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ORIGINAL PAPER

On the speed of convergence to the asymptotic cone for non-singular nilpotent groups

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Abstract
We study the speed of convergence to the asymptotic cone for a finitely generated nilpotent group endowed with a word metric. The first result on this theme is given by Burago who showed that an abelian group endowed with a word metric converges to the normed space with the speed $O\left(\frac{1}{n}\right)$ in the sense of Gromov–Hausdorff distance. Later Krat showed the same statement for the Heisenberg group, and Breuillard and Le Donne constructed an example, the direct product of the $\mathbb{Z}$ and the Heisenberg group with a specific word metric, whose speed of convergence is precisely $O\left(\frac{1}{\sqrt{n}}\right)$. For 2-step nilpotent groups, we show that if the asymptotic cone is non-singular, then the speed of convergence is $O\left(\frac{1}{n}\right)$ for any choice of generating set. Our argument can be applied to every nilpotent Lie group with a left-invariant sub-Finsler metric. In terms of sub-Finsler geometry, the condition being non-singular is equivalent to the strongly bracket generating condition, and also to absence of abnormal curves.

Keywords Nilpotent groups · Sub-Finsler geometry · Asymptotic cones · Abnormal curves

Mathematics Subject Classification (2000) 20F18 · 53C17

1 Introduction

Let Γ be a torsion-free nilpotent group generated by a finite symmetric subset $S \subset \Gamma$, and ρ_S the associated word metric. The asymptotic cone of (Γ, ρ_S, id) is the Gromov–Hausdorff limit of the sequence $\{(\Gamma, \frac{1}{n}\rho_S, id)\}_{n\in\mathbb{N}}$. In general, the existence and the uniqueness of the limit is not trivial, however, Pansu showed that the asymptotic cone of (Γ, ρ_S, id) is uniquely determined up to isometry in [12]. The limit space (N, d_∞, id) is a Carnot group endowed with a subFinsler metric (see Sect. 2.3). If Γ is 2-step, then N is isomorphic to the Mal'cev completion of Γ as a Lie group.

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The asymptotic cone and the original metric space are sometimes close in the following sense. Burago [4] showed that a Cayley graph of every free abelian group is $(1, C)$-quasi-isometric to its asymptotic cone for some $C > 0$. This implies that the unit ball of a scaled down Cayley graph centered at the identity, denote $B_{\frac{1}{n}\rho_S}(1)$, converges to that of the asymptotic cone rapidly. Namely,

$$d_{GH}(B_{\frac{1}{n}\rho_S}(1), B_{d_\infty}(1)) = O(n^{-1}),$$

where d_{GH} is the Gromov–Hausdorff distance.

Motivated by this result, Gromov [9] asked whether a Cayley graph of a nilpotent group is $(1, C)$-quasi-isometric to its asymptotic cone, and if not, what is the speed of convergence. The first result on non-abelian nilpotent groups is given by Krat [10], who showed that the discrete 3-Heisenberg group $H_3(\mathbb{Z})$ endowed with a word metric is $(1, C)$-quasi isometric to its asymptotic cone. For general cases, Breuillard and Le Donne first gave estimates in [3]. Later the result is sharpened by Gianella [8], who showed that

$$d_{GH}\left(B_{\frac{1}{n}\rho_S}(1), B_{d_\infty}(1)\right) = O\left(n^{-\frac{1}{r}}\right),$$

where r is the nilpotency class of Γ. Moreover, Breuillard and Le Donne also showed in [3] that for the 2-step nilpotent group $\mathbb{Z} \times H_3(\mathbb{Z})$, there is a generating set such that the estimates $O(n^{-\frac{1}{2}})$ is sharp. From this example, Fujiwara [7] asked the following question.

Question 1.1 (Question 4 in [7]) *Let Γ be a lattice in a simply connected non-singular nilpotent Lie group, and ρ a Γ-invariant proper coarsely geodesic pseudo metric. Then are (Γ, ρ) and its asymptotic cone $(1, C)$-quasi isometric for some $C > 0$?*

Here we do not pursue the assumption on the metric such as coarsely geodesic condition. We will mention the group theoretic condition on Γ.

Definition 1.1 A simply connected nilpotent Lie group N is called non-singular if for all z in the center $Z(N)$ and all $x \in N \setminus Z(N)$, there is $y \in N$ such that $[x, y] = z$.

We answer Question 1.1 in the following restricted case.

Theorem 1.1 *Let Γ be a lattice of a simply connected non-singular 2-step nilpotent group N, and ρ_S a word metric on Γ. Then there is $C > 0$ such that (Γ, ρ_S) is $(1, C)$-quasi isometric to its asymptotic cone.*

Remark 1.1 In 2-step case, the asymptotic cone is isomorphic to the Mal'cev completion as a Lie group. Hence the assumptions on the Mal'cev complation can be changed to that on the asymptotic cone.

Via the exponential map from the associated Lie algebra $\mathfrak{n}$ to N, the non-singular condition is equivalent to every bracket generating subspaces in the Lie algebra being *strongly bracket generating*. Here a subspace $V \subset \mathfrak{n}$ is called strongly bracket generating if for any $X \in V\setminus\{0\}$, $\mathfrak{n} = V \oplus [X, V]$. By Theorem A.1 in [11], strongly bracket generating condition, equivalently non-singular condition, is equivalent to the absence of abnormal curves.

Remark 1.2 After the first draft of this paper is completed, we are informed by Emmanuel Breuillard that Theorem 1.1 and an argument in the same line as our proof are known to specialists including him, but it does not exist in the literature yet, and we feel it is worth publishing it.

Theorem 1.1 is on a finitely generated group, which is related to a claim on a nilpotent Lie group by using the following result by Stoll.

Proposition 1.1 (Proposition 4.3 in [14]) *Let Γ be a finitely generated torsion-free 2-step nilpotent group, ρ_S a word metric on Γ, and N the Mal'cev completion of Γ. Then there is a left invariant subFinler metric d_S on N and $C > 0$ such that (Γ, ρ_S) is $(1, C)$-quasi isometric to (N, d_S) by the natural inclusion map.*

He constructed such a metric d_S explicitly, now called the *Stoll metric*.

It is easy to see that the asymptotic cones of (Γ, ρ_S, id) and (N, d_S, id) are isometric, hence the following theorem implies Theorem 1.1.

