term, in a homogeneous and isotropic background. We then look for Noether symmetry of this action, following the approach of de Ritis et al. [9]. In Sect. 3 we find the form of coupling parameter and that of the potential along with Noether conserved current. It turns out that such symmetry leads to explicit time dependence of the scale factor as well as of the scalar field. This is a new result that has never been expected and observed in any earlier work, not even in the work of Capozziello et al. [8]. It might be just a generic feature of higher order theory of gravity, and has been apparent only after the introduction of auxiliary variable.

It is known that Noether symmetry does not necessarily satisfy the field equations [10] . The reason for such uncanny behaviour is not clear, and it requires to check whether solutions generated by Noether symmetry are really the solutions of the field equations. This has been systematically carried out in all possible situations. In Sect. 4, we discuss some exciting and nevertheless important features of our work.

2 Classical field equations and the equations governing Noether symmetry

In the frame of scalar tensor theories of gravity a generic squared curvature action takes the form:

$$S=\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{16\pi G}\left(f(\phi)R+\frac{\beta}{6}R^2\right)-\frac{1}{2\pi^2}\left(\frac{1}{2}\phi_{,\mu}\,\phi'^{\mu}+V(\phi)\right)\right]. \tag{1}$$

Using the Robertson–Walker metric

$$\mathrm{d}s^2=e^{2\alpha}[-\mathrm{d}\eta^2+\mathrm{d}\chi^2+F(\chi)(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)] \tag{2}$$

we have,

$$S=\int\left[\frac{3\pi}{2G}\{f\alpha''e^{2\alpha}+f(\alpha'^2+k)e^{2\alpha}+\beta\alpha''^2+\beta(\alpha'^2+k)^2 +2\beta\alpha''(\alpha'^2+k)\}+\frac{1}{2}\phi'^2e^{2\alpha}-V(\phi)e^{4\alpha}\right]\mathrm{d}\eta, \tag{3}$$

where dash ($'$) denotes derivative with respect to η and $k=0,\pm1$. Removing total derivative terms and setting $\frac{3\pi}{2G}=M$, the action can be expressed as

$$S=\int[M\{f(k-\alpha'^2)e^{2\alpha}-f_{,\phi}\,\alpha'\phi'e^{2\alpha}+\beta(\alpha'^2+k)^2+\beta\alpha''^2\} +1/2\phi'^2e^{2\alpha}-V(\phi)e^{4\alpha}]\mathrm{d}\eta+\Sigma_1, \tag{4}$$

being $\Sigma_1=M[f\alpha'e^{2\alpha}+2\beta(\alpha'^3/3+k\alpha')]$ the surface term. According to earlier works [5–7] we define the auxiliary variable Q as

$$MQ=\frac{\partial S}{\partial\alpha''}=2M\beta\alpha'',\quad \text{i.e.,}Q=2\beta\alpha''. \tag{5}$$

In terms of Q the action can be written in the following canonical form:

$$S=\int\left[M\left\{f(k-\alpha'^2)e^{2\alpha}-f_{,\phi}\,\alpha'\phi'e^{2\alpha}+\beta(\alpha'^2+k)^2+Q\alpha''-\frac{Q^2}{4\beta}\right\} +\frac{1}{2}\phi'^2e^{2\alpha}-Ve^{4\alpha}\right]\mathrm{d}\eta+\Sigma_1, \tag{6}$$

or, finally, after removing total derivatives

$$S=\int\left[M\left\{f(k-\alpha'^2)e^{2\alpha}-f,_{\phi}\,\alpha'\phi'e^{2\alpha}+\beta(\alpha'^2+k)^2-Q'\alpha'-\frac{Q^2}{4\beta}\right\}\right.$$
$$\left.+\frac{1}{2}\phi'^2e^{2\alpha}-Ve^{4\alpha}\right]\mathrm{d}\eta+\Sigma, \tag{7}$$

where $\Sigma = \Sigma_1 + MQ\alpha'$. It is not difficult to see, in view of Eq. (7), that the auxiliary variable introduced in this manner makes the action canonical, since, the Hessian determinant $|\Sigma\frac{\partial^2 L}{\partial q_i{}'\partial q_j'}| = -M^2e^{2\alpha} \neq 0$. Thus the field equations are

$$4\beta(3\alpha'^2+k)\alpha''-2f(\alpha''+\alpha'^2+k)e^{2\alpha}-(\phi''f,_{\phi}+2\alpha'\phi'f,_{\phi}+\phi'^2f,_{\phi\phi})e^{2\alpha}$$
$$-Q''=\frac{1}{M}(\phi'^2-4V(\phi)e^{2\alpha})e^{2\alpha}. \tag{8}$$

$$Q=2\beta\alpha'', \tag{9}$$

$$f,_{\phi}(\alpha''+\alpha'^2+k)=\frac{1}{M}(\phi''+2\alpha'\phi'+V,_{\phi}e^{2\alpha}). \tag{10}$$

