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Automorphism group, Galois points and lines of the generalized Artin–Schreier–Mumford curve

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Abstract
An elementary proof of a theorem of Montanucci and Zini on the automorphism group of generalized Artin–Schreier–Mumford curves is presented, with the argument of Korchmáros and Montanucci for Artin–Schreier–Mumford curves being improved. Although the characteristic of a ground field is assumed to be *odd* in the article of Montanucci and Zini, the proof in the present article is applicable to the case of characteristic two as well. As an application of the theorem of Montanucci and Zini, the arrangement of Galois points or Galois lines for the generalized Artin–Schreier–Mumford curve is determined.

Keywords Automorphism group · Positive characteristic · Artin–Schreier curves

Mathematics Subject Classification 14H37 · 14H05

1 Introduction

The Artin–Schreier–Mumford (ASM) curve over an algebraically closed field k of characteristic $p > 0$ is the smooth model of the plane curve defined by

$$(x^{p^e} - x)(y^{p^e} - y) = c,$$

where $e > 0$, $p^e > 2$ and $c \in k \setminus \{0\}$. This curve is important in the study of the automorphism groups of algebraic curves, since this is an ordinary curve and its automorphism group is large compared to its genus (see [15]). For the case $e = 1$, the automorphism group was determined in [16], and a characterization according to its genus and automorphism group was given in [1]. The ASM curve is generalized as the smooth model X of the curve C defined by

$$L_1(x) \cdot L_2(y) + c = 0,$$

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where $c \in k\setminus\{0\}$ and L_1 and L_2 are linearlized polynomials of degree p^e, that is,

$$L_i = \alpha_{ie}x^{p^e} + \alpha_{ie-1}x^{p^{e-1}} + \cdots + \alpha_{i0}x$$

for some $\alpha_{ie}, \alpha_{ie-1}, \ldots, \alpha_{i0} \in k$ with $\alpha_{ie}\alpha_{i0} \neq 0$, for $i = 1, 2$. We can assume that $\alpha_{1e} = \alpha_{2e} = 1$ for a suitable system of coordinates. This curve was studied by the present author [4, 5] (for the case $L_1 = L_2$), and by Montanucci and Zini [12]. The curve X is called a *generalized Artin–Mumford curve* in [12]. The full automorphism group $\mathrm{Aut}(X)$ of X is completely determined by Montanucci and Zini [12, Theorems 1.1 and 1.2], as follows.

Fact 1 *Assume that $p > 2$. Let $\mathbb{F}_{p^k} = \bigcap_{i>0,\alpha_{1i}\neq 0}\mathbb{F}_{p^i} \cap \bigcap_{j>0,\alpha_{2j}\neq 0}\mathbb{F}_{p^j}$.*

(a) *If $L_1 = L_2$, then $\mathrm{Aut}(X) \cong \Sigma \rtimes D_{p^k-1}$, where Σ is an elementary abelian p-group of order p^{2e} and D_{p^k-1} is the dihedral group of order $2(p^k - 1)$.*
(b) *If $L_1 \neq L_2$, then $\mathrm{Aut}(X) \cong \Sigma \rtimes \mathbb{F}^*_{p^k}$.*

It is assumed that the characteristic is *odd* in [12]. One key point to prove is [12, Lemma 3.1 v) and Corollary 3.2], which asserts that a Sylow p-subgroup of $\mathrm{Aut}(X)$ is linear and acts on $\Omega_1 \cup \Omega_2$, where the set Ω_1 (resp. Ω_2) consists of all poles of x (resp. of y). This assertion relies on a theorem of Nakajima [13, Theorem 1] on relations between the p-rank and Sylow p-subgroups of the automorphism group of algebraic curves. Another key point is that the genus $(p^e - 1)^2$ of X is *even* if $p > 2$, because Montanucci and Zini used some group-theoretic lemmas from [9] concerning curves of *even* genus.