Theorem 1.2 (Precisely in Theorem 4.1) *Let N be a simply connected non-singular 2-step nilpotent Lie group endowed with a left invariant subFinsler metric d. Then there is $C > 0$ such that (N, d) is $(1, C)$-quasi isometric to its asymptotic cone.*

Remark 1.3 In the original setting of Question 1.1, that is the metric ρ is a coarsely geodesic metric, our method cannot be applied because of the following reason. Roughly speaking, we show the main result by constructing a path in (N, d_∞) from a geodesic in (N, d) and vice versa. The scheme is;

1. Project the geodesic c_0 in (N, d) onto its abelianized normed space, say c_1.
2. Construct a path c_2 in the abelianization of (N, d_∞) which is close to the c_1 in the (Gromov–)Hausdorff sense.
3. Lift up the c_2 to a path c_3 in (N, d_∞).
4. Slight variation of the c_3 can have the same endpoints with the c.

Finally we find that the length of c_3 is same to the c_0 up to constant.

In coarsely geodesic setting, the second step is impossible since geodesics on $\mathbb{R}^n$ with a $\mathbb{Z}^n$-invariant metric may be quite far from the straight segment in the Hausdorff sense (cf. [5] and [1]).

2 The asymptotic cone of a nilpotent Lie group endowed with a left invariant subFinsler metric

Let N be a simply connected 2-step nilpotent Lie group, and d a left invariant subFinsler metric on N. In this section, we shall construct the asymptotic cone of (N, d, id).

2.1 Nilpotent Lie groups and nilpotent Lie algebras

Let $\mathfrak{n}$ be the Lie algebra associated to N. It is known that the exponential map from $\mathfrak{n}$ to N is a diffeomorphism. By the Baker–Campbell–Hausdorff formula, the group operation on N is written by

$$\exp(X) \cdot \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y]).$$

In particular, we can identify the commutator on N and the Lie bracket on $\mathfrak{n}$ as

$$[\exp(X), \exp(Y)] = \exp([X, Y]).$$

Hence we sometimes identify elements in N and $\mathfrak{n}$ via the exponential map.

Let V_∞ be a subspace of $\mathfrak{n}$ such that

$$V_\infty \cap [\mathfrak{n}, \mathfrak{n}] = \{0\} \text{ and } V_\infty + [\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}.$$

Then $\mathfrak{n}$ is spanned by the direct sum $V_\infty \oplus [\mathfrak{n}, \mathfrak{n}]$ and any element in $\mathfrak{n}$ will be written by $X + Y$, where $X \in V_\infty$ and $Y \in [\mathfrak{n}, \mathfrak{n}]$.

To such a decomposition, we can define the following two endomorphisms of N and $\mathfrak{n}$. We may associate a Lie algebra automorphism $\delta_t : \mathfrak{n} \to \mathfrak{n}$ ($t \in \mathbb{R}_{>0}$) which is determined by

$$\delta_t(X + Y) = tX + t^2 Y.$$

This Lie algebra automorphism is called the *dilation*. It induces the diffeomorphism of N via the exponential map (we also denote that diffeomorphism by δ_t).

Set a mapping $\pi : \mathfrak{n} \to V_\infty$ by $\pi(X + Y) = X$. By the Baker–Campbell–Hausdorff formula, it is easy to see that $\pi \circ \log : N \to V_\infty$ is a surjective group homomorphism, as we see V_∞ an abelian Lie group. We will simply denote the homomorphism $\pi \circ \log$ by π.

2.2 Left invariant subFinsler metrics

Let N be a connected Lie group with the associated Lie algebra $\mathfrak{n}$. Suppose a vector subspace $V \subset \mathfrak{n}$ and a norm $\|\cdot\|$ on V are given. Then V induces the left invariant subbundle Δ of the tangent bundle of N. Namely, a vector v at a point $p \in N$ is an element of Δ if $(L_p)^* v \in V$. For such v, we set $\|v\| := \|(L_p)^* v\|$. This Δ is called a *horizontal distribution*.

One says that an absolutely continuous curve $c : [a, b] \to N$ with $a, b \in \mathbb{R}$ is *horizontal* if the derivative $\dot{c}(t)$ is in Δ for almost all $t \in [a, b]$. Then for $x, y \in N$, one may define a subFinsler metric as

$$d(x, y) = \inf \left\{ \int_a^b \|\dot{c}(t)\| dt \,\middle|\, c \text{ is horizontal }, c(a) = x, c(b) = y \right\}.$$

Note that such d is left invariant.

Chow showed that any two points in N are connected by a horizontal path if and only if V is *bracket generating*, that is,

$$V + [V, V] + \cdots + \underbrace{[V, [V, [\cdots] \cdots]}_{r} = \mathfrak{n}.$$

In particular, the subspace V_∞, given in Sect. 2.1, is bracket generating.

2.3 The asymptotic cone

Roughly speaking, an asymptotic cone is a metric space which describes how a metric space looks like when it is seen from very far. This is characterized by the Gromov–Hausdorff distance.

Definition 2.1 Let (X, d, p) be a pointed proper geodesic metric space. If the sequence of scaled metric spaces $\{(X, \frac{1}{n}d, p)\}_{n \in \mathbb{N}}$ converges to a metric space $(X_\infty, d_\infty, p_\infty)$ in the Gromov–Hausdorff topology, then $(X_\infty, d_\infty, p_\infty)$ is called the asymptotic cone of (X, d, p).

Remark 2.1 It is not trivial whether the limit exists or not. For nilpotent Lie groups endowed with left invariant subFinsler metrics, the existence and the uniqueness of the limit is shown in [2]. For more precise information, see [15]

Let us recall the definition of the Gromov–Hausdorff topology on the set of pointed proper geodesic metric spaces. A sequence of pointed proper metric spaces $\{(X_n, d_n, p_n)\}_{n\in\mathbb{N}}$ is said to converge to the pointed metric space $(X_\infty, d_\infty, p_\infty)$ if for any $R > 0$, the sequence of metric balls $\{B_{d_n}(p_n, R)\}_{n\in\mathbb{N}}$ converges to $B_{d_\infty}(p_\infty, R)$ in the Gromov–Hausdorff topology on the set of compact metric spaces.

