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ORIGINAL PAPER

Neumann boundary value problem for general curvature flow with forcing term

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Abstract
We consider the evolution of a strictly convex hypersurface by a class of general curvature. We prove that given some Neumann boundary condition, the flow exists for all time and converges to a solution with prescribed general curvature that satisfies the Neumann boundary condition. Our method also works for the corresponding elliptic setting.

Keywords Neumann boundary value problem · General curvature flow · Convergence analysis

Mathematics Subject Classification Primary 53C44; Secondary 35K20 · 53C42

1 Introduction

In this paper, we study the deformation of a strictly convex graph over a bounded, convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, to a convex graph with prescribed general curvature and Neumann boundary condition.

More precisely, let $\Sigma(t) = \{X := (x, u(x,t)) | (x,t) \in \Omega \times [0,T)\}$, we study the long time existence and convergence of the following flow problem

$$\begin{cases} \dot{u} = w\,(f(\kappa[\Sigma(t)]) - \Phi(x,u)) & \text{in } \Omega \times [0,T) \\ u_\nu = \varphi(x,u) & \text{on } \partial\Omega \times [0,T) \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Phi, \varphi : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ are smooth functions, ν denotes the outer unit normal to $\partial\Omega$, $w = \sqrt{1 + |Du|^2}$, $\kappa[\Sigma(t)] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of $\Sigma(t)$, and $u_0 : \bar{\Omega} \to \mathbb{R}$, the initial hypersurface, is a smooth, strictly convex function over Ω.

To guarantee that as long as the flow exits, $\Sigma(t)$ stays convex, the curvature function f has to satisfy some structure conditions. Accordingly, the function f is assumed to be defined in the convex cone $\Gamma_n^+ \equiv \{\lambda \in \mathbb{R}^n$: each component $\lambda_i > 0\}$ in $\mathbb{R}^n$ and satisfying the following conditions:

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$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \text{ in } \Gamma_n^+, \quad 1 \le i \le n, \tag{1.2}$$

and

$$f \text{ is a concave function.} \tag{1.3}$$

In addition, f will be assumed to satisfy some more technical assumptions. These include

$$f > 0 \text{ in } \Gamma_n^+, f = 0 \quad \text{on } \partial\Gamma_n^+, \tag{1.4}$$

$$f(1, \ldots, 1) = 1, \tag{1.5}$$

and

$$f \text{ is homogeneous of degree one.} \tag{1.6}$$

Moreover, for any $C > 0$ and every compact set $E \subset \Gamma_n^+$, there is $R = R(E, C) > 0$ such that

$$f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \ge C, \quad \forall \lambda \in E. \tag{1.7}$$

An example of functions satisfying all assumptions above is given by $f = \frac{1}{2}\left[H_n^{\frac{1}{n}} + (H_n/H_l)^{\frac{1}{n-l}}\right]$, where H_l is the normalized l-th elementary symmetric polynomial. However, we point out that the pure curvature quotient $(H_n/H_l)^{\frac{1}{n-l}}$ does not satisfy (1.7).

For a graph of u, the induced metric and its inverse matrix are given by

$$g_{ij} = \delta_{ij} + u_i u_j \quad \text{and} \quad g^{ij} = \delta_{ij} - \frac{u_i u_j}{w^2}, \tag{1.8}$$

where $w = \sqrt{1 + |Du|^2}$. Following [2], the principle curvature of graph u are eigenvalues of the symmetric matrix $A[u] = [a_{ij}]$:

$$a_{ij} = \frac{\gamma^{ik} u_{kl} \gamma^{lj}}{w}, \quad \text{where } \gamma^{ik} = \delta_{ij} - \frac{u_i u_k}{w(1+w)}. \tag{1.9}$$

The inverse of γ^{ij} is denoted by γ_{ij}, and

$$\gamma_{ij} = \delta_{ij} + \frac{u_i u_k}{1+w}. \tag{1.10}$$

Geometrically $[\gamma_{ij}]$ is the square root of the metric, i.e. $\gamma_{ik}\gamma_{kj} = g_{ij}$.

Now, for any positive definite symmetric matrix A, we define the function F by

$$F(A) = f(\lambda(A)),$$

where $\lambda(A)$ denotes the eigenvalues of A. We will use the notation

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}, \quad F^{ij,kl} = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A).$$

The matrix $[F^{ij}(A)]$ is symmetric and has eigenvalues $f_1, \ldots, f_n$. By (1.2), $[F^{ij}(A)]$ is positive definite. Moreover, by (1.3), F is a concave function of A, that is

$$F^{ij,kl}(A)\xi_{ij}\xi_{kl} \le 0,$$

for any $n \times n$ symmetric matrix $[\xi_{ij}]$.