An alternative proof of Fact 1 for the ASM curve was obtained by Korchmáros and Montanucci [10]. It was proved that the linear system induced by some embedding into $\mathbb{P}^3$ is complete, and asserted that $\mathrm{Aut}(X)$ acts on $\Omega_1 \cup \Omega_2$, by using its completeness. We will prove the same things for generalized Artin–Schreier–Mumford curves in a different order. It was pointed out by Garcia [8] (see also [2, 7]) that points of $\Omega_1 \cup \Omega_2$ are Weierstrass points (see Lemma 2 for a more precise statement), and this implies that $\mathrm{Aut}(X)$ acts on $\Omega_1 \cup \Omega_2$. We reprove it. We also present an elementary proof of the completeness of the linear system for generalized ASM curves (Lemma 3). With these two results combined, an inclusion $\mathrm{Aut}(X) \hookrightarrow PGL(4, k)$ is obtained (Corollary 2). More strongly:

Theorem 1 *There exists an injective homomorphism*

$$\mathrm{Aut}(X) \cong \mathrm{Bir}(C) \hookrightarrow PGL(3, k).$$

This is very close to the theorem of Montanucci and Zini. Therefore:

Theorem 2 *The same assertion as Fact 1 holds for the case where $p = 2$.*

As an application of the theorem of Montanucci and Zini, the arrangement of Galois points or Galois lines for the generalized Artin–Schreier–Mumford curve is determined in Sects. 4 and 5.

2 Preliminaries

The system of homogeneous coordinates on $\mathbb{P}^2$ is denoted by $(X : Y : Z)$ and the system of affine coordinates of $\mathbb{A}^2$ is denoted by (x, y) with $x = X/Z$ and $y = Y/Z$. In this section, we consider the generalized Artin–Schreier–Mumford curve X and its plane model C described in Sect. 1. Let $q = p^e$. The set of all poles of x (resp. of y) is denoted by Ω_1 (resp. by

Ω_2), which coincides with the set of all zeros of $L_2(y)$ (resp. of $L_1(x)$). The sets Ω_1 and Ω_2 consist of q points. The pole of x (resp. of y) corresponding to $y = \beta$ (resp. $x = \alpha$) for $L_2(\beta) = 0$ (resp. $L_1(\alpha) = 0$) is denoted by P_β (resp. by Q_α). For the point P_β, $t = \frac{1}{x}$ is a local parameter. Let $P' = (1 : 0 : 0)$ and $Q' = (0 : 1 : 0) \in \mathbb{P}^2$. Then $\mathrm{Sing}(C) = \{P', Q'\}$, and the point P' (resp. the point Q') is the image of Ω_1 (resp. of Ω_2) under the normalization.

The system of homogeneous coordinates on $\mathbb{P}^3$ is denoted by $(X : Y : Z : W)$. We consider the morphism

$$\varphi : X \to \mathbb{P}^3;\ (x : y : 1 : xy),$$

similar to the case of the ASM curve (see [10]). For the point $P_\beta \in \Omega_1$ defined by $L_2(\beta) = 0$, $t = \frac{1}{x}$ is a local parameter at P_β. It follows that

$$\varphi = (x : y : 1 : xy) = (tx : ty : t : txy) = (1 : ty : t : y),$$

and $\varphi(P_\beta) = (1 : 0 : 0 : \beta)$. Therefore, q points of $\varphi(\Omega_1)$ are contained in the line $\ell_1 : Y = Z = 0$ in $\mathbb{P}^3$. Similarly, q points of $\varphi(\Omega_2)$ are contained in the line $\ell_2 : X = Z = 0$. Note that

$$f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = \alpha_{10} L_2(y) \frac{dx}{dt} + \alpha_{20} L_1(x) \frac{dy}{dt} = 0$$

in $k(C)$, where $f(x, y) = L_1(x)L_2(y) + c$. This implies that

$$\frac{dy}{dt}(P_\beta) = 0.$$

The tangent line at $\varphi(P_\beta)$ is spanned by points $\varphi(P_\beta)$ and

$$\left(\frac{d(1)}{dt}(P_\beta) : \frac{d(ty)}{dt}(P_\beta) : \frac{dt}{dt}(P_\beta) : \frac{dy}{dt}(P_\beta)\right) = (0 : \beta : 1 : 0).$$