The Gromov–Hausdorff topology on the set of compact metric spaces is characterized by the Gromov–Hausdorff distance. For compact metric spaces (X, d_X) and (Y, d_Y), it is determined by

$$d_{GH}(X, Y) := \inf \left\{ d_{H,Z}(X, Y) \middle| Z = X \sqcup Y, d_Z|_X = d_X, d_Z|_Y = d_Y \right\},$$

Here $d_{H,Z}$ is the Hausdorff distance on compact subsets on Z, namely the smallest $r > 0$ such that X lies in the r-neighborhood of Y and Y lies in the r-neighborhood of X.

Suppose a left invariant subFinsler metric d on N is determined by a bracket generating subspace $V \subset \mathfrak{n}$ and a norm $\|\cdot\|$ on V. By using the homomorphism π, define a left invariant subFinsler metric d_∞ on N which is determined by the subspace $V_\infty \subset \mathfrak{n}$ and the norm $\|\cdot\|_\infty$ on V_∞ whose unit ball is $\pi(B_{\|\cdot\|}(1))$, where $B_{\|\cdot\|}(1)$ is the unit ball of the normed space $(V, \|\cdot\|)$ cntered at 0.

We will write $d(g) = d(id, g)$ and $d_\infty(g) = d_\infty(id, g)$ for $g \in N$.

Theorem 2.1 (Theorem 3.2 in [3]) *For any sequence $\{g_i\}_{i\in\mathbb{N}}$ on N such that $d(g_i) \to \infty$ as $i \to \infty$,*

$$\lim_{i\to\infty} \frac{d_\infty(g_i)}{d(g_i)} = 1.$$

In particular, the asymptotic cone of (N, d, id) is isometric to (N, d_∞, id).

The pair (N, V_∞) is an example of *Carnot group*. If a subFinsler metric is induced from a Carnot group, such as d_∞, then it satisfies the following properties.

Fact 2.1 *(a) For every horizontal path c,*

$$length(c) = length(\pi \circ c).$$

In particular,

$$\|\pi(g)\|_\infty \le d_\infty(g),$$

and the equality holds if $g \in \exp(V_\infty)$.

(b) For $x, y \in N$,

$$d_\infty(\delta_t(x), \delta_t(y)) = t d_\infty(x, y).$$

Notice that a general subFinsler metric, such as d, does not satisfies Fact 2.1.

Remark 2.2 (1) By its definition, $\pi|_V$ sends R-balls in $(V, \|\cdot\|)$ onto R-balls in $(V_\infty, \|\cdot\|_\infty)$.
(2) By Fact 2.1(a), π sends R-balls in (N, d_∞) onto R-balls in $(V_\infty, \|\cdot\|_\infty)$.
(3) In Lemma 3.3, we shall see that π sends R-balls in (N, d) onto R-balls in $(V_\infty, \|\cdot\|_\infty)$.

3 Geodesics in (N, d)

From now on we will assume that a Lie group N is a simply connected 2-step non-singular nilpotent Lie group, and d is a left invariant subFinsler metric on N determined by a subspace $V \subset \mathfrak{n}$ and a norm $\|\cdot\|$ on V. In this section, we study geodesics in (N, d).

For $g \in (N, d)$, let c be a geodesic from id to g with its length $t = d(g)$. Divide c into M pieces so that each lengths are $\frac{t}{M}$. In other words, c is the concatenation of paths $c_i : [0, \frac{t}{M}] \to N$, $i = 1, \dots, M$, which are geodesics from id to $h_i = c(\frac{i-1}{M}t)^{-1}c(\frac{i}{M}t)$. Notice that $g = h_1 \cdots h_M$. Set

$$I(c, M, R) = \left\{ i \in \{1, \dots M\} \;\middle|\; \|\pi(h_i)\|_\infty < Rd(h_i) = R\frac{t}{M} \right\}$$

for $0 < R \leq 1$. If an integer i is not in $I(c, M, R)$, then it implies that the projection of the subpath $\pi(c_i)$ has the length close to a straight segment. The goal of this section is to show the following proposition.

Proposition 3.1 *There exists a positive constant $K > 0$ such that for any $M \in \mathbb{N}$, any $g \in N$ with $d(g) \geq M$, and any geodesic c joining id and g,*

$$|I(c, M, R)| \leq \frac{K}{(1 - R)^2}.$$

Example 3.1 (The 3-Heisenberg Lie group with a subFinsler metric) The 3-Heisenberg Lie group $H_3(\mathbb{R})$ is the 2-step nilpotent Lie group diffeomorphic to $\mathbb{R}^3$ equipped with a group operation

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{x_1y_2 - x_2y_1}{2}\right).$$

The associated Lie algebra $\mathfrak{h}_3$ is spanned by three vectors $\{X, Y, Z\}$ such that $[X, Y] = Z$, and its derived Lie algebra is $[\mathfrak{h}_3, \mathfrak{h}_3] = Span(Z)$. Then $V_\infty = \langle X, Y\rangle \subset \mathfrak{h}_3$ and we can identify it to the plane $\{(x, y, 0)\} \subset H_3$ via the exponential map.

(1) Let $\|\cdot\|_1$ be the l^1 norm on a vector subspace V_∞, and d_1 the induced left invariant subFinsler metric on $(H_3, V_\infty, \|\cdot\|_1)$.
The shape of geodesics in (H_3, d_1) is given in [6]. For example, a geodesic c from $(0, 0, 0)$ to $\left(0, 0, \frac{t^2}{16}\right)$ is the concatenation of 4 linear paths as in Fig. 1. Here we say a curve is linear if it is represented by $c(t) = \exp(tX)$ for $X \in V_\infty$. We can catch precise shape of geodesics by projecting the curve to the plane $\{(x, y, 0)\}$. As in Fig. 2, it starts and ends at $(0, 0)$ forming the square.
Divide c into 4-pieces and denote them by c_i $(i = 1, 2, 3, 4)$. Then c_i's are the linear paths. It is easy to see that $length(c_i) = length(\pi \circ c_i) = \frac{t}{4}$ for all i. Hence $I(c, 4, 1) = 0$ independent of t.
(2) Let $\|\cdot\|_2$ be the l^2 norm on V_∞, and d_2 the induced subFinsler (subRiemannian) metric on $(H_3, V_\infty, \|\cdot\|_2)$. A geodesic c from $(0, 0, 0)$ to $(0, 0, \frac{t^2}{4\pi})$ is given as in Fig. 3. If the geodesic is projected to $\{(x, y, 0)\}$ by π, then the projected path starts and ends at $(0, 0)$ rounding the circle of radius $\frac{t}{2\pi}$. This curve is not a concatenation of linear paths, however Proposition 3.1 holds.