We rewrite Eq. (1.1) as following,

$$\begin{cases} \dot{u} = w\left[F\left(\dfrac{\gamma^{ik}u_{kl}\gamma^{lj}}{w}\right) - \Phi(x,u)\right] & \text{in } \Omega\times[0,T) \\ u_\nu = \varphi(x,u) & \text{on } \partial\Omega\times[0,T) \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

We will prove

Theorem 1.1 *Let Ω be a smooth bounded, strictly convex domain in $\mathbb{R}^n$. Let $\Phi, \varphi : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$, be smooth functions satisfying*

$$\Phi > 0 \quad \text{and} \quad \Phi_z \geq 0, \tag{1.12}$$

$$\varphi_z \leq c_\varphi < 0. \tag{1.13}$$

Let u_0 be a smooth, strictly convex function that satisfies the compatibility condition on $\partial\Omega$:

$$\nu^i u_i - \varphi(x,u)\Big|_{t=0} = 0. \tag{1.14}$$

We also assume

$$f(\kappa[\Sigma_0]) - \Phi(x,u_0) \geq 0, \tag{1.15}$$

where $\Sigma_0 = \{(x,u_0(x))|x \in \Omega\}$. Then there exists a solution $u \in C^\infty(\bar{\Omega}\times(0,\infty))$ of Eq. (1.11). Moreover, as $t\to\infty$, the function $u(x,t)$ smoothly converges to a smooth limit function u^∞, such that u^∞ solves the Neumann boundary value problem

$$\begin{cases} F\left(\dfrac{\gamma^{ik}u^\infty_{kl}\gamma^{lj}}{w}\right) = \Phi(x,u^\infty) & \text{in } \Omega \\ u^\infty_\nu = \varphi(x,u^\infty) & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where ν is the outer unit normal of $\partial\Omega$.

Remark 1.2 As it is explained in [11], in view of the compatibility assumption (1.14), the short time existence for Eq. (1.11) follows from Theorem 5.3 in [6] and the implicit function theorem. Moreover, the solution $u(\cdot,t)$ approaches u_0 in $C^2(\bar{\Omega})$ as $t\to 0$, this implies $\dot{u}$ is continuous up to $t=0$.

By applying short time existence theorem, we know that the flow exists for $t\in[0,T^*)$, for some $T^*>0$ very small. In the following sections, we fix $T<T^*$, and establish the uniform C^2 bounds for the solution u of (1.11) in $(0,T]$. Since our estimates are independent of T, repeating this process we obtain the longtime existence of Eq. (1.11).

Neumann boundary problem has attracted lots of attetions through the years. In particular, real Monge–Ampère equations in bounded uniformly convex domains are solved with Neumann boundary conditions by Lions, Trudinger, and Urbas in [8]. There, they built the foundation for C^2 a priori estimates of Neumman boundary problem, which departs completely from that of the Dirichlet problem. By adapting and developing the techniques in [8], Jiang et al. [5] proved the classical solvability of a generalized Monge–Ampère type equation with Neumann boundary condition. Recently, Ma and Qiu proved the existence of the solution to Hession equations with Neumann boundary condition in their beautiful paper [9], which confirms a longstanding conjecture by Trudinger.

The Neumann boundary problems for parabolic equations have been widely studied as well. For example, mean curvature flow with Neumann boundary condition have been studied in [1,3,10,14]; Guass curvature flow with Neumann boundary condition have been studied in [12,13].

Our paper is oganized as follows: In Sect. 2 we prove the uniform estimate for $\dot{u}$, which also implies the convexity of $u(\cdot, t)$. This estimate is used in Sect. 3 to derive the bounds for $|u|$ and $|Du|$. Section 4 is the most important section, in which we derive the C^2 estimates for u. Finally, in Sect. 5, we combine all results above to prove the convergence of the solution of (1.11) as $t \to \infty$.

2 Speed estimates

Lemma 2.1 *As long as a smooth convex solution of* (1.11) *exists, we have*

$$\min\{\min_{t=0} \dot{u}, 0\} \le \dot{u} \le \max\{\max_{t=0} \dot{u}, 0\}. \tag{2.1}$$

Proof If $(\dot{u})^2$ achieves a positive local maximum at $(x, t) \in \partial\Omega \times (0, T]$, then by (1.13) at this point we would have

$$(\dot{u})^2_\nu = 2\dot{u}\dot{u}_\nu = 2(\dot{u})^2\varphi_z < 0, \tag{2.2}$$

which leads to a contradiction. Thus, we assume $(\dot{u})^2$ achieves its maximum at an interior point. Now let's denote

$$\tilde{G}(D^2u, Du, u) = wF\left(\frac{\gamma^{ik}u_{kl}\gamma^{lj}}{w}\right) - w\Phi(x, u)$$

and $r = (\dot{u})^2$. A straightforward calculation gives us

$$\dot{r} = \tilde{G}^{ij}r_{ij} - 2\tilde{G}^{ij}\dot{u}_i\dot{u}_j + \tilde{G}^s r_s + 2\tilde{G}_u r, \tag{2.3}$$

where $\tilde{G}^{ij} = \frac{\partial \tilde{G}}{\partial u_{ij}}$, $\tilde{G}^s = \frac{\partial \tilde{G}}{\partial u_s}$, and $\tilde{G}_u = \frac{\partial \tilde{G}}{\partial u}$. Since

$$\tilde{G}_u := \frac{\partial \tilde{G}}{\partial u} = -w\Phi_u \le 0, \tag{2.4}$$

we have

$$\dot{r} - \tilde{G}^{ij}r_{ij} - \tilde{G}^s r_s \le 0. \tag{2.5}$$

By the maximum principle we know that a positive local maximum of $(\dot{u})^2$ can not occur at an interior point of $\Omega \times (0, T]$. Therefore, we proved this Lemma. □

Lemma 2.2 *A solution of* (1.11) *satisfies* $\dot{u} > 0$ *for* $t > 0$ *if* $0 \not\equiv \dot{u} \ge 0$ *for* $t = 0$.

