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CORRECTION

Correction to: Classification of generalized Wallach spaces

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The paper [2] is devoted to the classification of generalized Wallach spaces. A generalized Wallach space is a homogeneous spaces G/H of a connected compact semisimple Lie group G (H is a compact subgroup of G), such that there is a $\langle \cdot, \cdot \rangle$-orthogonal and $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$, where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively, $\langle \cdot, \cdot \rangle$ is the minus Killing form of $\mathfrak{g}$, the modules $\mathfrak{p}_i$ are $\mathrm{Ad}(H)$-irreducible, $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ for $i = 1, 2, 3$. Here we will make a correction to the obtained classification. In what follows, we use the notation from [2].

The main result of the paper [2] (Theorem 1) should be stated as follows (in fact, we just add the item (4)):

Theorem 1 *Let G/H be a connected and simply connected compact homogeneous space. Then G/H is a generalized Wallach space if and only if it is of one of the following types:*

(1) *G/H is a direct product of three irreducible symmetric spaces of compact type ($A = a_1 = a_2 = a_3 = 0$ in this case);*
(2) *The group G is simple and the pair $(\mathfrak{g}, \mathfrak{h})$ is one of the pairs in Table 1 of [2] (the embedding of $\mathfrak{h}$ to $\mathfrak{g}$ is determined by the following requirement: the corresponding pairs $(\mathfrak{g}, \mathfrak{k}_i)$ and $(\mathfrak{k}_i, \mathfrak{h})$, $i = 1, 2, 3$, in Table 2 of [2] are symmetric);*
(3) *$G = F \times F \times F \times F$ and $H = \mathrm{diag}(F) \subset G$ for some connected simply connected compact simple Lie group F, with the following description on the Lie algebra level:*

$$(\mathfrak{g}, \mathfrak{h}) = \big(\mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f} \oplus \mathfrak{f},\ \mathrm{diag}(\mathfrak{f}) = \{(X, X, X, X) \,|\, X \in f\}\big),$$

where $\mathfrak{f}$ is the Lie algebra of F, and (up to permutation) $\mathfrak{p}_1 = \{(X, X, -X, -X) \,|\, X \in f\}$, $\mathfrak{p}_2 = \{(X, -X, X, -X) \,|\, X \in f\}$, $\mathfrak{p}_3 = \{(X, -X, -X, X) \,|\, X \in f\}$ ($a_1 = a_2 = a_3 = 1/4$ in this case).

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(4) $H = \operatorname{diag}(K) \subset K \times K \subset F \times F = G$, *where* $(\mathfrak{f}, \mathfrak{k})$ *is a compact irreducible symmetric pair with simple* $\mathfrak{f}$ *and with simple or one-dimensional* $\mathfrak{k}$, *(up to permutation)* $\mathfrak{p}_1 = \{(X, X) \mid X \in \mathfrak{q}\}$, $\mathfrak{p}_2 = \{(X, -X) \mid X \in \mathfrak{q}\}$, $\mathfrak{p}_3 = \{(Y, -Y) \mid Y \in \mathfrak{k}\}$, and $\mathfrak{q}$ *is the orthogonal complement to* $\mathfrak{k}$ *in* $\mathfrak{f}$ *with respect to the Killing form of the Lie algebra* $\mathfrak{f}$.

The same item (4) should be added in the statement of Theorem 3 in [2]. The reason for the above correction is the fact that Corollary 2 in [2] is not correct in general, but it is true under some additional assumptions. The correct version of this corollary is as follows.

Corollary 1 *If* $p \geq 2$ *and at least one of the modules* $\mathfrak{p}_i$, $i = 1, 2, 3$, *is situated in some simple ideal* $\mathfrak{g}_j$ *of the Lie algebra* $\mathfrak{g}$, *then* $A = 0$, *consequently, G/H locally is a direct product of three irreducible symmetric spaces of compact type.*

Proof Without loss of generality we may suppose that $\mathfrak{p}_1 \subset \mathfrak{g}_1$, then $[\mathfrak{p}_1, \mathfrak{p}_2] \subset \mathfrak{p}_3 \cap \mathfrak{g}_1$ and $[\mathfrak{p}_1, \mathfrak{p}_3] \subset \mathfrak{p}_2 \cap \mathfrak{g}_1$. If $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$ or $[\mathfrak{p}_1, \mathfrak{p}_3] = 0$, we get $A = 0$. Otherwise, $\mathfrak{p}_2, \mathfrak{p}_3 \subset \mathfrak{g}_1$ (note that all the modules $\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_2 \cap \mathfrak{g}_1$, and $\mathfrak{p}_3 \cap \mathfrak{g}_1$ are $\operatorname{Ad}(H)$-irreducible), which implies $p = 1$. □

This result should be completed with the following proposition (that provides the case (4) for Theorems 1 and 3 in [2]).