Notice that the length of c is t, which is the circumference of the projected circle in V_∞. As in Fig. 4, divide c into 4 pieces, and denote them by c_i $(i = 1, 2, 3, 4)$. Each arc c_i's have

Fig. 1 (H_3, d_1)

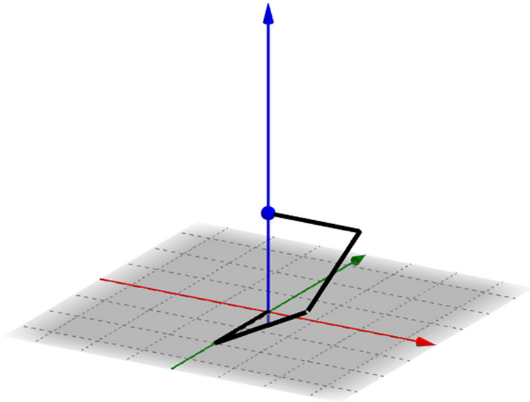


Fig. 2 $(W, \|\cdot\|_1)$

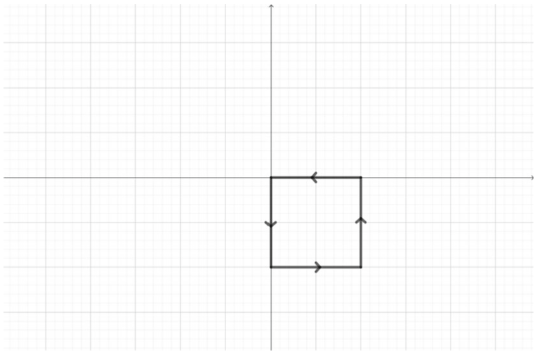
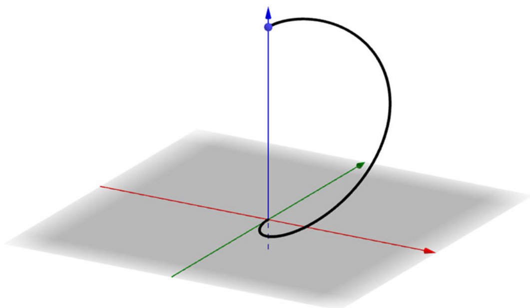


Fig. 3 (H_3, d_2)



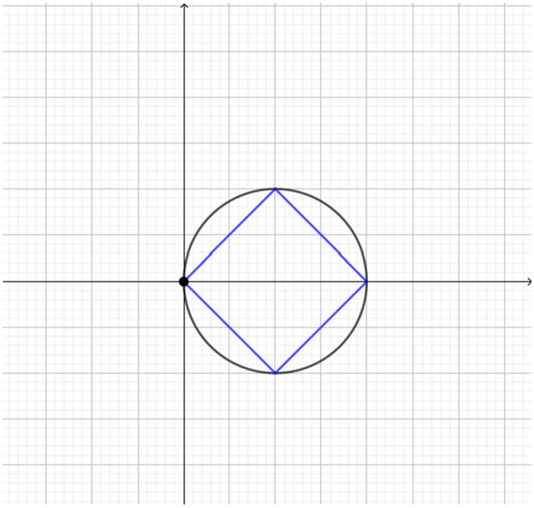
length $\frac{t}{4}$. On the other hand, each chords in Fig. 4 is a geodesic in $(W, \|\cdot\|_2)$ whose length is $2\frac{t}{2\pi}\sin(\frac{\pi}{4}) = \frac{t}{\pi\sqrt{2}}$. Hence h_i's, the endpoints of c_i's, satisfy

$$\|\pi(h_i)\|_2 = \frac{t}{\pi\sqrt{2}}.$$

It means that $I(c, 4, R) = 0$ for $R \leq \frac{2\sqrt{2}}{\pi}$.

We start to prove easy lemmas. Fix a norm $\|\cdot\|_{[N,N]}$ on $[N, N]$. We consider four kinds of closed r -balls. Let $B_d(r) \subset N$ (resp. $B_{d_\infty}(r) \subset N$) be the r-ball centered at id of the metric function d (resp. d_∞), $B_{\|\cdot\|_\infty}(r) \subset V_\infty$ the r-ball centered at 0 of the norm $\|\cdot\|_\infty$, and $B_{\|\cdot\|_{[N,N]}}(r) \subset [N, N]$ the r-ball centered at $id \in [N, N]$ of the fixed norm $\|\cdot\|_{[N,N]}$.

Fig. 4 $(W, \|\cdot\|_2)$



Lemma 3.1 *There exists* $K_1 > 0$ *such that for any* $r \geq 1$,

$$\sup\{\|g^{-1}h\|_{[N,N]} \mid g, h \in B_{d_\infty}(r), g^{-1}h \in [N, N]\} = K_1 r^2.$$

Proof Take $g, h \in B_{d_\infty}(r)$ so that $g^{-1}h \in [N, N]$. By Fact 2.1(b), $g' = \delta_{\frac{1}{r}}(g)$ and $h' = \delta_{\frac{1}{r}}(h)$ are in $B_{d_\infty}(1)$.

Set $X = X_1 + X_2 = \log(g)$ and $Y = X_1 + Y_2 = \log(h)$, where $X_1 \in V_\infty$ and $X_2, Y_2 \in [\mathfrak{n}, \mathfrak{n}]$. Here we can take the common X_1 since $g^{-1}h \in [N, N]$. By the definition of $\delta_{\frac{1}{r}}$, $\log(g') = \frac{1}{r}X_1 + \frac{1}{r^2}X_2$ and $\log(h') = \frac{1}{r}X_1 + \frac{1}{r^2}Y_2$.