Therefore, the tangent line $T_{\varphi(P_\beta)}\varphi(X)$ at $\varphi(P_\beta)$ is defined by

$$W - \beta X = Y - \beta Z = 0.$$

Similarly, the tangent line $T_{\varphi(Q_\alpha)}\varphi(X)$ at $\varphi(Q_\alpha)$ is defined by

$$W - \alpha Y = X - \alpha Z = 0.$$

3 Proofs of Theorems 1 and 2

Let $D \in \mathrm{Div}(X)$ be the divisor of X derived from the intersection of the plane model C and the line $\{Z = 0\}$ in $\mathbb{P}^2$. It is known that the genus g of X is equal to $(q-1)^2$ (see [12, Lemma 3.1], [14, III. 7.10]). Therefore, the degree of the canonical divisor is $2g - 2 = 2q(q - 2)$. We consider the linear space $\mathcal{L}((q-2)D)$ associated with the divisor $(q-2)D$. The following two lemmas were proved by Boseck [2] and Garcia [8] in a more general setting (see also [7]). We reprove them, for the convenience of the readers.

Lemma 1 *A divisor $(q-2)D$ is a canonical divisor, and*

$$\mathcal{L}((q-2)D) = \langle x^i y^j \mid 0 \le i, j \le q-2 \rangle.$$

Proof The monomials

$$x^i y^j \text{ with } i \leq q-2 \text{ and } j \leq q-2$$

are containd in $\mathcal{L}((q-2)D)$. They are linearly independent, since $\deg C = 2q$. Since $\deg(q-2)D = 2g-2$ and $\dim \mathcal{L}((q-2)D) \geq (q-1)^2 = g$, it follows from [14, I.6.2] that the assertion follows. □

Lemma 2 *The set* $\Omega_1 \cup \Omega_2$ *coincides with* $\{P \in X \mid q \in H(P)\}$, *where* $H(P)$ *is the Weierstrass semigroup of* P. *In particular, all points of* $\Omega_1 \cup \Omega_2$ *are Weierstrass points.*

Proof We consider the embedding ψ induced by the canonical linear system $|(q-2)D|$. Let $P \in \Omega_1$ and let $t = (1/x)$. Then t is a local parameter at P. Note that $\mathrm{ord}_P(y-\beta) = q$ for some $\beta \in k$. Considering the functions $t^k y^j \in \mathcal{L}((x^{q-2}) + (q-2)D)$, it follows that the orders $\mathrm{ord}_P \psi^* H$ for hyperplanes $H \ni P$ are

$$1, 2, \ldots, q-2, q, \ldots,$$

namely, $q-1+1$ is a *non-gap* (of pole numbers). On the other hand, $\mathrm{ord}_{(a,b)}(x-a)^{q-2}(y-b) = q-1$ for each point $(x, y) = (a, b)$ of $X \backslash (\Omega_1 \cup \Omega_2)$, since functions $x-a$ and $y-b$ are local parameters. □

Corollary 1 *The automorphism group* $\mathrm{Aut}(X)$ *preserves* $\Omega_1 \cup \Omega_2$.

Lemma 3 *The morphism* $\varphi : X \to \mathbb{P}^3$ *is an embedding, and the linear system induced by* φ *is complete.*

Proof The former assertion is derived from the fact that the set $\varphi(\Omega_1 \cup \Omega_2) = \varphi(X) \cap \{Z = 0\}$ consists of $2q$ points (this proof is similar to [10]). We consider the latter assertion. It follows that $1, x, y, xy \in \mathcal{L}(D)$. Each element of the linear space $\mathcal{L}(D)$ is represented by a polynomial of x and y, since each function $g \in \mathcal{L}(D)$ is regular on the affine open set $C \cap \{Z \neq 0\}$. Using the defining polynomial, since

$$x^q y^q = \sum_{i \leq q, j \leq q, (i,j) \neq (q,q)} a_{ij} x^i y^j$$

in $k(X)$, it follows that any $g \in \mathcal{L}(D)$ is represented as a linear combination of monomials

