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ORIGINAL PAPER

Scalar curvature along the Ricci flow

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Abstract
In this note, we prove a well-known conjecture on the Ricci flow under a curvature condition, which is a pinching between the Ricci and Weyl tensors divided by suitably translated scalar curvature, motivated by Cao's result (Commun Anal Geom 19(5):975–990, 2011).

Keywords Ricci flow · scalar curvature · Weyl curvature

Mathematics Subject Classification 53E20 · 58J35

1 Introduction

Consider the Ricci flow

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)}, \quad g(0) = g, \quad t \in [0, T), \tag{1.1}$$

on a given closed n-dimensional Riemannian manifold (M, g). Here T is the maximal time of (1.1) which is finite or infinite according to Hamilton's result [8]. In this paper, we assume

$$T \in (0, \infty). \tag{1.2}$$

In this case, we have

$$\lim_{t\to T} \max_M |\mathrm{Rm}_{g(t)}|_{g(t)} = \infty \tag{1.3}$$

by Hamilton [8], and

$$\lim_{t\to T} \max |\mathrm{Ric}_{g(t)}|_{g(t)} = \infty \tag{1.4}$$

by Sesum [10] (for another proof see [9]). For scalar curvature, Cao [3] proved that

$$\textit{either} \ \lim_{t\to T} \max_M R_{g(t)} = \infty \quad \textit{or} \quad \lim_{t\to T} \max_M R_{g(t)} < \infty \text{ and } \lim_{t\to T} \frac{|W_{g(t)}|_{g(t)}}{R_{g(t)} + C} = \infty, \tag{1.5}$$

where C is a positive constant such that $\min_M R_g + C > 0$ (hence, $R_{g(t)} + C \ge R \ge \min_M R_{g(t)} + C > 0$ by the evolution equation of $R_{g(t)}$) and, $W_{g(t)}$ is the Weyl tensor of $g(t)$. A well-know conjecture on scalar curvature is

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Conjecture 1.1 *Under the condition* (1.2)*, the Ricci flow* (1.1) *has the following property*

$$\lim_{t\to T}\max_M R_{g(t)} = \infty. \tag{1.6}$$

The above conjecture was proved for Kähler-Ricci flow by Zhang[11] and for type-I maximal solution of Ricci flow by Enders, Müller and Topping [7].

In a very recent paper, Buzano and Di Matteo obtained an important result about Conjecture 1.1 in [2], where they showed that (see Corollary 1.12 in [2]) under an extra condition on injective radius bound of Ricci flow (i.e., $\mathrm{inj}(M, g(t)) \geq \alpha(\sup_{M\times[0,t]}|\mathrm{Ric}_{g(s)}|_{g(s)})^{-1/2}$ for some $\alpha > 0$) and $n < 8$, Conjecture 1.1 is true. When $n \geq 8$, they also studied the singularities (see Theorem 1.13, [2]).

On the other hand, under the condition of boundedness of scalar curvature and finite T, Bamler [1] proved that there exists an open subset Σ of M such that $g(t)$ converges in $C^\infty(\Sigma)$ to a Riemannian metric g_T on Σ as $t \to T$, and the Hausdorff dimension of $M \setminus \Sigma$, with respect to some pseudo-length metric d_T (i.e., the limit of the induced length metric d_t of $g(t)$) on M, is not greater than $n - 4$.

In this paper, we give a partial answer of Conjecture 1.1. Given an arbitrary positive number ϵ, we choose a positive constant $C := C_\epsilon$ such that $\min_M R_g + C \geq \epsilon > 0$, and then $R_{g(t)} + C \geq \epsilon$. Define two quantities along the Ricci flow

$$f := \frac{|W|^2}{(R+C)^2} = \frac{|W_{g(t)}|^2_{g(t)}}{(R_{g(t)}+C)^2}, \quad h := \frac{|\mathrm{Ric}|^2}{(R+C)^2} = \frac{|\mathrm{Ric}_{g(t)}|^2_{g(t)}}{(R_{g(t)}+C)^2}. \tag{1.7}$$

Cao [3] proved that, for any $T' \in (0, T)$, the inequality

$$h \leq C_1 + \frac{1}{\epsilon}\max_{M\times[0,T']} f^{1/2} \tag{1.8}$$

holds on $M \times [0, T']$, where C_1 is a universal constant depending only on M, g, ϵ, and n. Using (1.8) we can easily deduce (1.5).