Finally the Hamilton constraint equation is

$$[f(\alpha'^2+k)+f,_{\phi}\,\alpha'\phi']e^{2\alpha}-\beta(\alpha'^2+k)(3\alpha'^2-k)+Q'\alpha'-\frac{Q^2}{4\beta}$$
$$=\frac{1}{M}\left[\frac{1}{2}\phi'^2+V(\phi)e^{2\alpha}\right]e^{2\alpha}. \tag{11}$$

The above set of field Eqs. (8) through (11) is underdetermined. Thus, some sort of physically reasonable assumptions are to be imposed for finding exact solutions. In the present work this is achieved by demanding Noether symmetry. In the above dynamical system the configuration space is three dimensional and each point of it is described by (α, Q, ϕ); whose tangent space is specified by the variables $(\alpha, Q, \phi, \alpha', Q', \phi')$. At this stage, following the approach of de-Ritis et al. [9] we assume the infinitesimal generator of the Noether symmetry as

$$\mathbf{X}=A\frac{\partial}{\partial\alpha}+B\frac{\partial}{\partial Q}+C\frac{\partial}{\partial\phi}+A'\frac{\partial}{\partial\alpha'}+B'\frac{\partial}{\partial Q'}+C'\frac{\partial}{\partial\phi'}, \tag{12}$$

where A, B, C are function of α, Q, ϕ. The existence of Noether symmetry in the action implies the existence of the vector field $\mathbf{X}$ such that the Lie derivative of the Lagrangian with respect to the vector field vanishes i.e.

$$\mathcal{L}_{\mathbf{X}}L=0. \tag{13}$$

The conserved quantity corresponding to the Noether symmetry is

$$F=A\frac{\partial L}{\partial\alpha'}+B\frac{\partial L}{\partial Q'}+C\frac{\partial L}{\partial\phi'}. \tag{14}$$

Taking into account Eqs. (7) and (12), the explicit form of Eq. (13) is

$$\begin{aligned}&A[2M\{f(k-\alpha'^2)-f_{,\phi}\,\alpha'\phi'\}e^{2\alpha}+(\phi'^2-4Ve^{2\alpha})e^{2\alpha}]\\&\quad+\left(\frac{\partial A}{\partial\alpha}\alpha'+\frac{\partial A}{\partial Q}Q'+\frac{\partial A}{\partial\phi}\phi'\right)M[(-2f\alpha'-f_{,\phi}\,\phi')e^{2\alpha}\\&\quad+4\beta\alpha'(\alpha'^2+k)-Q']+B\left(-\frac{MQ}{2\beta}\right)+\left(\frac{\partial B}{\partial\alpha}\alpha'+\frac{\partial B}{\partial Q}Q'+\frac{\partial B}{\partial\phi}\phi'\right)\\&\quad(-M\alpha')+C[M(f_{,\phi}\,(k-\alpha'^2)-f_{,\phi\phi}\,\alpha'\phi')e^{2\alpha}-V_{,\phi}\,e^{4\alpha}]\\&\quad+\left(\frac{\partial C}{\partial\alpha}\alpha'+\frac{\partial C}{\partial Q}Q'+\frac{\partial C}{\partial\phi}\phi'\right)(-Mf_{,\phi}\,\alpha'e^{2\alpha}+\phi'e^{2\alpha})=0\end{aligned}\tag{15}$$