$$x^i y^j \text{ with } i < q \text{ or } j < q.$$

Assume that $g = \sum_{i=0}^m a_i(y) x^i$ with $m \geq q$ and $a_m(y) \neq 0$. Then $\deg a_m(y) < q$. This implies that there exists a pole P of x such that $\mathrm{ord}_P a_m(y) = 0$. Then $\mathrm{ord}_P g = -m \leq -q < -1$. This is a contradiction to $g \in \mathcal{L}(D)$. It follows that any $g \in \mathcal{L}((D)$ is represented as a linear combination of monomials

$$x^i y^j \text{ with } i < q \text{ and } j < q.$$

Let $g = \sum_{i=0}^m a_i(y) x^i$ with $m < q$ and $a_m(y) \neq 0$. Since $\deg a_m(y) < q$, there exists a pole P of x such that $\mathrm{ord}_P a_m(y) = 0$. Then $\mathrm{ord}_P g = -m$. By the condition $g \in \mathcal{L}(D)$, $-m = \mathrm{ord}_P g \geq -1$. It follows that any $g \in \mathcal{L}(D)$ is represented as a linear combination of monomials $1, x, y, xy$. □

Corollary 2 *There exists an injective homomorphism*

$$\mathrm{Aut}(X) \hookrightarrow PGL(4, k).$$

Proof By Corollary 1, $\sigma^* D = D$ for each $\sigma \in \mathrm{Aut}(X)$. By Lemma 3, $\dim |D| = 3$. The assertion follows. □

Using Corollaries 1 and 2, we prove Theorem 1.

Proof of Theorem 1 It is proved that $\varphi(\Omega_1)$ and $\varphi(\Omega_2) \subset \mathbb{P}^3$ are contained in lines $Y = Z = 0$ and $X = Z = 0$ respectively, in Sect. 2. The point $(0 : 0 : 0 : 1)$ given by the intersection of such lines is fixed by each element of $\mathrm{Aut}(X)$. Then $\mathrm{Aut}(X)$ acts on the sublinear system of $|D|$ corresponding to the linear subspace $\langle x, y, 1\rangle \subset \mathcal{L}(D)$. □

The image of the injective homomorphism described in Theorem 1 is denoted by $\mathrm{Lin}(X)$. Let

$$\Sigma := \{\sigma_{\alpha,\beta} : (x, y) \mapsto (x + \alpha, y + \beta) \mid L_1(\alpha) = 0, L_2(\beta) = 0\} \subset PGL(3, k),$$
$$\Gamma := \{\theta_\lambda : (x, y) \mapsto (\lambda x, \lambda^{-1} y) \mid \lambda \in \mathbb{F}^*_{p^k}\} \subset PGL(3, k),$$

and let $\tau \in PGL(3, k)$ be defined by $\tau(x, y) = (y, x)$. It follows that $\langle\Sigma, \Gamma\rangle \subset \mathrm{Lin}(X)$. If $L_1 = L_2$, then $\langle\Sigma, \Gamma, \tau\rangle \subset \mathrm{Lin}(X)$.

Proof of Fact 1 We prove that $\langle\Sigma, \Gamma\rangle = \mathrm{Lin}(X)$ if $L_1 \neq L_2$, and that $\langle\Sigma, \Gamma, \tau\rangle = \mathrm{Lin}(X)$ if $L_1 = L_2$. Let $P' = (1 : 0 : 0)$ and let $Q' = (0 : 1 : 0)$. Then $\mathrm{Lin}(X)$ acts on $\mathrm{Sing}(C) = \{P', Q'\}$. Note that all tangent lines at P' (resp. at Q') are defined by $Y - \beta Z = 0$ (resp. $X - \alpha Z = 0$) for some $\beta \in k$ with $L_2(\beta) = 0$ (resp. for some $\alpha \in k$ with $L_1(\alpha) = 0$). Since $\mathrm{Lin}(X)$ acts on the set of tangent lines at P' or at Q', it follows that there exists $\tau' \in \mathrm{Lin}(X)$ such that $\tau'(P') = Q'$ and $\tau'(Q') = P'$ if and only if $L_1 = L_2$. Therefore, we prove that if $\sigma \in \mathrm{Lin}(X)$, $\sigma(P') = P'$ and $\sigma(Q') = Q'$, then $\sigma \in \langle\Sigma, \Gamma\rangle$.