According to Proposition 1.1 in [4], we can obtain

$$\Box|W|^2 = -2|\nabla W|^2 + \frac{8}{n-2}W_{ijk\ell}R^{ik}R^{j\ell} + 8(W^{ijk\ell} + W^{ikj\ell})W_{pijq}W^p{}_{k\ell}{}^q \tag{1.9}$$

where $\Box = \Box_{g(t)} := \partial_t - \Delta_{g(t)} = \partial_t - \Delta$. There exists a positive constant C_n, depending only on n, so that $C_n > \frac{\epsilon}{4}$ and

$$8(W^{ijk\ell} + W^{ikj\ell})W_{pijq}W^p{}_{k\ell}{}^q \leq C_n|W|^3. \tag{1.10}$$

Motivated by (1.8), we make the following assumption: for any $T' \in [0, T)$, the inequality

$$h \geq C_{n,\epsilon}\max_{[0,T']} f^{1/2} - C_2, \tag{1.11}$$

holds on $M \times [0, T']$, for some ϵ and universal constants C_2 and $C_{n,\epsilon}$ with $\frac{1}{\epsilon} > C_{n,\epsilon} > \frac{1}{4}C_n$.

Theorem 1.2 *Under the condition* (1.11)*, Conjecture* (1.1) *holds. More precisely, there exist constants $C_{n,\epsilon}$ and C_2 such that if a Ricci flow* (1.1) *is singular at a finite time T, and satisfying the pinching condition* (1.11)*, then its scalar curvature must blow up at T.*

Going through the following proof of Theorem 1.2, we observe that (1.11) can be replaced by the following condition

$$|\mathrm{Ric}|^2 \geq C_{n,\epsilon}(R + C)|W| \tag{1.12}$$

along the Ricci flow (1.1), where $C = C_\epsilon$ satisfies $\min_M R_g + C \geq \epsilon > 0$.

Remark 1.3 We now suppose $R \geq R_{\min} := \min_M R \in (0, 4/C_n)$, where C_n is determined by (1.10). In this case, we can take $\epsilon = R_{\min}$ and $C_\epsilon = 0$. Then the conclusion (1.5) implies that

$$\frac{|W_{g(t)}|_{g(t)}}{R_{g(t)}} \leq C \Longrightarrow \lim_{t\to T} \max_M R_{g(t)} = \infty, \tag{1.13}$$

while (1.12) becomes

$$\frac{|W_{g(t)}|_{g(t)}}{R_{g(t)}} \leq C'_n \frac{|\mathrm{Ric}_{g(t)}|^2_{g(t)}}{R^2_{g(t)}} \Longrightarrow \lim_{t\to T} \max_M R_{g(t)} = \infty. \tag{1.14}$$

Here C'_n is a constant satisfying $R_{\min} < C'_n < 4/C_n$. In our situation, it is clear that the condition in (1.13) is stronger than that in (1.14), e.g., $|W_{g(t)}|_{g(t)}/R_{g(t)} \leq C$ (for some C satisfying $nC \leq C'_n$) implies $|W_{g(t)}|_{g(t)}/R_{g(t)} \leq C'_n |\mathrm{Ric}_{g(t)}|^2_{g(t)}/R^2_{g(t)}$. Choosing normal coordinates we can assume that $\mathrm{Ric}_{g(t)} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. From

$$R^2_{g(t)} = \left(\sum_{1\leq i\leq n} \lambda_i\right)^2 \leq n \sum_{1\leq i\leq n} \lambda_i^2 = n|\mathrm{Ric}_{g(t)}|^2_{g(t)}$$

we can conclude that $|\mathrm{Ric}_{g(t)}|^2_{g(t)}/R^2_{g(t)} \geq 1/n$.

2 Proof of Theorem 1.2

We start from an elementary identity.