Proof Differentiating

$$\dot{u} = \tilde{G}(D^2u, Du, u), \tag{2.6}$$

with respect to t we get,

$$\frac{d}{dt}u_t = \tilde{G}^{ij}(u_t)_{ij} + \tilde{G}^s(u_t)_s + \tilde{G}_u u_t. \tag{2.7}$$

Then, for any constant λ we have

$$\frac{d}{dt}(u_t e^{\lambda t}) = \tilde{G}^{ij}(u_t e^{\lambda t})_{ij} + \tilde{G}^s(u_t e^{\lambda t})_s + \tilde{G}_u(u_t e^{\lambda t}) + \lambda u_t e^{\lambda t}. \tag{2.8}$$

Now we fix $t_0 > 0$, and choosing a constant λ such that $\lambda + \tilde{G}_u > 0$ for $(x, t) \in \bar{\Omega} \times [0, t_0]$. If $u_t e^{\lambda t} = 0$ at some interior point $(x_1, t_1) \in \Omega \times (0, t_0]$, then by the strong maximum principle we would have $u_t e^{\lambda t}$ vanishes identically in $\bar{\Omega} \times [0, t_0]$, which leads to a contradiction.

Assuming $u_t e^{\lambda t} = 0$ at a boundary point $(x_1, t_1) \in \partial\Omega \times (0, t_0]$, then we would have

$$(u_t e^{\lambda t})_\nu = \varphi_z (u_t e^{\lambda t}) = 0. \tag{2.9}$$

This contradicts the Hopf Lemma. □

Remark 2.3 Lemma 2.2 implies that, if we start from a strictly convex hypersurface Σ_0 that satisfies the inequality (1.15), then as long as the flow exists, the flow surfaces $\Sigma(t)$ are strictly convex and satisfying $f(\kappa[\Sigma(t)]) - \Phi(x, u) > 0$.

3 C^0 and C^1 estimates

Recall that $u_\nu = \varphi(x, u)$ on $\partial\Omega$, the strict convexity of u and the fact that $\varphi(\cdot, z) \to -\infty$ uniformly as $z \to \infty$ implies that u is uniformly bounded from above. By Lemma 2.2 we also have,

$$u(x, t) = u(x, 0) + \int_0^t \dot{u}(x, \tau) d\tau \geq u(x, 0). \tag{3.1}$$

This yields u is bounded from below. To conclude, we have

Theorem 3.1 (C^0 estimates) *Under our assumption* (1.15) *on* u_0, *a solution of equation* (1.11) *satisfies*

$$|u| \leq C_0, \tag{3.2}$$

where $C_0 = C_0(u_0, \varphi)$.

Theorem 3.2 (C^1 estimates) *For a convex solution u of Eq.* (1.11)*, the gradient of u remains bounded during the evolution,*

$$|Du| \leq C_1, \tag{3.3}$$

where $C_1 = C_1(|u|_{C^0}, \Omega, \varphi)$.

Proof The proof is the same as Theorem 2.2 in [8], for readers convenience we include it here. By the convexity of u we have for any $t \in [0, T]$

$$\max_{\Omega} |Du(\cdot, t)| = \max_{\partial\Omega} |Du(\cdot, t)|. \tag{3.4}$$

Let $x_0 \in \partial\Omega$ and let τ be a direction such that $\nu \cdot \tau = 0$ at x_0. Let $B = B_R(z)$ be an interior ball at x_0, L be the line through x_0 in the direction of $-\nu$, and L intersects ∂B at y_0. Then $z = \frac{1}{2}(x_0 + y_0)$, we also let $y \in \partial B$ be the unique point such that $\frac{y-z}{|y-z|} = \tau$.

Now let ω be an affine function such that $\omega(x_0) = u(x_0, t)$ and $D\omega = Du(x_0, t)$. Then $\omega \leq u(x, t), \; x \in \Omega$ and

$$\begin{aligned} \omega(z) &= \omega(x_0) + D\omega(x_0) \cdot (z - x_0) \\ &= u(x_0, t) + Du(x_0, t) \cdot \frac{z - x_0}{|z - x_0|} \cdot |z - x_0| \\ &\geq u(x_0, t) - M_1 R, \end{aligned} \tag{3.5}$$

where we assume $|\varphi(x, u)| \leq M_1$ in $\bar{\Omega} \times [-C_0, C_0]$. Therefore,

$$D_\tau u(x_0, t) = D_\tau \omega(x_0) = \frac{\omega(y) - \omega(z)}{|y - z|} \leq \frac{u(y, t) - u(x_0, t) + M_1 R}{R} \leq \frac{2C_0}{R} + M_1. \tag{3.6}$$

Since τ, x_0, and t are arbitrary, we are done. □

4 C^2 estimates

First of all, we will list some evolution equations that will be used later. Since the calculations are straightforward, we will only state our results here.