Then

$$\begin{aligned} g^{-1}h &= \exp(-X_1 - X_2)\exp(X_1 + Y_2) \\ &= \exp(-X_2 + Y_2) \\ &= \exp(r^2 \frac{1}{r^2}(-X_2 + Y_2)) \\ &= \exp(\frac{1}{r^2}(-X_2 + Y_2))^{r^2} \\ &= (g'^{-1}h')^{r^2}. \end{aligned}$$

We obtain the desired equality

$$\begin{aligned} &\sup\left\{\|g^{-1}h\|_{[N,N]} \mid g, h \in B_{d_\infty}(r), g^{-1}h \in [N, N]\right\} \\ &= r^2 \sup\{\|x^{-1}y\|_{[N,N]} \mid x, y \in B_{d_\infty}(1), x^{-1}y \in [N, N]\} \\ &= K_1 r^2, \end{aligned}$$

where $K_1 = \sup\{\|x^{-1}y\|_{[N,N]} \mid x, y \in B_{d_\infty}(1), x^{-1}y \in [N, N]\} < \infty$. □

The following lemma reflects the non-singular condition. Notice that the derived subgroup $[N, N]$ is contained in the center $Z(N)$ if the nilpotent group N is 2-step.

Lemma 3.2 *There exists* $L_0 > 0$ *such that for all* $r_1, r_2 \in \mathbb{R}_{>0}$ *and all* $g \in \pi^{-1}(\partial B_{\|\cdot\|_\infty}(0, r_1))$,

$$B_{\|\cdot\|_{[N,N]}}(L_0 r_1 r_2) \subset [g, B_{d_\infty}(r_2)].$$

Proof First of all, we show that a subset $[g, B_{d_\infty}(r_2)]$ is a compact star convex neighborhood around $id \in [N, N]$.

The subset $[g, B_{d_\infty}(r_2)]$ is compact since the mapping $[g, \cdot] : N \to [N, N]$ is continuous and the ball $B_{d_\infty}(r_2)$ is compact. Moreover it is a neighborhood around the identity since $[g, \cdot]$ is a submersion by non-singular condition.

Next we check the star convexity. For any $h \in [g, B_{d_\infty}(r_2)]$, we can choose $Y \in B_{\|\cdot\|_\infty}(r_2)$ such that $h = [g, \exp(Y)]$. By Campbell–Baker–Haudorff formula, for $s \in [0, 1]$,

$$h^s = [g, \exp(Y)]^s = [g, \exp(sY)].$$

By Fact 2.1(b),

$$d_\infty(\exp(sY)) = s d_\infty(\exp(Y)) \le r_2.$$

Hence $h^s \in [g, B_{d_\infty}(r_2)]$, that is, $[g, B_{d_\infty}(r_2)]$ is star convex.

We will compute the positive number $L_0 > 0$. By the star convexity, there exists $L(g) > 0$ such that

$$B_{\|\cdot\|_{[N,N]}}(L(g)) \subset [g, B_{d_\infty}(1)].$$

We can assume $L(g_1) = L(g_2)$ if $\pi(g_1) = \pi(g_2)$, since $[g_1, B_{d_\infty}(1)] = [g_2, B_{d_\infty}(1)]$ for such g_i's. Since the commutating operator $[\cdot, \cdot]$ is continuous, we may take $L(g)$ continuously. Hence

$$L_0 = \min\{L(g) \mid g \in \pi^{-1}(\partial B_{\|\cdot\|_\infty}(1))\}$$

exists and is positive.

We will check that such L_0 is the desired constant. Since $[g, B_{d_\infty}(r_2)]$ is star convex, we only need to show that all points at the boundary of $[g, B_{d_\infty}(r_2)]$ are at least $L_0 r_1 r_2$ away from the identity. It is easily observed that every element h at the boundary $\partial[g, B_{d_\infty}(r_2)]$ is written by $h = [\exp(\pi(g)), \exp(Y)]$, where $Y \in \partial B_{\|\cdot\|_\infty}(r_2)$. Set $X' = \frac{1}{r_1}\pi(g)$ and $Y' = \frac{1}{r_2}Y$, then we have

$$\|h\|_{[N,N]} = \|[\exp(\pi(g)), \exp(Y)]\|_{[N,N]} = r_1 r_2 \|[\exp(X'), \exp(Y')]\|_{[N,N]} \ge L_0 r_1 r_2.$$

□

Remark 3.1 We can replace $[g, B_{d_\infty}(r_2)]$ to $[g, B_d(r_2)]$ in Lemma 3.2, since $\pi(B_d(r_2)) = \pi(B_{d_\infty}(r_2))$ implies

$$[g, B_{d_\infty}(r_2)] = [g, B_d(r_2)].$$

Next we study a length preserving translation of a element in $(V_\infty, \|\cdot\|_\infty)$ to $(V, \|\cdot\|)$ and vice versa.

Lemma 3.3 *For any* $g \in N$, *there exists* $Y_g \in \pi|_V^{-1}(\pi(g))$ *such that*

(1) $\|Y_g\| = \|\pi(g)\|_\infty = d(\exp(Y_g)) = \inf\{d(h) \mid h \in \pi^{-1}(\pi(g))\}$,
(2) *An infinite path* $c : \mathbb{R}_{\ge 0} \to N, t \mapsto \exp\left(t\frac{Y_g}{\|Y_g\|}\right)$ *is a geodesic ray i.e. for any* $t_1, t_2 \in \mathbb{R}_{\ge 0}$, $d(c(t_1), c(t_2)) = |c_1 - c_2|$.

Proof From the construction of the asymptotic cone of (N, d, id), $\pi|_V(B_{\|\cdot\|}(R)) = B_{\|\cdot\|_\infty}(R)$ for any $R > 0$. Thus for any $g \in N$, we can take Y_g in V such that $\|Y_g\| = \|\pi(g)\|_\infty$.

We shall see that this Y_g is the desired one. Clearly $\|Y_g\| \geq d\left(\exp(Y_g)\right)$ since the curve $c : [0, \|Y_g\|] \to N$, $c(t) = \exp\left(t\frac{Y_g}{\|Y_g\|}\right)$ is a horizontal path from id to $\exp(Y_g)$ such that $length(c) = \|Y_g\|$.