Assume that $\sigma \in \mathrm{Lin}(X)$, $\sigma(P') = P'$ and $\sigma(Q') = Q'$. Then σ is represented by a matrix

$$A_\sigma = \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & 1 \end{pmatrix}$$

for some $a, b, c, d \in k$. Let $\beta \in k$ (resp. $\alpha \in k$) be a root of L_2 (resp. L_1). Then the image of the tangent line $Y - \beta Z = 0$ (resp. $X - \alpha Z = 0$) under σ is some tangent line $Y - \beta' Z = 0$ (resp. $X - \alpha' Z = 0$). Then an automorphism $\sigma_1 := \sigma_{\alpha-\alpha',\beta-\beta'} \circ \sigma$ fixes the tangent lines $Y - \beta Z = 0$ and $X - \alpha Z = 0$. Therefore, this automorphism is represented by

$$A_{\sigma_1} = \begin{pmatrix} a & 0 & \alpha(1 - a) \\ 0 & b & \beta(1 - b) \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$\sigma_1^*(L_1(x) \cdot L_2(y) + c) = \left(\sum_i a^{p^i} \alpha_{1i} x^{p^i} + c_1\right)\left(\sum_j b^{p^j} \alpha_{2j} y^{p^j} + c_2\right) + c$$
$$= L_1(x) \cdot L_2(y) + c$$

up to a constant for some $c_1, c_2 \in k$, it follows that $c_1 = c_2 = 0$, $ab = 1$ and $a, b \in \mathbb{F}^*_{p^k}$. For an automorphism $\theta_{a^{-1}} \in \Gamma$, $\sigma_2 := \theta_{a^{-1}} \circ \sigma_{\alpha-\alpha',\beta-\beta'} \circ \sigma$ is represented by

$$A_{\sigma_2} = \begin{pmatrix} 1 & 0 & \alpha(a^{-1} - 1) \\ 0 & 1 & \beta(a - 1) \\ 0 & 0 & 1 \end{pmatrix}.$$

This implies that $\theta_{a^{-1}} \circ \sigma_{\alpha-\alpha',\beta-\beta'} \circ \sigma \in \Sigma$, namely, $\sigma \in \langle \Sigma, \Gamma \rangle$. □

4 Application to the arrangement of Galois points

A point $R \in \mathbb{P}^2 \setminus C$ is called an outer Galois point for a plane curve $C \subset \mathbb{P}^2$ if the function field extension $k(C)/\pi_R^* k(\mathbb{P}^1)$ induced by the projection π_R from R is Galois (see [11, 17]). Furthermore, an outer Galois point is said to be extendable if each element of the Galois group is the restriction of some linear transformation of $\mathbb{P}^2$ (see [4]). The number of outer Galois points (resp. of extendable outer Galois points) is denoted by $\delta'(C)$ (resp. by $\delta'_0(C)$). As we are investigating Galois points or Galois lines, the following fact regarding Galois extensions is required in this and the next sections (see [14, III.7.1, III.7.2]).

Fact 2 *Let $\pi : C \to C'$ be a morphism between smooth projective curves C and C' with $\pi(C) = C'$. Assume that the field extension $k(C)/\pi^* k(C')$ is Galois. Then the following hold.*

(a) *The Galois group acts on each fiber of π transitively.*
(b) *For points $Q_1, Q_2 \in C$ with $\pi(Q_1) = \pi(Q_2)$, the ramification indices are the same.*

In this section, we consider outer Galois points for the plane model C of the generalized Artin–Schreier–Mumford curve X. According to Theorem 1, $\delta'(C) = \delta'_0(C)$. It was proved by the present author that for the case where $L_1 = L_2$, $\delta'_0(C) \geq p^k - 1$ and the equality holds if $p = 2$ (see [4, 5]). Therefore, it has been proved that $\delta'(C) = \delta'_0(C) = p^k - 1$ if $p = 2$. The same assertion holds for the case where $p > 2$.