Lemma 4.1 *Let u be a solution to the general curvature flow (1.11). Then we have the following evolution equations:*

(i) $\frac{d}{dt} g_{ij} = -2(F - \Phi)h_{ij}$,
(ii) $\frac{d}{dt}\mathbf{n} = -g^{ij}(F - \Phi)_i \tau_j$,
(iii) $\frac{d}{dt}\mathbf{n}^{n+1} = -g^{ij}(F - \Phi)_i u_j$,
(vi) $\frac{d}{dt} h_i^j = (F - \Phi)_i^j + (F - \Phi)h_i^k h_k^j$,

where g_{ij}, h_{ij} are the first and second fundamental forms, $\mathbf{n}$ is the upward unit normal to $\Sigma(t)$, $\mathbf{n}^{n+1} = \langle \mathbf{n}, e^{n+1}\rangle$, and $h_i^j = g^{jk}h_{kj}$.

4.1 C^2 interior estimates

In this subsection, we will prove the following theorem.

Theorem 4.2 *Let $\Sigma(t) = \{(x, u(x, t))|x \in \Omega, t \in [0, T]\}$ be the flow surfaces, where $u(x, t)$ satisfies Eq. (1.11) and*

$$\mathbf{n}^{n+1} \geq 2a > 0 \quad on\ \Sigma(t), \quad \forall t \in [0, T].$$

For $X \in \Sigma(t)$, let $\kappa_{\max}(X)$ be the largest principle curvature of $\Sigma(t)$ at X. Then

$$\max_{\bar{\Omega}_T} \frac{\kappa_{\max}}{\mathbf{n}^{n+1} - a} \leq C_2(\Phi, |u|_{C^1}) \left(1 + \max_{\partial\Omega_T} \kappa_{\max}\right), \tag{4.1}$$

where $\Omega_T = \Omega \times (0, T]$.

Proof Let's consider

$$M_0 = \max_{\bar{\Omega}_T} \frac{\kappa_{\max}}{\mathbf{n}^{n+1} - a},$$

we assume $M_0 > 0$ is attained at an interior point $(x_0, t_0) \in \Omega \times (0, T]$. We can choose a local coordinate at (x_0, t_0) such that $\kappa_1 = \kappa_{\max}$, $h_i^j = \kappa_i \delta_{ij}$, and $g_{ij} = \delta_{ij}$.

At (x_0, t_0), $\psi = \frac{h_1^1}{\mathbf{n}^{n+1} - a}$ achieves its local maximum. Therefore, at this point we have

$$\frac{h_{1i}^1}{h_1^1} - \frac{\nabla_i \mathbf{n}^{n+1}}{\mathbf{n}^{n+1} - a} = 0. \tag{4.2}$$

Moreover, by Lemma 4.1

$$\begin{aligned}\frac{\partial}{\partial t}\psi &= \frac{\dot{h_1^1}}{\mathbf{n}^{n+1}-a} - \frac{h_1^1\dot{\mathbf{n}}^{n+1}}{(\mathbf{n}^{n+1}-a)^2} \\ &= \frac{1}{\mathbf{n}^{n+1}-a}\left\{\nabla_{11}F - \nabla_{11}\Phi + (F-\Phi)\kappa_1^2\right\} + \frac{h_1^1}{(\mathbf{n}^{n+1}-a)^2}(F-\Phi)_i u_i.\end{aligned} \tag{4.3}$$

Since

$$\nabla_{11}\Phi = \Phi_{x_1x_1}(x,u) + 2\Phi_{x_1z}u_1 + \Phi_z u_{11}, \tag{4.4}$$

$$\nabla_{11}u = \langle X, e_{n+1}\rangle_{11} = \langle h_{11}\mathbf{n}, e_{n+1}\rangle = h_{11}\mathbf{n}^{n+1}, \tag{4.5}$$

$$\nabla_{ii}\mathbf{n}^{n+1} = \nabla_i\left\langle -h_{ik}\tau_k, e_{n+1}\right\rangle = \langle -h_{iik}\tau_k, e_{n+1}\rangle - h_{ii}^2\left\langle \mathbf{n}, e_{n+1}\right\rangle, \tag{4.6}$$

and

$$\begin{aligned}\nabla_{11}F &= F^{ij}h_{ij11} + F^{ij,rs}h_{ij1}h_{rs1} \\ &= F^{ij}(h_{11ij} - h_{11}^2h_{ij} + h_{ik}h_{kj}h_{11}) + F^{ij,rs}h_{ij1}h_{rs1}.\end{aligned} \tag{4.7}$$

In view of Eqs. (4.3), (4.6), and (4.7), we get at (x_0, t_0)

$$\begin{aligned}0 &\le \frac{\partial}{\partial t}\psi - F^{ii}\nabla_{ii}\psi \\ &= \frac{1}{\mathbf{n}^{n+1}-a}\left\{F^{ii}h_{ii11} + F^{ij,rs}h_{ij1}h_{rs1} - \nabla_{11}\Phi + (F-\Phi)\kappa_1^2\right\} \\ &+ \frac{h_{11}}{(\mathbf{n}^{n+1}-a)^2}(F-\Phi)_i u_i - \frac{F^{ii}h_{11ii}}{\mathbf{n}^{n+1}-a} + \frac{h_{11}}{(\mathbf{n}^{n+1}-a)^2}F^{ii}\mathbf{n}_{ii}^{n+1} \\ &= \frac{1}{\mathbf{n}^{n+1}-a}F^{ii}(h_{ii}^2h_{11} - h_{11}^2h_{ii}) + \frac{F^{ij,rs}h_{ij1}h_{rs1}}{\mathbf{n}^{n+1}-a} \\ &- \frac{\nabla_{11}\Phi}{\mathbf{n}^{n+1}-a} + \frac{(F-\Phi)\kappa_1^2}{\mathbf{n}^{n+1}-a} + \frac{h_{11}}{(\mathbf{n}^{n+1}-a)^2}(F-\Phi)_i u_i \\ &+ \frac{h_{11}}{(\mathbf{n}^{n+1}-a)^2}F^{ii}\left(-\nabla_k h_{ii}u_k - h_{ii}^2\mathbf{n}^{n+1}\right).\end{aligned} \tag{4.8}$$