Equation (15) is satisfied if a set of equations that are obtained on collecting the co-efficients of $\alpha'^4, Q'\alpha'^3, \phi'\alpha'^3, Q'\phi', Q'\alpha', \phi'^2, \alpha'^2, \alpha'\phi'$ and $Q'\phi'$ from Eq. (15) are satisfied. Now in view of the coefficients of $\alpha'^4, Q'\alpha'^3$ and $\phi'\alpha'^3$, we get

$$A=A_0,\tag{16}$$

where A_0 is a constant. Coefficient of $Q'\phi'$ gives

$$\frac{\partial C}{\partial Q}=0,\tag{17}$$

i.e., C is not a function of Q. Further the co-efficients of $\alpha'Q'$ gives

$$f_{,\phi}\,\frac{\partial C}{\partial Q}+\frac{\partial B}{\partial Q}e^{-2\alpha}=0,\tag{18}$$

i.e., B also does not depend on Q, what, indeed, should be, since Q is an auxiliary variable only. Coefficient of ϕ'^2 gives

$$\frac{\partial C}{\partial\phi}+A=0,\tag{19}$$

which implies, in view of the solution 16

$$C=-A_0\phi+g_1(\alpha).\tag{20}$$

Finally, co-efficients of $\alpha'^2, \phi'\alpha'$ and $Q'\phi'$ give

$$2Af+f_{,\phi}\left(C+\frac{\partial C}{\partial\alpha}\right)+\frac{\partial B}{\partial\alpha}e^{-2\alpha}=0,\tag{21}$$

$$f_{,\phi}\left(2A+\frac{\partial C}{\partial\phi}\right)+Cf_{,\phi\phi}-\frac{1}{M}\frac{\partial C}{\partial\alpha}+\frac{\partial B}{\partial\phi}e^{-2\alpha}=0\tag{22}$$

and

$$k(2Af+Cf_{,\phi}\,)e^{2\alpha}-\frac{BQ}{2\beta}-\frac{1}{M}(CV_{,\phi}\,+4AV)e^{4\alpha}=0.\tag{23}$$

The solutions of $A, B, C, V(\phi)$ and $f(\phi)$ satisfying all these Eqs. (16)–(23) yield Noether symmetry, that we shall take up in the following section.

3 Solutions

This section is dedicated in finding the solutions of $A, B, C, f(\phi)$ and $V(\phi)$ in view of Eqs. (16)–(23). It has already been pointed out that once we can find the functional forms of A, B and C, Noether conserved current can be found explicitly in view of Eq. (14). For the purpose mentioned, we assume that the solution of B admits separation of variables in the form $B = B_1(\alpha)B_2(\phi)$. Thus, Eq. (21) gives

$$A_0(2f - \phi f,_\phi) + f,_\phi \left(g_1 + \frac{\mathrm{d}g_1}{\mathrm{d}\alpha}\right) + B_2 \frac{\mathrm{d}B_1}{\mathrm{d}\alpha} e^{-2\alpha} = 0. \tag{24}$$

Differentiating above Eq. (24) with respect to ϕ, we get

$$A_0 f,_\phi - A_0 \phi f,_{\phi\phi} + \left(g_1 + \frac{\mathrm{d}g_1}{\mathrm{d}\alpha}\right) f,_{\phi\phi} + B_{2,\phi} \frac{\mathrm{d}B_1}{\mathrm{d}\alpha} e^{-2\alpha} = 0. \tag{25}$$

Eliminating, $g_1 + \frac{\mathrm{d}g_1}{\mathrm{d}\alpha}$ between Eqs. (24) and 25 we get (N being an arbitary constant)

$$A_0 \frac{[(2f - \phi f,_\phi),_\phi f,_\phi - (2f - \phi f,_\phi) f,_{\phi\phi}]}{(f,_\phi B_{2,\phi} - B_2 f,_{\phi\phi})} = \frac{\mathrm{d}B_1}{\mathrm{d}\alpha} e^{-2\alpha} = N. \tag{26}$$

Since, the left-hand side of Eq. (26) is a function of ϕ and the right hand side of it is that of α, therefore, both sides are equated to a constant N. Hence

$$B_1 = \frac{N}{2} e^{2\alpha} + b_0, \tag{27}$$

where b_0 is a constant of integration, and

$$2f - \phi f,_\phi = N_1 f,_\phi - \frac{N}{A_0} B_2, \tag{28}$$

N_1 being yet another constant. In view of Eqs. (28) and (24) is

$$g_1 + \frac{\mathrm{d}g_1}{\mathrm{d}\alpha} + A_0 N_1 = 0, \tag{29}$$

for $f,_\phi \neq 0$. Hence g_1 can be solved to find C as

$$C = \alpha_0 e^{-\alpha} - A_0(\phi + N_1). \tag{30}$$

In view of which, Eq. (22) takes the following form:

$$A_0[f,_\phi - (\phi + N_1) f,_{\phi\phi}] + \alpha_0 \left(f,_{\phi\phi} + \frac{1}{M}\right) e^{-\alpha} + \left(\frac{N}{2} e^{2\alpha} + b_0\right) e^{-2\alpha} B_{2,\phi} = 0. \tag{31}$$