Proposition 1 *If* $p \geq 2$ *and no one module* $\mathfrak{p}_i$, $i = 1, 2, 3$, *is in some simple ideal* $\mathfrak{g}_j$ *of* $\mathfrak{g}$, then $p = 2$ and $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{f} \oplus \mathfrak{f}, \operatorname{diag}(\mathfrak{k}))$, *where* $(\mathfrak{f}, \mathfrak{k})$ *is a compact irreducible symmetric pair with simple* $\mathfrak{f}$ *and with simple or one-dimensional* $\mathfrak{k}$. *Moreover, up to permutation of indices, we have* $\mathfrak{p}_1 = \{(X, X) \mid X \in \mathfrak{q}\}$, $\mathfrak{p}_2 = \{(X, -X) \mid X \in \mathfrak{q}\}$, $\mathfrak{p}_3 = \{(Y, -Y) \mid Y \in \mathfrak{k}\}$, *where* $\mathfrak{q}$ *is the orthogonal complement to* $\mathfrak{k}$ *in* $\mathfrak{f}$ *with respect to the Killing form of the Lie algebra* $\mathfrak{f}$.

Proof Recall that $\varphi_i(\mathfrak{h})$ is the $\langle \cdot, \cdot \rangle$-orthogonal projection of $\mathfrak{h}$ to $\mathfrak{g}_i$. Let $\mathfrak{q}_i$ be the $\langle \cdot, \cdot \rangle$ -orthogonal complement to $\varphi_i(\mathfrak{h})$ in $\mathfrak{g}_i$, $1 \leq i \leq p$. It is clear that $\mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \cdots \oplus \mathfrak{q}_p \subset \mathfrak{p}$. Obviously, we have $p \leq 3$. If $p = 3$, then all $\mathfrak{q}_i$, $i = 1, 2, 3$, are $\operatorname{Ad}(H)$-irreducible and $\varphi_1(\mathfrak{h}) \oplus \varphi_2(\mathfrak{h}) \oplus \varphi_2(\mathfrak{h}) \subset \mathfrak{h}$. Since $[\varphi_i(\mathfrak{h}), \mathfrak{q}_i] \neq 0$ and $[\varphi_i(\mathfrak{h}), \mathfrak{q}_j] = 0$ for $i \neq j$, the $\operatorname{Ad}(H)$ -modules $\mathfrak{q}_i$, $i = 1, 2, 3$, are pairwise non-isomorphic, hence, they coincides with the corresponding modules $\mathfrak{p}_i$, $i = 1, 2, 3$. By the above corollary we have $A = 0$ in this case.

If $p = 2$, then there are some isomorphic $\operatorname{Ad}(H)$-irreducible submodules $\mathfrak{q}_1' \subset \mathfrak{q}_1$ and $\mathfrak{q}_2' \subset \mathfrak{q}_2$. Therefore, by the above arguments, $\varphi_1(\mathfrak{h}) \not\subset \mathfrak{h}$ and $\varphi_2(\mathfrak{h}) \not\subset \mathfrak{h}$ (otherwise, $\mathfrak{q}_1'$ is not isomorphic to $\mathfrak{q}_2'$). Hence, $\mathfrak{h} \subsetneq \varphi_1(\mathfrak{h}) \oplus \varphi_2(\mathfrak{h})$, $\mathfrak{q}_1' = \mathfrak{q}_1$, and $\mathfrak{q}_2' = \mathfrak{q}_2$. Without loss of generality, we may assume that $\mathfrak{p}_1 \oplus \mathfrak{p}_2 = \mathfrak{q}_1 \oplus \mathfrak{q}_2$, and $\mathfrak{h} \oplus \mathfrak{p}_3 = \varphi_1(\mathfrak{h}) \oplus \varphi_2(\mathfrak{h})$. Therefore, $\big(\varphi_1(\mathfrak{h}) \oplus \varphi_2(\mathfrak{h}), \mathfrak{h}\big)$ is a compact irreducible symmetric pair, which has the form $\big(\mathfrak{k} \oplus \mathfrak{k}, \operatorname{diag}(\mathfrak{k})\big)$, where $\mathfrak{k}$ is a compact simple Lie algebra or $\mathbb{R}$ [1, Theorem 7.81]. Hence, φ_1 and φ_2 determine Lie algebra isomorphisms between $\mathfrak{h}$ and $\varphi_i(\mathfrak{h})$, $i = 1, 2$. Let us consider $\theta : \varphi_1(\mathfrak{h}) \mapsto \varphi_2(\mathfrak{h})$, such that $\theta = \varphi_2 \circ \varphi_1^{-1}$. It is clear that $\mathfrak{h} = \{(Y, \theta(Y)) \mid Y \in \varphi_1(\mathfrak{h})\}$.