Lemma 2.1 *For any functions F, G we have*

$$\Box\left(\frac{F}{G}\right) = \frac{\Box F}{G} - \frac{F\Box G}{G^2} + 2\frac{\langle\nabla F, \nabla G\rangle}{G^2} - 2\frac{F}{G^3}|\nabla G|^2. \tag{2.1}$$

Proof Compute

$$\begin{aligned}
\Box\left(\frac{F}{G}\right) &= (\partial_t - \Delta)\left(\frac{F}{G}\right) \\
&= \frac{\partial_t F \cdot G - F\cdot \partial_t G}{G^2} - \nabla^i\left(\frac{\nabla^i F\cdot G - F\cdot\nabla_i G}{G^2}\right) \\
&= \frac{\partial_t F}{G} - \frac{F}{G^2}\partial_t G - \nabla^i\left(\frac{\nabla_i F}{G} - \frac{F}{G^2}\nabla_i G\right) \\
&= \frac{\partial_t F}{G} - \frac{F}{G^2}\Box G - \frac{\Delta F\cdot G - \langle\nabla F, \nabla G\rangle}{G^2} + \frac{\nabla^i F\cdot G^2 - 2FG\nabla^i G}{G^4}\nabla_i G
\end{aligned}$$

which yields (2.1). □

Now we choose

$$F = |W|^2, \quad G := (R+C)^2 \tag{2.2}$$

where as before $C = C_\epsilon$ is a positive constant so that $\min_M R + C \geq \epsilon > 0$. We then get from (2.1) that

$$\Box\left(\frac{|W|^2}{(R+C)^2}\right) = \frac{\Box|W|^2}{(R+C)^2} - \frac{|W|^2}{(R+C)^4}\Box(R+C)^2 + 2\frac{\langle\nabla|W|^2, \nabla(R+C)^2\rangle}{(R+C)^4}$$

$$- 2\frac{|W|^2}{(R+C)^6}|\nabla(R+C)^2|^2. \tag{2.3}$$

Thanks to

$$\begin{aligned}
\nabla(R+C)^2 &= 2(R+C)\nabla(R+C),\\
\Box(R+C)^2 &= (\partial_t - \Delta)(R+C)^2 \;=\; 2(R+C)\partial_t R - \nabla^i[2(R+C)\nabla_i(R+C)]\\
&= 2(R+C)\partial_t R - 2|\nabla(R+C)|^2 - 2(R+C)\Delta(R+C),
\end{aligned}$$

we arrive at

$$\begin{aligned}
\Box\left(\frac{|W|^2}{(R+C)^2}\right) &= \frac{\Box|W|^2}{(R+C)^2} - \frac{|W|^2}{(R+C)^4}\left[2(R+C)\Box R - 2|\nabla(R+C)|^2\right]\\
&\quad + 4\frac{\langle\nabla|W|^2, (R+C)\nabla(R+C)\rangle}{(R+C)^4} - 8\frac{|W|^2}{(R+C)^4}|\nabla(R+C)|^2
\end{aligned}$$

or

$$\begin{aligned}
\Box\left(\frac{|W|^2}{(R+C)^2}\right) &= \frac{\Box|W|^2}{(R+C)^2} - 2\frac{|W|^2\Box R}{(R+C)^3} - 6\frac{|W|^2}{(R+C)^2}|\nabla\ln(R+C)|^2\\
&\quad + \frac{4}{(R+C)^4}\langle\nabla|W|^2, (R+C)\nabla(R+C)\rangle.
\end{aligned} \tag{2.4}$$

On the other hand,

$$\begin{aligned}
\nabla\left(\frac{|W|^2}{(R+C)^2}\right) &= \frac{\nabla|W|^2\cdot(R+C)^2 - 2|W|^2(R+C)\nabla(R+C)}{(R+C)^4}\\
&= \frac{\nabla|W|^2}{(R+C)^2} - 2\frac{|W|^2}{(R+C)^3}\nabla(R+C)\\
&= \frac{\nabla|W|^2}{(R+C)^2} - 2\frac{|W|^2}{(R+C)^2}\nabla\ln(R+C).
\end{aligned}$$