By our assumptions (1.6) and (1.3), we know that at (x_0, t_0),

$$F^{ii}h_{ii} = f_i\kappa_i = F \tag{4.9}$$

and

$$F^{ij,rs}h_{ij1}h_{rs1} \le 0. \tag{4.10}$$

Substituting (4.4) and (4.5) into (4.8), then combining with (4.9) and (4.10) we get

$$\begin{aligned}0 &\le \frac{\partial}{\partial t}\psi - F^{ii}\nabla_{ii}\psi \\ &\le \frac{-ah_{11}}{(\mathbf{n}^{n+1}-a)^2}f_i\kappa_i^2 - \frac{\Phi\kappa_1^2}{\mathbf{n}^{n+1}-a} + \frac{C}{\mathbf{n}^{n+1}-a} \\ &\quad - \frac{\Phi_z\kappa_1\mathbf{n}^{n+1}}{\mathbf{n}^{n+1}-a} - \frac{\kappa_1}{(\mathbf{n}^{n+1}-a)^2}(\Phi_i + \Phi_z u_i)u_i,\end{aligned} \tag{4.11}$$

which implies,

$$0 \le \frac{-a\kappa_1}{(\mathbf{n}^{n+1}-a)^2}f_i\kappa_i^2 - \frac{\left(\inf\limits_{\bar{\Omega}\times[-C_0,C_0]}\Phi\right)\kappa_1^2}{\mathbf{n}^{n+1}-a} + C\kappa_1, \tag{4.12}$$

thus

$$\kappa_1 \leq C = C(\Phi, |u|_{C^1}). \tag{4.13}$$

Note that the constants C in (4.12) and (4.13) also depend on a; since our choice of a depends on $|u|_{C^1}$, we omit the dependency on a. Therefore, we conclude that

$$\max_{\bar{\Omega}_T} \frac{\kappa_{\max}}{\mathbf{n}^{n+1} - a} \leq C_2 \left(1 + \max_{\partial\Omega_T} \kappa_{\max}\right). \tag{4.14}$$

□

4.2 C^2 boundary estimates

We use ν for the outer unit normal of $\partial\Omega$ and τ for a direction that tangential to $\partial\Omega$. By the exactly same argument as Lemma 4.1 of [13] we have

Lemma 4.3 (Mixed C^2 estimates at the boundary) *Let u be the solution of our flow Eq. (1.11). Then the absolute value of $u_{\tau\nu}$ remains a priori bounded on $\partial\Omega$ during the evolution.*

Now we consider the function

$$V(x, \xi, t) := u_{\xi\xi} - 2(\xi \cdot \nu)\xi_i'(D_i\varphi - D_k u D_i \nu^k), \tag{4.15}$$

where $\xi \in \mathbb{S}^{n-1}$ is a unit vector and $\xi' = \xi - (\xi \cdot \nu)\nu$. By Theorem 4.2, we may assume $V(x, \xi, t)$ achieves its maximum at $(x_0, t_0) \in \partial\Omega \times (0, T]$, otherwise, we would be done.

We will devide it into 3 cases.

(i) ξ *is tangential* We will compute the second tangential derivatives of the boundary condition. The proof is the same as in [9], for readers convenience, we will include it here. Following the notation in [9], we denote $c^{ij} = \delta_{ij} - \nu^i\nu^j$. Differentiating the boundary condition with respect to the tangential direction twice we obtain

$$u_{li}\nu^l = c^{ij} D_j\varphi - c^{ij} u_l D_j \nu^l + \nu^i\nu^j\nu^l u_{lj},$$

and

$$u_{lip}\nu^l = c^{pq} D_q \left(c^{ij} D_j\varphi - c^{ij} u_l D_j \nu^l + \nu^i\nu^j\nu^l u_{lj}\right) + \nu^p\nu^q\nu^l u_{liq} - c^{pq} u_{li} D_q \nu^l.$$

Summing with $\xi^i\xi^p$ yields

$$\begin{aligned} u_{\xi\xi\nu} = &-2\xi^p\xi^i u_{li} D_p\nu^l - u_l\xi^p D_{ip}\nu^l\xi^i + u_{\nu\nu}\sum_i \xi^p D_p\nu^i\xi^i \\ &- \sum_i \xi^p\xi^i\nu^j D_p\nu^i D_j\varphi + \varphi_z u_{\xi\xi} + \xi^p\xi^i\varphi_{ip} \\ &+ \varphi_{zz}u_\xi^2 + 2u_\xi\xi^i\varphi_{zi}, \end{aligned}$$

where we have used $\sum_j \nu^j D_l \nu^j = 0$.
Therefore, at (x_0, t_0) we have

$$D_{\xi\xi\nu}u \leq -2(D_i\nu^k)D_{jk}u\xi_i\xi_j + (D_i\nu^j)\xi_i\xi_j D_{\nu\nu}u + \varphi_z D_{ij}u\xi_i\xi_j + C, \tag{4.16}$$