Theorem 3 *If $L_1 = L_2$, then $\delta'(C) = \delta'_0(C) = p^k - 1$.*

Proof Let $R \in \mathbb{P}^2 \setminus C$ be an outer Galois point. Note that the line $\overline{RP'}$ corresponds to the fiber of the projection π_R, where $P' = (1 : 0 : 0) \in \mathrm{Sing}(C)$. If $R \notin \{Z = 0\}$, then π_R is ramified at each point of Ω_1, by Corollary 1 and Fact 2 (a). However, the directions of the tangent lines at P' are different. This is a contradiction. Therefore, $R \in \{Z = 0\}$.

We can assume that $p > 2$. Since $|G_R| = 2q$, there exists an involution $\tau' \in G_R \subset \mathrm{Lin}(X)$. If $\tau'(P') = (P')$, then τ' fixes some point of Ω_1. This is a contradiction to the transitivity of G_R on fibers in Fact 2 (a). Therefore, $\tau'(P') = Q'$ and $\tau'(Q') = P'$, where $Q' = (0 : 1 : 0) \in \mathrm{Sing}(C)$. Since each elements of $\langle \Sigma, \Gamma \rangle \subset \mathrm{Lin}(X)$ are represented by

$$(X : Y : Z) \mapsto (\lambda^{-1}(X + \alpha Z) : \lambda(Y + \beta Z) : Z)$$

for some $\lambda \in \mathbb{F}_{p^k}^*$ and some $\alpha, \beta \in k$ with $L_1(\alpha) = L_1(\beta) = 0$ as described in the previous section, it follows that τ' is given by

$$(X : Y : Z) \mapsto (\lambda(Y + \beta Z) : \lambda^{-1}(X + \alpha Z) : Z).$$

Then fixed points of τ' on $\{Z = 0\}$ are $(\lambda : 1 : 0)$ and $(-\lambda : 1 : 0)$. Note that any element of G_R fixes R, since G_R preserves any line passing through R. Therefore, $R = (\lambda : 1 : 0)$ or $(-\lambda : 1 : 0)$. The claim follows. □

Remark 1 Let $R_1, \ldots, R_{p^k-1}$ be all outer Galois points for C and let $G_{R_1}, \ldots, G_{R_{p^k-1}}$ be their Galois groups. Then $\mathrm{Aut}(X) = \langle G_{R_1}, \ldots, G_{R_{p^k-1}} \rangle$.

For the case where $L_1 \neq L_2$, the following holds.

Theorem 4 *If $L_1 \neq L_2$, then $\delta'(C) = 0$.*

Proof Let $R \in \mathbb{P}^2 \setminus C$ be an outer Galois point. By Corollary 1 and Fact 2, the projection π_R is ramified at each point of Ω_1. However, the directions of the tangent lines at $P' \in \varphi(\Omega_1)$ are different. This is a contradiction. □

Remark 2 The generalized Artin–Schreier–Mumford curve with $L_1 \neq L_2$ does not belong to families studied by the present author in [4, 5], since a linear subgroup of the automorphism group of the families acts on the set defined by $Z = 0$ transitively.

5 Application to the arrangement of Galois lines

A line $\ell \subset \mathbb{P}^3$ is called a Galois line for a space curve $X \subset \mathbb{P}^3$ if the function field extension $k(X)/\pi_\ell^* k(\mathbb{P}^1)$ induced by the projection π_ℓ from ℓ is Galois (see [3, 18]). In this section, we consider Galois lines for a space model $\varphi(X) \subset \mathbb{P}^3$ of the generalized Artin–Schreier–Mumford curve X, where

$$\varphi : X \to \mathbb{P}^3;\ (x : y : 1 : xy).$$

The following lemma is required to determine Galois lines for $\varphi(X)$.