If we introduce the tensor $Z_{aijk\ell} := (R+C)\nabla_a W_{ijk\ell} - \nabla_a R\cdot W_{ijk\ell}$, then

$$|Z|^2 = (R+C)^2|\nabla W|^2 + |\nabla R|^2|W|^2 - \langle\nabla|W|^2, (R+C)\nabla(R+C)\rangle.$$

For any $\gamma\in[0,4]$, we obtain from (2.4) and the evolution equation for the scalar curvature that

$$\begin{aligned}
\Box f &= \frac{\Box|W|^2}{(R+C)^2} - 4f\frac{|\mathrm{Ric}|^2}{R+C} - 6f|\nabla\ln(R+C)|^2 + (4-\gamma)\langle\nabla f, \nabla\ln(R+C)\rangle\\
&\quad + 2(4-\gamma)f|\nabla\ln(R+C)|^2 + \gamma\frac{|\nabla W|^2}{(R+C)^2} + \gamma\frac{|\nabla R|^2|W|^2}{(R+C)^4} - \gamma\frac{|Z|^2}{(R+C)^4}.
\end{aligned} \tag{2.5}$$

It then follows from (1.9) and (2.5) that

$$\begin{aligned}
\Box f &= (\gamma-2)\frac{|\nabla W|^2}{(R+C)^2} - 4f\frac{|\mathrm{Ric}|^2}{R+C} + (4-\gamma)\langle\nabla f, \nabla\ln(R+C)\rangle\\
&\quad + (2-\gamma)f|\nabla\ln(R+C)|^2 - \gamma\frac{|Z|^2}{(R+C)^4}\\
&\quad + \frac{8(W^{ijk\ell}+W^{ikj\ell})W_{pijq}W^{p}{}_{k\ell}{}^{q} + \frac{8}{n-2}W_{ijk\ell}R^{ik}R^{j\ell}}{(R+C)^2}.
\end{aligned} \tag{2.6}$$

Choosing $\gamma = 2$ and using (1.7), (1.10), we have

$$\Box f \le -4fh(R+C) - 2\frac{|Z|^2}{(R+C)^4} + 2\langle \nabla f, \nabla \ln(R+C)\rangle + C_n f^{3/2}(R+C) + \frac{8}{n-2} f^{1/2} h(R+C). \tag{2.7}$$

Proof of Theorem 1.2 Given $T' \in (0, T)$ and consider the time interval $[0, T']$. Suppose f achieves its maximum at a point $(x_0, t_0) \in M \times [0, T']$. The condition (1.11) now implies

$$h \ge C_{n,\epsilon} f_0^{1/2} - C_2, \quad f_0 := f(x_0, t_0),$$

at this point (x_0, t_0). Plugging it into (2.7) and using (1.8), we find

$$\begin{aligned} 0 &\le -4f_0(C_{n,\epsilon} f_0^{1/2} - C_2)(R+C) + C_n f_0^{3/2}(R+C) \\ &\quad + \frac{8}{n-2} f_0^{1/2}\left(C_1 + \frac{1}{\epsilon} f_0^{1/2}\right)(R+C) \\ &= (C_n - 4C_{n,\epsilon}) f_0^{3/2}(R+C) + \left(4C_2 + \frac{8}{n-2}\frac{1}{\epsilon}\right) f_0(R+C) \\ &\quad + \frac{8C_1}{n-2} f_0^{1/2}(R+C) \end{aligned}$$

at (x_0, t_0). Because $R + C \ge \epsilon > 0$ and $C_{n,\epsilon} > \frac{1}{4}C_n$, we can conclude that

$$(4C_{n,\epsilon} - C_n) f_0^{3/2} \le \left(4C_2 + \frac{8}{n-2}\frac{1}{\epsilon}\right) f_0 + \frac{8C_1}{n-2} f_0^{1/2}$$

at (x_0, t_0). Hence $f_0 \le C(n, \epsilon)$ at (x_0, t_0); explicitly,

$$f_0 \le \max\left\{1, \left[\frac{C_2 + \frac{2}{n-2}(C_1 + \frac{1}{\epsilon})}{C_{n,\epsilon} - \frac{1}{4}C_n}\right]^2\right\} =: C(n, \epsilon).$$

Consequently, we get $f \le C(n, \epsilon)$ in $M \times [0, T']$ and hence in $M \times [0, T)$. According to (1.5), we must have $\lim_{t\to T} \max_M R_{g(t)} = \infty$. □