This Eq. (31) is satisfied, provided $f,_{\phi\phi} + \frac{1}{M} = 0$ or $\alpha_0 = 0$ along with b_0 or $B_{2,\phi} = 0$. The first case ie. $f,_{\phi\phi} + \frac{1}{M} = 0$, leads to some interesting results, which are presently under consideration and will be communicated in a future article. Now, for the other choice, i.e., $\alpha_0 = b_0 = 0$, the above Eq. (31) reads

$$A_0[f,_\phi - (\phi + N_1) f,_{\phi\phi}] + \frac{N}{2} B_{2,\phi} = 0. \tag{32}$$

Comparison of Eq. (32) with Eq. (28) being differentiated with respect to ϕ implies that these two equations are consistent either for $N = 0$ or for $B_2 =$ a constant. The first choice leads to inconsistency. So finally, we are left with only one option, i.e., $\alpha_0 = 0 = B_{2,\phi}$, ie., $B_2 = b_2$, a constant. For this choice, Eq. (31) is

$$f,_{\phi\phi}(\phi + N_1) - f,_{\phi} = 0. \tag{33}$$

Further, Eq. (28) gives

$$(\phi + N_1)f,_{\phi} = 2f + \frac{Nb_2}{2A_0}. \tag{34}$$

Equations (33) and (34) are thus consistent and yield the following solution:

$$f = f_0(\phi + N_1)^2 - \frac{Nb_2}{4A_0}, \tag{35}$$

along with

$$A = A_0 \quad B = b_2\left(\frac{N}{2}e^{2\alpha} + b_0\right) \quad C = -A_0(\phi + N_1). \tag{36}$$

In view of the solutions (35) and (36), Eq. (23) reads

$$kNb_2e^{-2\alpha} + \frac{b_2}{2\beta}\left(\frac{N}{2}e^{2\alpha} + b_0\right)Qe^{-4\alpha} = \frac{A_0}{M}[(\phi + N_1)V,_{\phi} - 4V]. \tag{37}$$

It is clear that we are left with only one equation, viz. (37), that has to be satisfied for the existence of Noether symmetry and that would eventually lead to a functional form of the potential $V(\phi)$. Moreover, while the left-hand side of equation is a function of α and the right hand side is only a function of ϕ, both sides must be separately equal to a constant (that may be chosen to be zero as a special case). As a consequence, the only request of finding Noether symmetry for the system under consideration allow to find at least in principle an explicit form of the auxiliary variable, and eventually lead to the temporal evolution of the scale factor. It should be mentioned that only those temporal behaviours of the scale factor which are consistent with the field Eqs. (8) through (11) can be selected as physically acceptable solutions. As mentioned earlier, we have encountered situation [10], where Noether solution does not satisfy the field equations. Unfortunately, it is almost impossible to find the general solution of Eq. (37) as far as the left hand side is concern. Therefore depending on different choice of integration constants appearing in Eq. (37) we study the following different cases.

3.1 **Case 1** $b_0 = 0,\ \ b_2 \neq 0,\ \ (\phi + N_1)V,_{\phi} = 4V$

So, here we have considered the separation constant to be zero. Under this situation Eq. (37) yields

$$Q = -4k\beta, \quad V(\phi) = V_0(\phi + N_1)^4. \tag{38}$$

The scale factor e^{α} can be obtained easily from Eq. (38) and it can be used in the Noether constant of motion (14) to find solution for ϕ. Equation (14) takes the form

$$\frac{F}{A_0 M} = 4\beta\alpha'(k+\alpha'^2) - Q' - \left[2f\alpha' + f_{,\phi}\phi' + \frac{b_2 N}{2A_0}\alpha' - (\phi+N_1)\alpha' f_{,\phi} + \frac{\phi+N_1}{M}\phi'\right]e^{2\alpha}. \quad (39)$$