where $C = C(\|u\|_{C^1}, \|\partial\Omega\|_{C^3}, \|\varphi\|_{C^2})$.
Next, since V attains its maximum at (x_0, t_0), we get

$$0 \leq D_\nu V = u_{\xi\xi\nu} - a_k D_{k\nu}u - (D_\nu a_k)D_k u - D_\nu b, \tag{4.17}$$

where $a_k = 2(\xi \cdot \nu)(\varphi_z \xi'_k - \xi'_i D_i \nu^k)$ and $b = 2(\xi \cdot \nu)\xi'_k \varphi_k$. Thus, applying Lemma 4.3

$$u_{\xi\xi\nu} \geq a_\nu D_{\nu\nu} u - C(\|\varphi\|_{C^2}, \|u\|_{C^1}, \|\partial\Omega\|_{C^3}) = -C, \tag{4.18}$$

where we have used $a_\nu = 0$. Combine with (4.16) and condition (1.13) yields

$$-2(D_i \nu^k) D_{jk} u \xi_i \xi_j + (D_i \nu^j)\xi_i \xi_j u_{\nu\nu} + c_\varphi D_{ij} u \xi_i \xi_j + C \geq -C. \tag{4.19}$$

By virtue of the uniformly convexity of the domain Ω, we have $[D_i \nu^k] \geq c_0 I$, for some $c_0 > 0$. This gives

$$D_{\xi\xi} u(x_0, t_0) \leq C(1 + D_{\nu\nu} u(x_0, t_0)). \tag{4.20}$$

(ii) *ξ is non-tangential* We write $\xi = \alpha\tau + \beta\nu$, where $\alpha = \xi \cdot \tau$, $\beta = \xi \cdot \nu \neq 0$. Then

$$\begin{aligned} D_{\xi\xi} u &= \alpha^2 D^2_{\tau\tau} u + \beta^2 D_{\nu\nu} u + 2\alpha\beta D_{\tau\nu} u \\ &= \alpha^2 D_{\tau\tau} u + \beta^2 D_{\nu\nu} u + V'(x, \xi), \end{aligned} \tag{4.21}$$

where $V' = 2(\xi \cdot \nu)\xi'_i (D_i \varphi - D_k u D_i \nu^k)$. Thus we get,

$$\begin{aligned} V(x_0, \xi, t_0) &= \alpha^2 V(x_0, \tau, t_0) + \beta^2 V(x_0, \nu, t_0) \\ &\leq \alpha^2 V(x_0, \xi, t_0) + \beta^2 V(x_0, \nu, t_0), \end{aligned} \tag{4.22}$$

which yeilds

$$u_{\xi\xi}(x_0, t_0) \leq C(1 + u_{\nu\nu}(x_0, t_0)). \tag{4.23}$$

(iii) *Double normal C^2-estimates at the boundary* Let's recall our evolution equation

$$\begin{cases} \dot{u} = w\left[F\left(\dfrac{\gamma^{ik} u_{kl} \gamma^{lj}}{w}\right) - \Phi(x, u)\right] \\ u_\nu = \varphi(x, u) \end{cases} \tag{4.24}$$

In the following we denote

$$G(D^2 u, Du) = F\left(\frac{\gamma^{ik} u_{kl} \gamma^{lj}}{w}\right),$$

then G satisfies similar structure conditions to those of F. We have

$$G^{ij} := \frac{\partial G}{\partial u_{ij}} = \frac{1}{w} F^{kl} \gamma^{ik} \gamma^{lj}, \tag{4.25}$$

and it's easy to see that

$$\frac{1}{w^3} \sum F^{ii} \leq \sum G^{ii} \leq \frac{1}{w} \sum F^{ii}. \tag{4.26}$$

By a straightforward calculation we get (for details see the proof of Lemma 2.3 in [4]),

$$G^s := \frac{\partial G}{\partial u_s} = -\frac{u_s}{w^2} F - \frac{2}{w(1+w)} F^{ij} a_{ik}(w u_k \gamma^{sj} + u_j \gamma^{ks}), \tag{4.27}$$

where we have used $\sum f_i \kappa_i = f(\kappa)$. Since $[a_{ij}]$ is positive definite, we obtain

$$\sum |G^i| \leq CF \leq \tilde{C}_0. \tag{4.28}$$

Now, let

$$\Omega_\mu := \{x \in \bar{\Omega} : 0 < d(x) = \mathrm{dist}(x, \partial\Omega) < \mu\}.$$

Consider $q(x) = -d(x) + Nd^2(x)$, then $q \in C^\infty$ in Ω_μ for some constant μ satisfies $\mu \le \tilde{\mu}$ and $N\mu \le \frac{1}{8}$, where $\tilde{\mu}$ is a small constant depending on Ω. Since

$$-Dd(y_0) = \nu(x_0)$$

where $x_0 \in \partial\Omega$ and $\text{dist}(y_0, \partial\Omega) = \text{dist}(x_0, y_0)$, q satisfies the following properties in Ω_μ :

$$-\mu + N\mu^2 \le q \le 0 \quad \text{and} \quad \frac{1}{2} \le |Dq| \le 2. \tag{4.29}$$

Moreover, $\frac{Dq}{|Dq|} = \nu$ in Ω_μ, here ν is the unit outer normal to the boundary $\partial\Omega$. Next, let

$$M = \max_{\partial\Omega\times[0,T]} u_{\nu\nu} \tag{4.30}$$

and $Q(x, t) = Q(x) = (A + \frac{1}{2}M)q(x)$ in Ω_μ, where μ, A, N are positive constants to be chosen later. We consider the following function

$$P(x, t) := Du \cdot Dq - \varphi - Q \tag{4.31}$$

Lemma 4.4 *For any* $(x, t) \in \bar{\Omega}_\mu \times [0, T]$, *if we choose* $A, N > 0$ *large,* $\mu > 0$ *small, then we have* $P(x, t) \ge 0$.