We claim the converse by showing the inequality

$$\|\pi(g)\|_\infty \leq d\left(\exp(Y_g)\right). \tag{1}$$

Let $c_1 : \left[0, d\left(\exp(Y_g)\right)\right] \to N$ be a geodesic from id to $\exp(Y_g)$ in (N, d). Then we obtain the horizontal path c_2 in (N, d_∞) by letting the derivative $c_2'(t) = \pi(c_1'(t))$ for each $t \in [0, d\left(\exp(Y_g)\right)]$. Since π is distance non-increasing, $length(c_2) \leq length(c_1)$. By using Fact 2.1(a), $\pi \circ c_2$ is a path in V_∞ from id to $\pi(Y_g) = \pi(g)$ whose length equals that of c_2. Now we have constructed the path $\pi \circ c_2$ in $(V_\infty, \|\cdot\|_\infty)$ from id to $\pi(g)$ whose length is shorter than $length(c_1)$, which yields the inequality (1).

The construction of $\pi \circ c_2$ from c_1 is applied to any $h \in \pi^{-1}(\pi(g))$ and any geodesic c_1 from id to h. Hence the inequality $d(h) \geq \|\pi(g)\|_\infty$ holds. This argument yields the last part of the equality.

The second part of this lemma follows in the same way. The above arguments imply that $c : [0, d(\exp(Y_g))] \to N$, $c(t) = \exp\left(t\frac{Y_g}{\|Y_g\|}\right)$ is a geodesic from id to $\exp(Y_g)$. By the choice of Y_g, we can show the second part of this lemma if $\|tY_g\| = \|t\pi(g)\|_\infty$ for $t \in \mathbb{R}_{\geq 0}$. It is trivial since the mapping π is a linear homomorphism. □

Lemma 3.4 (Proposition 2.13 in [3]) *There is $K_2 > 0$ such that for any $g \in N$,*

$$\frac{1}{K_2}d(g) - K_2 \leq d_\infty(g) \leq K_2 d(g) + K_2.$$

Now we pass to the proof of Proposition 3.1.

Proof *(Proof of Proposition 3.1)* Fix $M \in \mathbb{N}$ and $0 < R \leq 1$. Let c be a geodesic from id to $g \in N$ with $length(c) = t \geq M$. We consider an upper bound of the cardinality of $I = I(c, M, R)$. Divide c into M pieces, and denote each by c_i. Let h_i be the endpoint of c_i, that is, $h_i = c\left(\frac{t}{M}(i-1)\right)^{-1} c\left(\frac{t}{M}i\right)$. Deform c and c_i as follows (Figs. 5 and 6).

(1) If $i \in I$, set $\tilde{c}_i : [0, \|Y_{h_i}\|] \to N$,

$$\tilde{c}_i(t) = \frac{Y_{h_i}}{\|Y_{h_i}\|}t,$$

where Y_{h_i} are given as in Lemma 3.3

(2) If $i \notin I$, set $\tilde{c}_i = c_i$.

(3) Set $\tilde{c}$ to be the concatenation of $\tilde{c}_i$'s starting at the identity.

This $\tilde{c}$ is a horizontal path in (N, d). Let $\tilde{g}$ be the endpoint of $\tilde{c}$, and $\tilde{h}_i$ the endpoint of $\tilde{c}_i$. Hence $\tilde{h}_i = Y_{h_i}$ for $i \in I$ and $\tilde{h}_i = h_i$ for $i \notin I$. By the triangle inequality, $d\left(\tilde{g}\right)$ is bounded above by

$$\sum_{i \in I} d\left(\tilde{h}_i\right) + \sum_{i \notin I} \frac{t}{M}. \tag{2}$$

By using (2) and Lemma 3.3,

$$d(g) - d\left(\tilde{g}\right) \geq \sum_{i \in I}\left(\frac{t}{M} - d\left(\tilde{h}_i\right)\right)$$

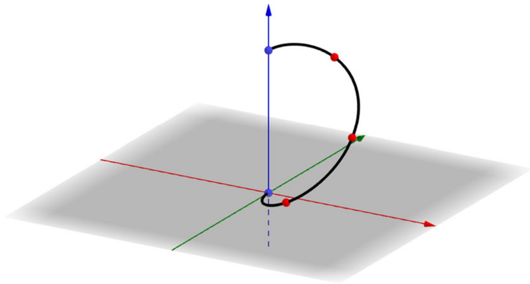


Fig. 5 The path c

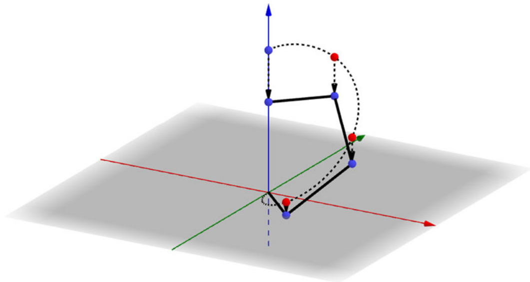


Fig. 6 The path $\tilde{c}$

$$
\begin{aligned}
&= \sum_{i \in I} \left(\frac{t}{M} - d\left(Y_{h_i}\right) \right) \\
&= \sum_{i \in I} \left(\frac{t}{M} - \|\pi(h_i)\|_\infty \right) \\
&\geq \frac{t}{M} (1 - R) |I|.
\end{aligned}
$$

We shall see that $d(g) - d(\tilde{g}) = O(t)$ as $t \to \infty$.

Set $h = \tilde{g}^{-1} g$. By the triangle inequality,

$$d(g) - d(\tilde{g}) \leq d(h).$$

Since each $\tilde{h}_i^{-1} h_i$ is in the center of N,

$$h = \tilde{g}^{-1} g = \tilde{h}_M^{-1} \cdots \tilde{h}_1^{-1} h_1 \cdots h_M = \prod_{i \in I} \tilde{h}_i^{-1} h_i = \prod_{i \in I} Y_{h_i}^{-1} h_i \in [N, N].$$

By Lemma 3.2, we can choose $X, Y \in \partial(B_{\|\cdot\|}(1))$ such that

(1) $[\exp(X), \exp(Y)] = [X, Y] \in h^{\mathbb{R}_{>0}}$, and
(2) $\|[X, Y]\|_{[N,N]} \geq L_0$.