To find a simple solution, we choose $k = 0$, for which

$$e^{\alpha} = e^{g\eta}, \quad (40)$$

where g is a constant of integration given by $g = [\frac{F}{A_0}\{\frac{(1+2f_0 M)^2}{4V_0} + 4M\beta\}]^{-1/3}$. Accordingly, Eq. (39) yields

$$(\phi+N_1)^2 = \phi_0^2 e^{-2g\eta} + \frac{C_1 + b_2 MNg/2}{A_0(1+2Mf_0)/2}, \quad (41)$$

where C_1 is a constant and $\phi_0^2 = \frac{F/2g - 2A_0 M\beta g^2}{A_0(1+2Mf_0)/2}$. It is to be noted that the solution for α and ϕ presented here, are obtained from Noether symmetry conditions and these solutions (40) and (41) satisfy the field Eqs. (8)–(10) trivially under a simple restriction on the integration constants $C_1 = -b_2 MNg/2$ and $V_0 = \frac{g^2(1+2Mf_0)}{4\phi_0^2}$. This solution represents a power law inflation, as the scale factor in proper time is $e^{\alpha} = gt$. Further, the solution of ϕ given by Eq. (41) reduces to $\phi = \frac{\phi_0}{gt} - N_1$. It is observed that the rate of expansion turns out to be independent of β, i.e. inflation continues even in the absence of higher order curvature invariant term. However, the evolution of the scalar field depends on β along with some other parameters like V_0 and f_0, etc. It is further observed that asymptotically ie. at sufficiently large t, ϕ becomes a constant $(-N_1)$, as a result f given by Eq. (35) also becomes a constant $(-\frac{Nb_2}{4A_0})$, that can be chosen to be one without any loss of generality. So one can recover Einstein's gravity, asymptotically.

3.2 **Case 2** $b_0 = 0, \quad b_2 \neq 0, \quad (\phi+N_1)V_{,\phi} - 4V = r_0 = \text{constant}$

This choice is less restrictive as it considers both sides of Eq. (37) to a constant, so that one obtains the following equations:

$$Q = 2\beta\omega_0^2 e^{2\alpha} - 4k\beta, \quad (42)$$

$$V = V_0(\phi+N_1)^4 + r_0, \quad (43)$$

where $\omega_0^2 = \frac{2A_0 r_0}{MNb_2}$. Now using (9) (the definition of Q) in (42) we get

$$\alpha'^2 = \omega_0^2 e^{2\alpha} - 4k\alpha + q^2 \quad (44)$$

whose integral gives, for $k = 0$, $q \neq 0$

$$e^{-\alpha} = \frac{\omega_0}{q}\sinh(q\eta), \quad (45)$$

where q is an integration constant. The solution (45) can be used in the Noether constant of motion to find ϕ and is given by

$$\left(f_0+\frac{1}{2M}\right)(\phi+N_1)^2=\frac{F\omega_0^2}{2Mq^2A_0}\left(\eta-\frac{\sinh(2q\eta)}{2q}\right)-2\beta\omega_0^2\sinh^2(q\eta)$$
$$-\frac{Nb_2}{2A_0}\ln|\sinh(q\eta)|+C_2, \tag{46}$$

where C_2 is a constant. Here again we point out that the solutions (45) and (46) are obtained from the Noether symmetry only. To justify consistency of the solutions given above, e^{α} and ϕ have to satisfy the field Eqs. (8)–(10).

Another simple solution of the above equations is obtained for $q=0$ and it is

$$e^{-\alpha}=\omega_0\eta \tag{47}$$

and as a consequence solution of ϕ, as obtained from Eq. (14) is

$$\left(f_0+\frac{1}{2M}\right)(\phi+N_1)^2=C_2-\frac{F\eta^3}{3A_0M}-\frac{Nb_2}{2A_0\omega_0^2}\ln\eta. \tag{48}$$

The solutions (47) and (48) obtained from the Noether symmetry are not consistent with the field equations.

3.3 **Case 3** $f=$ constant

It is also possible to study a totally different, nevertheless important case viz., $f=$ constant $=1$ (say).