3 A remark on four-dimensional case

When $n = 4$, we can make the constant $C_{n,\epsilon}$ in (1.11) explicitly. Recall first the following property for closed 4-manifold (M, g) in [6]. The Weyl tensor W defines a symmetric operator $\mathcal{W} : \wedge^2 M \to \wedge^2 M$, that is,

$$(\mathcal{W}\alpha)_{k\ell} := \frac{1}{2}\alpha^{ij} W_{ijk\ell},$$

and then, by the Hodge star operator, splits into two operators $\mathcal{W}^\pm : \wedge^{2,\pm} M \to \wedge^{2,\pm} M$ with $\mathrm{tr}\mathcal{W}^\pm = 0$, which induce tensors $W^\pm$. In this notation, we can write $\mathcal{W} = \mathrm{diag}(\mathcal{W}^+, \mathcal{W}^-)$.

For any point $x \in M$, we can choose an oriented orthogonal basis $\omega^+, \eta^+, \theta^+$ (resp. $\omega^-, \eta^-, \theta^-$) of $\wedge_x^{2,+} M$ (resp. $\lambda_x^{2,-} M$), consisting of eigenvectors of $\mathcal{W}^\pm$ that $||\omega^\pm|| = ||\eta^\pm|| = ||\theta^\pm|| = \sqrt{2}$ and $(\lambda^\pm \le \mu^\pm \le \nu^\pm)$

$$W^\pm = \frac{1}{2}\left(\lambda^\pm \omega^\pm \otimes \omega^\pm + \mu^\pm \eta^\pm \otimes \eta^\pm + \nu^\pm \theta^\pm \otimes \theta^\pm\right),$$

$$\mathcal{W}^{\pm} = \begin{bmatrix} \lambda^{\pm} & 0 & 0 \\ 0 & \mu^{\pm} & 0 \\ 0 & 0 & \nu^{\pm} \end{bmatrix}, \quad 0 = \lambda^{\pm} + \mu^{\pm} + \nu^{\pm}.$$

Here $||T||^2 := \frac{1}{p} T_{i_1 \cdots i_p} T^{i_1 \cdots i_p} = \frac{1}{p}|T|^2$ for $T \in \wedge^p M$. Moreover, $\omega^{\pm}, \eta^{\pm}, \theta^{\pm}$ form a quaternionic structure on $T_x M$:

$$g^{pq} \omega^{\pm}_{ip} \omega^{\pm}_{qj} = g^{pq} \eta^{\pm}_{ip} \eta^{\pm}_{qj} = g^{pq} \theta^{\pm}_{ip} \theta^{\pm}_{qj} = -g_{ij}$$

and

$$g^{pq} \omega^{\pm}_{ip} \eta^{\pm}_{qj} = \theta^{\pm}_{ij}, \quad g^{pq} \eta^{\pm}_{ip} \theta^{\pm}_{qj} = \omega^{\pm}_{ij}, \quad g^{pq} \theta^{\pm}_{ip} \omega^{\pm}_{qj} = \eta^{\pm}_{ij}.$$

Using this decomposition, we can prove (see for example [5])

$$W_{ijk\ell} W^{i}{}_{p}{}^{k}{}_{q} W_{j}{}^{p\ell q} = \frac{1}{2} W_{ijk\ell} W^{ij}{}_{pq} W^{k\ell pq}, \quad W_{ipjq} W_{k\ell}{}^{pq} = \frac{1}{2} W_{ijpq} W_{k\ell}{}^{pq}. \tag{3.1}$$

Now we can simplify the evolution Eq. (1.9) in dimension $n = 4$:

$$\Box |W|^2 = -2|\nabla W|^2 + 4 W_{ijk\ell} R^{ik} R^{j\ell} + 8(W^{ijk\ell} + W^{ikj\ell}) W_{pijq} W^{p}{}_{k\ell}{}^{q}. \tag{3.2}$$