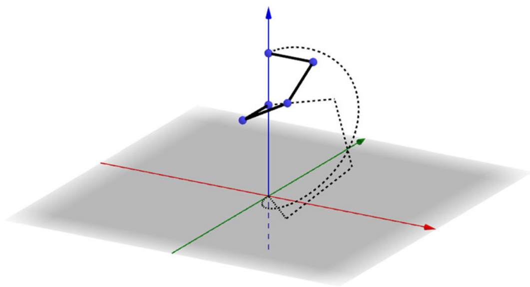
Set $r \in \mathbb{R}_{\geq 0}$ such that

$$[\sqrt{r}X, \sqrt{r}Y] = r[X, Y] = h.$$

Then we can construct a horizontal path from id to h (equivalently, can construct a path from $\tilde{g}$ to g by translating the starting point) by connecting the following four paths: $c_1(s) = -Xs$, $c_2(s) = -Ys$, $c_3(s) = Xs$ and $c_4(s) = Ys$ for $s \in [0, \sqrt{r}]$ (Fig. 7).
By the triangle inequality, we obtain

$$d(h) \leq 4\sqrt{r}.$$

Fig. 7 The path from $\tilde{g}$ to g



By the definition of X, Y and r, $\|h\|_{[N,N]} = \|r[X,Y]\|_{[N,N]} = r\|[X,Y]\|_{[N,N]}$. Hence we obtain

$$r \leq \frac{\|h\|_{[N,N]}}{L_0}.$$

Finally we can estimate $\|h\|_{[N,N]}$ by using Lemma 3.1 and Lemma 3.4,

$$\begin{aligned}
\|h\|_{[N,N]} &= \|\prod_{i\in I} Y_{h_i}^{-1} h_i\|_{[N,N]} \\
&\leq \sum_{i\in I} \|Y_{h_i}^{-1} h_i\|_{[N,N]} \\
&\leq |I| K_1 \left(\max\left\{d_\infty(h_i), d_\infty(Y_{h_i})\right\}\right)^2 \\
&\leq |I| K_1 \left(\max\left\{K_2 d(h_i) + K_2, K_2 d(Y_{h_i}) + K_2\right\}\right)^2 \\
&\leq 4K_1K_2^2 \frac{t^2}{M^2}|I|.
\end{aligned}$$

To be summarized,

$$\begin{aligned}
\frac{t}{M}(1-R)|I| &\leq d(g) - d(\tilde{g}) \\
&\leq d(h) \\
&\leq 4\sqrt{r} \\
&\leq 4\sqrt{\frac{\|h\|_{[N,N]}}{L_0}} \\
&\leq 8K_2 \frac{t}{M}\sqrt{\frac{K_1|I|}{L_0}}.
\end{aligned}$$

Solve the quadratic inequality for $\sqrt{|I|}$, then we have

$$|I| \leq \frac{64K_1K_2^2}{L_0(1-R)^2} = \frac{K}{(1-R)^2},$$

where $K = \frac{64K_1K_2^2}{L_0}$. □

Remark 3.2 Another choice of a norm may inherit another constant $K > 0$, however it does not affect the later arguments. If necessary, we can take the infimum one among obtained K since our method can be applied to any norm.

4 Proof of the main theorem

By using the facts in the previous section, we show the following theorem, which is a precise statement of Theorem 1.2.

Theorem 4.1 *Let N be a simply connected non-singular 2-step nilpotent Lie group endowed with a left invariant subFinsler metric d, and (N, d_∞, id) the asymptotic cone of (N, d, id). Then there is $C > 0$ such that for any $g \in N$,*

$$|d(g) - d_\infty(g)| < C.$$

Proof First we show that $d_\infty(g) - d(g)$ is uniformly bounded above. Fix $0 < R < 1$ and $M > 0$ sufficiently large so that $M - |I(c, M, R)| \neq 0$ for any geodesic c with $length(c) \geq M$. It is possible by Proposition 3.1. It suffices to show the case where $g \in N \setminus B_d(M)$, since d and d_∞ are proper metrics on N.

Let $t = d(g)$ and c a geodesic from id to g in (N, d). We will construct a horizontal path $\check{c}$ in (N, d_∞) which starts at the identity and ends at g, and show that the length of $\check{c}$ is not so long relative to that of c.

It needs two steps to construct a path $\check{c}$. First, Deform c into a horizontal path $\tilde{c}$ in (N, d_∞) as follows.

(1) Divide c into M pieces of geodesics c_i as in Proposition 3.1. Since d is left invariant, we may see each c_i a geodesic from id to $h_i \in N$ with $h_1 \cdots h_M = g$.
(2) Divide c_i into $m = \left[\frac{t}{M}\right]$ pieces of geodesics c_{ij} and set h_{ij} in the same way, where $[\ \cdot\ ]$ is the Gaussian symbol.
(3) Set $\tilde{c}$ the concatenation of $\tilde{c_{ij}}(s) = s\frac{\pi(h_{ij})}{\|\pi(h_{ij})\|_\infty}$, $s \in [0, \|\pi(h_{ij})\|_\infty]$. By Lemma 3.3, $length\ (\tilde{c}) = \sum \|\pi(h_{ij})\|_\infty \leq \sum d(h_{ij}) = d(g)$.
(4) Let $\tilde{c}_i$ be the concatenation of paths $\tilde{c}_{i1}, \ldots, \tilde{c}_{im}$.

Let $\tilde{g}$ be the endpoint of $\tilde{c}$ and set $h = \tilde{g}^{-1}g$. Since $\pi(h_{ij})^{-1}h_{ij} \in [N, N]$,

$$h = \pi(h_{Mm})^{-1} \cdots \pi(h_{11})^{-1}h_{11} \cdots h_{Mm} \in [N, N].$$

By using the path $\tilde{c}$, we shall construct a horizontal path $\check{c}$. By definition of I, $\|\pi(h_i)\|_\infty \geq Rd(h_i) = R\frac{t}{M}$ for $i \notin I$. In particular, $h_i \notin [N, N]$ for $i \notin I$. By Lemma 3.2, there exists $X_i \in \partial\pi(B_{\|\cdot\|_\infty}(1)))$ such that $[h_i, X_i] \in h^{\mathbb{R}_{>0}}$ and that $\|[h_i, X_i]\|_{[N,N]} \geq L_0$. Set $r \in \mathbb{R}_{\geq 0}$ so that

$$\prod_{i \notin I}[h_i, rX_i] = (\prod_{i \notin I}[h_i, X_i])^r = h.$$

Define $\check{c}$ as the concatenation of $\check{c}_i$, $i = 1, \ldots, M$, given as follows (Figs. 8, 9, 10 and 11).