Under this assumption, Eq. (21) gives

$$B=-A_0e^{2\alpha}+B_2(\phi). \tag{49}$$

Equation (22) becomes

$$MB_{2,\phi}=\frac{\mathrm{d}g_1}{\mathrm{d}\alpha}e^{2\alpha}=N, \tag{50}$$

where N is the separation constant. Equation (50) is solved to yield

$$B_2=\frac{N}{M}\phi+B_0\quad g_1=-\frac{N}{2}e^{-2\alpha}+C_0. \tag{51}$$

As a result Eq. (23) takes the following form:

$$A_0e^{2\alpha}\left[2k+\frac{Q}{2\beta}+\frac{N}{2MA_0}V_{,\phi}\right]-\frac{N}{2M\beta}Q\phi-\frac{B_0}{2\beta}Q$$
$$+\frac{1}{M}[(A_0\phi-C_0)V_{,\phi}-4A_0V]e^{4\alpha}=0. \tag{52}$$

This is the last equation that has to be satisfied to obtain Noether symmetry and as a consequence this will yield a form of $V(\phi)$. Equation (52) can be solved only under certain simplified assumptions, e.g., the choice $Q=Q_0e^{2\alpha}$. However, it leads to inconsistency. The other choice may be $B_0 = 0 = N$, under which Eq. (50) becomes

$$Q = -4\beta k \quad V = V_0(A_0\phi - C_0)^4 \tag{53}$$

together with

$$A = A_0 \quad B = -A_0 e^{2\alpha} \quad C = C_0 - A_0\phi. \tag{54}$$

The conserved current is

$$\frac{F}{A_0 M} = 4\beta\alpha'(\alpha'^2 + k) - Q' - \alpha' e^{2\alpha} + \frac{C_0}{A_0 M}\phi' e^{2\alpha} - \frac{1}{M}\phi\phi' e^{2\alpha}. \tag{55}$$

Now from Eq. (53)

$$\alpha = -k\eta^2 + g\eta + h, \tag{56}$$

where g, h are integration constant. This solution (56) can be used in Eq. (55) to find the scalar field and is given by

$$\phi^2 = \frac{F}{gA_0} e^{-2g\eta}, \tag{57}$$

where we have assumed $k = 0, h = 0$. Further, one has to check the consistency of solutions (56) and (57) with the field equations. They are found to satisfy the field equations under restriction on the integration constants $g^2 = \frac{1}{4\beta}$, $V_0 = \frac{g^2}{4A_0^4\phi_0^2}$ and $C_0 = 0$. This solution also leads to a power law inflation.

4 Concluding remarks

It is well known that the higher order gravity theory plays an important role in the physics of the early universe. However it is extremely difficult to generate solutions of higher order theory of gravity, due to the presence of fourth degree of nonlinearity in the corresponding field equations. The aim of this paper is motivated in finding a suitable technique to generate a class of such solutions.

In a series of earlier works it has been observed that properly chosen auxiliary variable leads to correct and transparent quantum dynamics of the theory. In view of such result, we were led to inspect how such auxiliary variable helps in solving corresponding classical field equations. Further, demanding Noether symmetry, one can fix up the coupling parameter and the potential of a nonminimally coupled scalar field. It also gives a conserved current that relates the scale factor and the scalar field variable under consideration, along with their time derivatives, in a spatially homogeneous and isotropic background. A conserved quantity leads to a cyclic variable that simplifies in finding the solutions of the classical field equations. Thus, we were motivated in finding the solutions of the field equations corresponding to an action, containing curvature squared term, in addition to a nonminimally coupled scalar field, in spatially homogeneous and isotropic background, introducing auxiliary variable and demanding Noether symmetry.

We explored an excellent and remarkable feature of Noether symmetry in the context of higher order theory of gravity, according to which it directly yields a class of solution without handling the fourth degree nonlinearity of the field equations. Only a very few of such solutions are presented here, just to show, how the technique works. It requires mentioning that not all solutions generated in this method satisfy classical field equations. The reason of such uncanny behaviour is presently not known.

Solutions obtained with this method, and which satisfy the field equations, are nevertheless interesting. They admit power law inflation and at least in one of the situations it is possible to recover Einstein's gravity, asymptotically.

Thus, the technique of choosing such auxiliary variable now reveals new direction in the classical context also, as Noether symmetry has been found to be a powerful tool in generating a class of solutions to the field equations in highly nonlinear dynamics.

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