Proof First, let's assume $P(x, t)$ attains its minimum at $(x_0, t_0) \in \Omega_\mu \times (0, T]$. Let's choose a local coordinate such that $a_{ij}(x_0, t_0) = \kappa_i(x_0, t_0)\delta_{ij}$, $1 \le i, j \le n$. Then at this point we have $F^{ij} = \frac{\partial f}{\partial \kappa_i}\delta_{ij}$. Differentiating P twice, we get

$$P_i = \sum_l u_{li}q_l + \sum_l u_l q_{li} - \varphi_i - Q_i, \tag{4.32}$$

and

$$P_{ij} = \sum_l u_{lij}q_l + 2\sum_l u_{li}q_{lj} + \sum_l u_l q_{lij} - \varphi_{ij} - Q_{ij}. \tag{4.33}$$

Moreover,

$$\begin{aligned} P_t &= Du_t \cdot Dq - \varphi_t \\ &= \sum_l [w(F - \Phi)]_l q_l - \varphi_z u_t = \sum_l [w(F - \Phi)]_l q_l - \varphi_z w(F - \Phi). \end{aligned} \tag{4.34}$$

Therefore, at (x_0, t_0) we have

$$\begin{aligned} &\frac{1}{w}P_t - G^{ij}P_{ij} \\ &\quad = \frac{1}{w}[w(F - \Phi)]_l q_l - \varphi_z(F - \Phi) - G^{ij}\left(\sum_l u_{lij}q_l + 2\sum_l u_{li}q_{lj}\right. \\ &\qquad \left. + \sum_l u_l q_{lij} - \varphi_{ij}\right) + \left(A + \frac{1}{2}M\right)G^{ij}q_{ij} \\ &\quad = \sum_l \frac{1}{w}[w(F - \Phi)]_l q_l - \varphi_z(F - \Phi) - G^{ij}\sum_l u_{lij}q_l \\ &\qquad - 2\sum_l G^{ij}u_{li}q_{lj} - \sum_l G^{ij}u_l q_{lij} + G^{ij}\varphi_{ij} + \left(A + \frac{1}{2}M\right)G^{ij}q_{ij}. \end{aligned} \tag{4.35}$$

This implies at (x_0, t_0)

$$\begin{aligned} 0 \geq & \frac{1}{w} P_t - G^{ij} P_{ij} \\ = & \sum_l \frac{(F-\Phi)}{w} \cdot \frac{u_s u_{sl} q_l}{w} + \sum_l F_l q_l - \sum_l \Phi_l q_l - \varphi_z (F-\Phi) \\ & - G^{ij} \sum_l u_{lij} q_l - 2 \sum_l G^{ij} u_{li} q_{lj} - \sum_l G^{ij} u_l q_{lij} \\ & + G^{ij} (\varphi_{x_i x_j} + 2\varphi_{x_i z} u_j + \varphi_{zz} u_i u_j + \varphi_z u_{ij}) + \left(A + \frac{1}{2} M\right) G^{ij} q_{ij}. \end{aligned} \tag{4.36}$$

Since $G(D^2 u, Du) = F$ we have

$$G^{ij} u_{ijl} + G^s u_{sl} = F_l, \tag{4.37}$$

which gives us

$$F_l q_l - G^{ij} u_{ijl} q_l = G^s u_{sl} q_l. \tag{4.38}$$

By (4.28) and (4.29) we have

$$|G^s u_{sl} q_l| \leq \tilde{C}_1 (M+1). \tag{4.39}$$

Furthermore, by the speed estimate (2.1), height estimate (3.2), and the gradient estimate (3.3), we obtain

$$|\Phi_l q_l| + \left| \frac{F-\Phi}{w} \cdot \frac{u_s u_{sl} q_l}{w} + \varphi_z G^{ij} u_{ij} \right| \leq \tilde{C}_2 M. \tag{4.40}$$