Now, let us consider the $\langle \cdot, \cdot \rangle$-orthogonal projections $\pi_i : \mathfrak{p}_1 \to \mathfrak{q}_i$, $i = 1, 2$. We may assume that π_1 is a bijection (otherwise, we can take $\mathfrak{p}_2$ instead of $\mathfrak{p}_1$). Now, let us consider the $\operatorname{Ad}(H)$-equivariant linear map $\psi := \pi_2 \circ \pi_1^{-1} : \mathfrak{q}_1 \mapsto \mathfrak{q}_2$. We have $\mathfrak{p}_1 = \{(X, \psi(X)) \mid X \in \mathfrak{q}_1\}$. Since $[\mathfrak{h}, \mathfrak{p}_1] \subset \mathfrak{p}_1$ and $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{h}$, we get $[\mathfrak{q}_1, \mathfrak{q}_1] \subset \varphi_1(\mathfrak{h})$, $\psi([Y, X]) = [\theta(Y), \psi(X)]$ and $\theta([X, Z]) = [\psi(X), \psi(Z)]$ for every $Y \in \varphi_1(\mathfrak{h})$ and for every

$X, Z \in \mathfrak{q}_1$. In particular, $(\mathfrak{q}_1, \varphi_1(\mathfrak{h}))$ is a compact irreducible symmetric pair with simple $\mathfrak{q}_1$ and and with simple or one-dimensional $\varphi_1(\mathfrak{h})$.

If we extend the linear map ψ from $\mathfrak{q}_1$ to $\mathfrak{g}_1$ setting $\psi(X) := \theta(X)$ for any $X \in \varphi_1(\mathfrak{h})$, we obtain the isomorphism ψ between $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Indeed, $\psi([X, Y]) = [\psi(X), \psi(Y)]$ for every $X, Y \in \mathfrak{g}_1$, $\varphi_2(\mathfrak{h}) \subset \psi(\mathfrak{g}_1)$, and $\mathfrak{g}_1$ is simple. Therefore, $\psi(\mathfrak{g}_1)$ is a simple Lie subalgebra in $\mathfrak{g}_2$, and, moreover, $\psi(\mathfrak{g}_1) = \mathfrak{g}_2$, since $\varphi_2(\mathfrak{h}) \subset \psi(\mathfrak{g}_1)$ and $\mathfrak{p}_2$ is $\operatorname{ad}(\mathfrak{h})$-irreducible. Note that $\mathfrak{p}_2 = \{(X, -\psi(X)) \mid X \in \mathfrak{q}_1\}$ and $\mathfrak{p}_3 = \{(Y, -\psi(Y)) \mid Y \in \varphi_1(\mathfrak{h})\}$. Therefore, we may consider $\mathfrak{g}_2$ as the copy $\mathfrak{g}_1$ under the isomorphism ψ. The proposition is proved. □

The list of all generalized Wallach spaces of the type as in the Proposition 1 follows directly from the list of compact irreducible symmetric spaces, see e. g. [1, 7.102]. Using structure of symmetric spaces and the Casimir operators for the isotropy representations (see e. g. [1, Chapter 7]), one can easily compute the values A, a_1, a_2, and a_3 (see the formulas (5) and (6) in [2]) for the spaces in Proposition 1: $A = \frac{1}{4}\big(\dim(\mathfrak{f}) - \dim(\mathfrak{k})\big) = \frac{1}{4}\dim(\mathfrak{p}_1) = \frac{1}{4}\dim(\mathfrak{p}_2)$, $a_1 = \frac{A}{\dim(\mathfrak{p}_1)} = a_2 = \frac{A}{\dim(\mathfrak{p}_2)} = 1/4$, and $a_3 = \frac{A}{\dim(\mathfrak{p}_3)} = \frac{\dim(\mathfrak{f}) - \dim(\mathfrak{k})}{4\dim(\mathfrak{k})} \le 1/2$.

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