Lemma 4 *Let $P \in \Omega_1 \cup \Omega_2$ and let $H \supset T_{\varphi(P)}\varphi(X)$ be a tangent hyperplane in $\mathbb{P}^3$ at $\varphi(P)$. Then $\operatorname{ord}_P \varphi^* H \geq q$.*

Proof Let $P = P_\beta \in \Omega_1$. Then H is defined by

$$a(W - \beta X) + b(Y - \beta Z) = 0$$

for some $a, b \in k$ with $(a, b) \neq (0, 0)$. It follows that

$$\operatorname{ord}_{P_\beta} \varphi^* H = \operatorname{ord}_{P_\beta}(a(y - \beta) + b(y - \beta)t),$$

where $t = 1/x$. Since $\operatorname{ord}_{P_\beta}(y - \beta) = q$, the claim follows. □

The following is an analog of [6, Lemma 1 (b)] and can be proved in the same way.

Lemma 5 *Let $H \subset \mathbb{P}^3$ be a hyperplane with $H \neq \{Z = 0\}$. If $H \supset \varphi(\Omega_1)$, then H contains the tangent line at some point of $\varphi(\Omega_1)$, or q points of $(\varphi(X) \cap H) \setminus \varphi(\Omega_1)$ are collinear. For both cases, the defining equation of the tangent line or of the line spanned by the q points of $(\varphi(X) \cap H) \setminus \varphi(\Omega_1)$ is of the form*

$$W - aX = Y - aZ = 0$$

for some $a \in k$.

For the case where $\mathbb{F}_{p^k} = \mathbb{F}_{p^e}$, that is, $L_1 = L_2 = x^q + x$ (for a suitable system of coordinates), the arrangement of Galois lines was determined in [6]. We can assume that $k < e$.

Theorem 5 *Assume that $L_1 = L_2$ and $k < e$. Let $\ell \subset \mathbb{P}^3$ be a line. Then ℓ is a Galois line for $\varphi(X)$ if and only if ℓ is one of the following:*

(a) *an $\mathbb{F}_{p^k}$-line contained in $\{Z = 0\}$ and passing through $(0 : 0 : 0 : 1)$, or*
(b) *the line defined by $W - aX = Y - aZ = 0$ or $W - aY = X - aZ = 0$ for some $a \in k$.*

Proof The proof of the if-part is similar to [6], and is easily verified by a direct computation. Assume that ℓ is a Galois line. The proof for the nonexistence of the case where $\ell \cap \varphi(X) = \emptyset$ and $\ell \not\ni (0 : 0 : 0 : 1)$ is similar to [6, Case (i)], according to Montanucci–Zini's theorem (and Theorem 2). If $\ell \cap \varphi(X) = \emptyset$ and $\ell \ni (0 : 0 : 0 : 1)$, then such Galois lines correspond to Galois points in $\mathbb{P}^2$. According to Theorem 3, ℓ is an $\mathbb{F}_{p^k}$-line, that is, ℓ is of type (a) in the claim. If the degree of the projection from ℓ is $2q - 1$, then ℓ is not a Galois line, by considering the orders $|G_\ell|$ and $|\mathrm{Aut}(X)|$.

Assume that ℓ is a tangent line or $\ell \cap \varphi(X)$ consists of at least two points. If $\ell \subset \{Z = 0\}$ and $\ell \neq \ell_1, \ell_2$, where ℓ_1 and ℓ_2 are lines spanned by $\varphi(\Omega_1)$ and $\varphi(\Omega_2)$ respectively, then there exist points $P \in \varphi(\Omega_1)$ and $Q \in \varphi(\Omega_2)$ such that $\ell = \overline{PQ}$, where $\overline{PQ} \subset \mathbb{P}^3$ is a line passing through P and Q. Then $\deg \pi_\ell = 2q - 2 = 2(q - 1)$. Since $|\mathrm{Aut}(X)| = 2q^2(p^k - 1)$ by Montanucci–Zini's theorem (and Theorem 2), ℓ must not become a Galois line.