We may take normal coordinates. Then

$$\begin{aligned} (W^{ijk\ell} + W^{ikj\ell}) W_{pijq} W^{p}{}_{k\ell}{}^{q} &= (W_{ijk\ell} + W_{ikj\ell}) W_{pijq} W_{pk\ell q} \\ &= W_{ikj\ell} W_{ipjq} W_{kp\ell q} + W_{ijk\ell} W_{pijq} W_{pk\ell q} \\ &= \frac{1}{2} W_{ijk\ell} W_{ijpq} W_{k\ell pq} + W_{ijk\ell} W_{pijq} W_{pk\ell q} \end{aligned}$$

by the first identity in (3.1). For $A := W_{ijk\ell} W_{pijq} W_{pk\ell q}$ we have

$$\begin{aligned} A = -W_{ijk\ell} W_{pk\ell q}(W_{ijpq} + W_{jpiq}) &= W_{ijk\ell} W_{kp\ell q} W_{ijpq} - W_{ijk\ell} W_{pk\ell q} W_{jpiq} \\ &= \frac{1}{2} W_{ijk\ell} W_{ijpq} W_{k\ell pq} + W_{ijk\ell} W_{kp\ell q} W_{jpiq} \\ &= \frac{1}{2} W_{ijk\ell} W_{ijpq} W_{k\ell pq} + \left(\frac{1}{2} W_{pqk\ell} W_{ijk\ell} \right) W_{qipj} \\ &= \frac{1}{2} W_{ijk\ell} W_{ijpq} W_{k\ell pq} + \frac{1}{2} W_{pqk\ell} \left(\frac{1}{2} W_{qpij} W_{k\ell ij} \right) \\ &= \frac{1}{2} W_{ijk\ell} W_{ijpq} W_{k\ell pq} - \frac{1}{4} W_{pqk\ell} W_{k\ell ij} W_{ijpq} = \frac{1}{4} W_{ijk\ell} W_{k\ell pq} W_{pqij}. \end{aligned}$$

Hence

$$(W^{ijk\ell} + W^{ikj\ell}) W_{pijq} W^{p}{}_{k\ell}{}^{q} = (W_{ijk\ell} + W_{ikj\ell}) W_{pijq} W_{pk\ell q} = \frac{3}{4} W_{ijk\ell} W_{ijpq} W_{k\ell pq}$$

and

$$\Box |W|^2 = -2|\nabla W|^2 + 6 W_{ijk\ell} W^{ijpq} W_{pq}{}^{k\ell} + 4 W_{ijk\ell} R^{ik} R^{j\ell}, \quad n = 4. \tag{3.3}$$

It is clear that

$$6 W_{ijk\ell} W^{ijpq} W_{pq}{}^{k\ell} = \frac{6}{8} \left[(\lambda^+)^3 + (\mu^+)^3 + (\nu^+)^3 + (\lambda^-)^3 + (\mu^-)^3 + (\nu^-)^3 \right]$$

and

$$\frac{1}{4}|W|^2 = (\lambda^+)^2 + (\mu^+)^2 + (\nu^+)^2 + (\lambda^-)^2 + (\mu^-)^2 + (\nu^-)^2.$$

Then

$$\left|6W_{ijk\ell}W^{ijpq}W_{pq}{}^{k\ell}\right| \le \frac{6}{8}\times 6\times \frac{1}{4^{3/2}}|W|^3 = \frac{36}{64}|W|^3.$$

Therefore we can take $C_4 = 1$ in (1.10) and then any constant $C_{4,\epsilon}$ in $(1/4, 1/\epsilon)$.

4 A remark on the proof of Theorem 1.2

If we choose $F = |W|^2$ and $G = (R+C)^\alpha$ with $\alpha > 0$ in (2.1), then

$$\Box\left(\frac{|W|^2}{(R+C)^\alpha}\right) = \frac{\Box|W|^2}{(R+C)^\alpha} - \frac{|W|^2}{(R+C)^{2\alpha}}\Box(R+C)^\alpha + 2\frac{\langle\nabla|W|^2, \nabla(R+C)^\alpha\rangle}{(R+C)^{2\alpha}} - 2\frac{|W|^2}{(R+C)^{3\alpha}}|\nabla(R+C)^\alpha|^2. \quad (4.1)$$