(1) For $i \notin I$, let $\check{c}_i$ be a concatenation of three paths; $\check{c}_{i1}(s) = -sX_i \quad (s \in [0, r])$, $\tilde{c}_i$, and $\check{c}_{i2}(s) = sX_i \quad (s \in [0, r])$. Hence the length of $\check{c}_i$ is $2r + \sum_j \|\pi(h_{ij})\|_\infty$.
(2) For $i \in I$, set $\check{c}_i = \tilde{c}_i$.

This path $\check{c}$ starts at the identity and ends at g by the Campbell–Baker–Hausdorff formula. The length of $\check{c}$ is

$$length\ (\check{c}) = \sum \|\pi(h_{ij})\|_\infty + 2r(M - |I|) \leq d(g) + 2r(M - |I|).$$

The rest of the proof is to find an upper bound of $r(M - |I|)$.

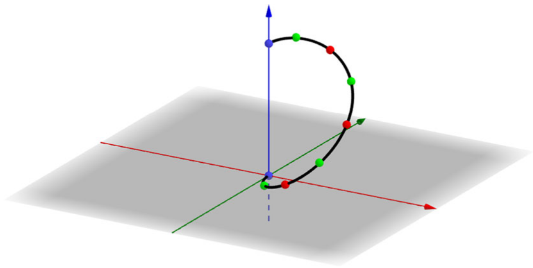


Fig. 8 The path c

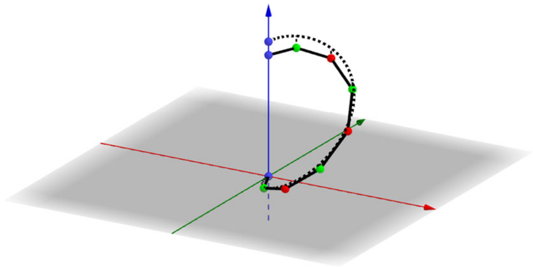


Fig. 9 The path $\tilde{c}$

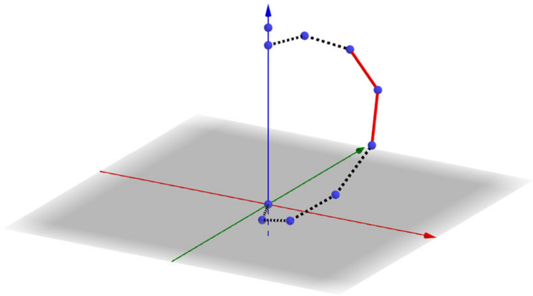


Fig. 10 The subpath $\tilde{c}_i$ for $i \notin I$

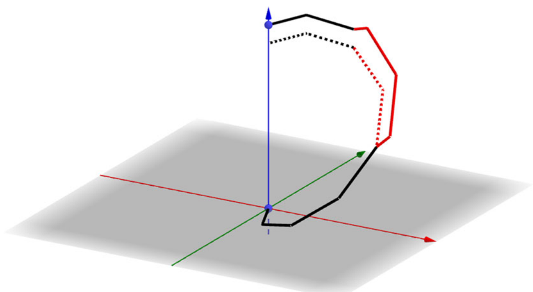


Fig. 11 The path $\breve{c}$

By Lemma 3.2,

$$\|[h_i, X_i]\|_{[N,N]} = \|[\pi(h_i), X_i]\|_{[N,N]} \geq R\frac{t}{M}L_0.$$

Since each $[h_i, rX_i]$ is in $h^{\mathbb{R}_{>0}}$, we obtain

$$\|h\|_{[N,N]} = \left\|\prod_{i \notin I}[h_i, rX_i]\right\|_{N,N]} = r\sum_{i \notin I}\|[h_i, X_i]\|_{[N,N]} \geq r(M - |I|)R\frac{t}{M}L_0. \quad (3)$$

Hence our goal is changed to find an upper bound of $\|h\|_{[N,N]}$. By using Lemmas 3.1 and 3.4,

$$\begin{aligned}
\|h\|_{[N,N]} &= \left\|\pi(h_{mM})^{-1}\cdots\pi(h_{11})^{-1}h_{11}\cdots h_{mM}\right\|_{[N,N]} \\
&= \left\|\prod \pi(h_{ij})^{-1}h_{ij}\right\|_{[N,N]} \\
&\leq \sum \|\pi(h_{ij})^{-1}h_{ij}\|_{[N,N]} \\
&\leq mM \sup\left\{\|g^{-1}h\|_{[N,N]} \;\middle|\; g,h \in B_d\left(\frac{t}{mM}\right), g^{-1}h \in [N,N]\right\} \\
&\leq mM \sup\left\{\|g^{-1}h\|_{[N,N]} \;\middle|\; g,h \in B_{d_\infty}\left(K_2\frac{t}{mM}+K_2\right), g^{-1}h \in [N,N]\right\} \\
&\leq mMK_1\left(K_2\frac{t}{mM}+K_2\right)^2 \\
&\leq 4mMK_1K_2^2\frac{t^2}{m^2M^2} \\
&\leq 4tK_1K_2^2.
\end{aligned}$$

Hence $\|h\|_{[N,N]}$ is linearly bounded by t.

Combining with the Eq. (3), we obtain

$$r(M-|I|) \leq \frac{4K_1K_2^2M}{RL_0}.$$

We have constructed a path $\check{c}$ which is sufficiently short relative to the original path c, hence we have

$$d_\infty(g) \leq d(g) + \frac{8K_1K_2^2M}{RL_0}.$$

The other side of the inequality follows in a similar way. The difference is only the construction of $\tilde{c}$ and $\check{c}$. In the construction of $\tilde{c}$, we let $\tilde{c}_{ij}(s) = s\frac{Y_{h_{ij}}}{\|Y_{h_{ij}}\|}$. In the construction of $\check{c}$, we let $\check{c}_{i1}(s) = -sY_{X_i}$ and $\check{c}_{i2}(s) = sY_{X_i}$. The rest of the proof follows in the same way. □

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Declarations

Conflict of interest The main result of this manuscript was announced in the paper "On the rate of convergence to the asymptotic cone for nilpotent groups and subFinsler geometry" without proof. I write this manuscript since it deserves publishing its proof, and already discussed the author Breuillard and Le Donne.

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