Assume that $\ell \cap \{Z = 0\} = \{R\} \not\subset \ell_1 \cup \ell_2$. If there does not exist a pair of points $P \in \varphi(\Omega_1)$ and $Q \in \varphi(\Omega_2)$ with $R \in \overline{PQ}$, then G_ℓ fixes $\Omega_1 \cup \Omega_2$ pointwise. This is a contradiction, since $\mathrm{Aut}(X)$ acts on $\Omega_1 \cup \Omega_2$ faithfully. Therefore, there exist points $P \in \varphi(\Omega_1)$ and $Q \in \varphi(\Omega_2)$ such that $R \in \overline{PQ}$. If H contains the tangent lines at P and at Q, then by Lemma 4, $\ell \cap \varphi(X)$ is an empty set. Therefore, by Fact 2 (b), it follows that H is not a tangent hyperplane at P or at Q. Since the Galois group G_ℓ acts on $\{P, Q\}$ by Fact 2 (a) and Corollary 1, it follows that $\deg \pi_\ell = 2$, namely, X is hyperelliptic. This is a contradiction to Lemma 1.

Assume that $\ell \cap \{Z = 0\} = \{R\} \subset \ell_1 \cup \ell_2$. We can assume that $R \in \ell_1$. The plane H spanned by ℓ_1 and ℓ contains $\varphi(\Omega_1)$. By Fact 2 (a) and Corollary 1, $H \cap \varphi(X) \subset \ell_1 \cup \ell$. Since ℓ contains two points or is a tangent line, it follows from Lemma 5 that ℓ is defined by $W - aX = Y - aZ = 0$ for some $a \in k$, that is, ℓ is of type (b) in the claim. □

Theorem 6 *Assume that $L_1 \neq L_2$. Let $\ell \subset \mathbb{P}^3$ be a line. Then ℓ is a Galois line for $\varphi(X)$ if and only if ℓ is defined by $aW - bX = aY - bZ = 0$ or $aW - bY = aX - bZ = 0$ for some $a, b \in k$ with $(a, b) \neq (0, 0)$.*

Proof The proof of the if-part is similar to [6], and is easily verified by a direct computation. Assume that $\ell \subset \mathbb{P}^3$ is a Galois line. The proof for the nonexistence of the case where $\ell \cap \varphi(X) = \emptyset$ and $\ell \not\ni (0 : 0 : 0 : 1)$ is similar to [6, Case (i)], according to Montanucci–Zini's theorem (and Theorem 2). If $\ell \cap \varphi(X) = \emptyset$ and $\ell \ni (0 : 0 : 0 : 1)$, then such Galois lines correspond to Galois points in $\mathbb{P}^2$. According to Theorem 4, this is a contradiction. If the degree of the projection from ℓ is $2q - 1$, then ℓ is not a Galois line, by considering the orders $|G_\ell|$ and $|\mathrm{Aut}(X)|$.

Assume that ℓ is a tangent line or $\ell \cap \varphi(X)$ consists of at least two points. If $\ell \subset \{Z = 0\}$ and $\ell \neq \ell_1, \ell_2$, then there exist points $P \in \varphi(\Omega_1)$ and $Q \in \varphi(\Omega_2)$ such that $\ell = \overline{PQ}$. Then $\deg \pi_\ell = 2q - 2 = 2(q - 1)$. Since $|\mathrm{Aut}(X)| = q^2(p^k - 1)$ by Montanucci–Zini's theorem (and Theorem 2), ℓ must not become a Galois line.

Assume that $\ell \cap \{Z = 0\} = \{R\} \not\subset \ell_1 \cup \ell_2$. Since $\mathrm{Aut}(X)$ acts on Ω_1, G_ℓ fixes Ω_1 pointwise. The same claim holds for Ω_2. This is a contradiction, since $\mathrm{Aut}(X)$ acts on $\Omega_1 \cup \Omega_2$ faithfully.

Assume that $\ell \cap \{Z = 0\} = \{R\} \subset \ell_1 \cup \ell_2$. We can assume that $R \in \ell_1$. The plane H spanned by ℓ_1 and ℓ contains $\varphi(\Omega_1)$. By Fact 2 (a) and Corollary 1, $H \cap \varphi(X) \subset \ell_1 \cup \ell$. Since ℓ contains two points or is a tangent line, it follows from Lemma 5 that ℓ is defined by $W - aX = Y - aZ = 0$ for some $a \in k$, that is, ℓ is a line described in the claim. □

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