From $\nabla(R+C)^\alpha = \alpha(R+C)^{\alpha-1}\nabla R$ and

$$\Box(R+C)^\alpha = \alpha(R+C)^{\alpha-1}\Box R - \alpha(\alpha-1)(R+C)^{\alpha-2}|\nabla R|^2$$

we from (4.1) that

$$\Box\left(\frac{|W|^2}{(R+C)^\alpha}\right) = \frac{\Box|W|^2}{(R+C)^\alpha} - \alpha\frac{|W|^2\Box R}{(R+C)^{\alpha+1}} - \alpha(\alpha+1)\frac{|W|^2}{(R+C)^{\alpha+2}}|\nabla \ln(R+C)|^2 + \frac{2\alpha}{(R+C)^{2\alpha}}\langle\nabla|W|^2, (R+C)^{\alpha-1}\nabla(R+C)\rangle. \quad (4.2)$$

With the same tensor Z as in Sect. 2, The following two identities

$$\nabla\left(\frac{|W|^2}{(R+C)^\alpha}\right) = \frac{\nabla|W|^2}{(R+C)^\alpha} - \alpha\frac{|W|^2}{(R+C)^\alpha}\nabla\ln(R+C),$$

$$|Z|^2 = (R+C)^2|\nabla W|^2 + |\nabla R|^2|W|^2 - \langle\nabla|W|^2, (R+C)\nabla(R+C)\rangle$$

show that

$$\Box\left(\frac{|W|^2}{(R+C)^\alpha}\right) = \gamma\left\langle\nabla\left(\frac{|W|^2}{(R+C)^\alpha}\right), \nabla\ln(R+C)\right\rangle + \frac{\Box|W|^2}{(R+C)^\alpha} - \alpha\frac{|W|^2\Box R}{(R+C)^{\alpha+1}} + (2\alpha-\gamma)\frac{|\nabla W|^2}{(R+C)^\alpha} - (2\alpha-\gamma)\frac{|Z|^2}{(R+C)^{\alpha+2}} - (\alpha-\gamma)(\alpha-1)\frac{|W|^2}{(R+C)^\alpha}|\nabla\ln(R+C)|^2, \quad (4.3)$$

where $0 \le \gamma \le 2\alpha$. In particular, choosing $\gamma = 0$,

$$\Box\left(\frac{|W|^2}{(R+C)^\alpha}\right) = \frac{\Box|W|^2}{(R+C)^\alpha} - \alpha\frac{|W|^2\Box R}{(R+C)^{\alpha+1}} + 2\alpha\frac{|\nabla W|^2}{(R+C)^\alpha} - 2\alpha\frac{|Z|^2}{(R+C)^{\alpha+2}} - \alpha(\alpha-1)\frac{|W|^2}{(R+C)^\alpha}|\nabla\ln(R+C)|^2. \quad (4.4)$$

Putting $\alpha = 1$ in (4.4) yields

$$\Box\left(\frac{|W|^2}{R+C}\right) = \frac{\Box|W|^2 + 2|\nabla W|^2}{R+C} - 2\frac{|W|^2|\mathrm{Ric}|^2}{(R+C)^2} - 2\frac{|Z|^2}{(R+C)^3}. \quad (4.5)$$

If we choose $\gamma = 2\alpha - 2$ with $\alpha \geq 1$ in (4.3), we get

$$\begin{aligned}\Box\left(\frac{|W|^2}{(R+C)^\alpha}\right) &= \frac{\Box|W|^2 + 2|\nabla W|^2}{(R+C)^\alpha} - 2\alpha\frac{|W|^2|\mathrm{Ric}|^2}{(R+C)^{\alpha+1}} - 2\frac{|Z|^2}{(R+C)^{\alpha+2}} \\ &\quad + 2(\alpha-1)\left\langle\nabla\left(\frac{|W|^2}{(R+C)^\alpha}\right), \nabla\ln(R+C)\right\rangle \\ &\quad + (\alpha-1)(\alpha-2)\frac{|W|^2}{(R+C)^\alpha}|\nabla\ln(R+C)|^2. \qquad (4.6)\end{aligned}$$

Author Contributions Yi Li wrote the manuscript.

Declarations

Conflict of interest The authors declare no Conflict of interest.

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