Now, by the convexity of $\partial\Omega$, we may assume

$$\kappa[2k_0 \delta_{\alpha\beta}] \leq \kappa[-d_{\alpha\beta}] \leq \kappa[k_1 \delta_{\alpha\beta}], \;\; 1 \leq \alpha, \beta \leq n-1, \tag{4.41}$$

for some $k_0, k_1 > 0$ depending on $\partial\Omega$. Thus, in Ω_μ we have

$$\kappa[(k_1 + 3N)\delta_{ij}] \geq \kappa[q_{ij}] = \kappa[-d_{ij} + 2N d d_{ij} + 2N d_i d_j] \geq \kappa[k_0 \delta_{ij}], \tag{4.42}$$

where $1 \leq i, j \leq n$ and $\kappa[A]$ denotes the eigenvalue of the matrix A. This gives

$$|2G^{ij} u_{li} q_{lj}| \leq \tilde{C}_3 (k_1 + 3N), \tag{4.43}$$

where $\tilde{C}_3$ depends on F. Next, an easy calculations yields

$$q_{ijl} = -d_{ijl} + 2N d_l d_{ij} + 2N d d_{ijl} + 4N d_{il} d_j, \tag{4.44}$$

which implies

$$|q_{ijl}| \leq C(|\partial\Omega|_{C^3}) + 6Nk_1. \tag{4.45}$$

Therefore,

$$|G^{ij} u_l q_{lij}| \leq \big(C(|\partial\Omega|_{C^3}) + 6Nk_1\big)\, C_1 \sum G^{ii}, \tag{4.46}$$

where we have used $G^{ij} = \frac{1}{w} F^{kl} \gamma^{ik} \gamma^{lj} \leq C \sum F^{kk} \leq C \sum G^{ii}$. Consequently, we have

$$|G^{ij} u_l q_{lij} + G^{ij} (\varphi_{x_i x_j} + 2\varphi_{x_i z} u_j + \varphi_{zz} u_i u_j)| \leq (\tilde{C}_4 + \tilde{C}_5 N k_1) \sum G^{ii}. \tag{4.47}$$

To conclude, we obtained

$$\begin{aligned} 0 &\geq \frac{1}{w}P_t - G^{ij}P_{ij} \\ &\geq -\tilde{C}_2 M - \tilde{C}_1(M+1) - \tilde{C}_3(k_1+3N) - (\tilde{C}_4 + \tilde{C}_5 N k_1)\sum G^{ii} \\ &+ \left(\frac{A}{2}+\frac{M}{4}\right)k_0 \sum G^{ii} + \left(\frac{A}{2}+\frac{M}{4}\right)G(D^2 q, Du), \end{aligned} \tag{4.48}$$

Note that here we used the inequality $G^{ij}(D^2u, Du)q_{ij} \geq G(D^2q, Du)$, which follows from the concavity of f. By Lemma 2.2 of Guan and Spruck [4], we may choose N sufficiently large such that

$$\frac{1}{4}G(D^2q, Du) \geq \tilde{C}_1 + \tilde{C}_2 + 1, \tag{4.49}$$

then we choose A such that

$$\frac{k_0}{2}A\sum G^{ii} > \tilde{C}_3(k_1+3N) + (\tilde{C}_4 + N\tilde{C}_5 k_1)\sum G^{ii}, \tag{4.50}$$

here we have used $\sum G^{ii} \geq c_0(|u|_{C^1}) > 0$, which follows from (4.26) and the assumptions (1.3), (1.6) of f. Substituting (4.49) and (4.50) into (4.48) we get

$$\frac{1}{w}P_t - G^{ij}P_{ij} > 0 \tag{4.51}$$

at (x_0, t_0), which leads to a contradiction.

Finally, note that for any $(x, t) \in \partial\Omega \times [0, T]$ we have

$$P(x, t) = 0.$$

For $(x, t) \in \partial\Omega_\mu \setminus \partial\Omega \times [0, T]$ we have

$$P(x,t) \geq -\tilde{C}_6 + (A + \frac{1}{2}M)\cdot\frac{1}{2}\mu > 0,$$

when $A \geq \frac{2\tilde{C}_6}{\mu}$. Moreover, when $A \geq \tilde{C}_7 = \tilde{C}_7(|u_0|_{C^2}, |\varphi|_{C^1}, |\partial\Omega|_{C^2})$, we have for $x \in \Omega_\mu$

$$P(x, 0) \geq 0.$$

Thus, choosing

$$A = \frac{2[\tilde{C}_3(k_1+2N) + \tilde{C}_4 + N\tilde{C}_5 k_1]}{k_0 c_0} + \frac{2\tilde{C}_6}{\mu} + \tilde{C}_7$$

we have $P(x,t) \geq 0$ in $\Omega_\mu \times [0, T]$. Here $c_0 = \min\{1, \sum G^{ii}\} > 0$. □

Theorem 4.5 *Let Ω be a smooth bounded, strictly convex domain in $\mathbb{R}^n$, u is a smooth solution of (1.11), ν is the outer unit normal vector of $\partial\Omega$. Then we have*

$$\max_{\partial\Omega\times[0,T]} u_{\nu\nu} \leq C. \tag{4.52}$$

Proof Assume $(z_0, t_0) \in \partial\Omega \times [0, T]$ is the maximum point of $u_{\nu\nu}$ on $\partial\Omega \times [0, T]$. By Lemma 4.4 we have

$$\begin{aligned} 0 \geq P_\nu(z_0, t_0) &= \left(\sum_l u_{l\nu}q_l + u_l q_{l\nu} - \varphi_\nu\right) - \left(A + \frac{1}{2}M\right)q_\nu \\ &\geq u_{\nu\nu} - C(|u|_{C^1}, N, |\partial\Omega|_{C^2}, |\varphi|_{C^1}) - \left(A + \frac{1}{2}M\right), \end{aligned} \tag{4.53}$$

Therefore we have,

$$\max_{\partial\Omega\times[0,T]} u_{\nu\nu} \le C + \frac{1}{2}M. \tag{4.54}$$

Inequality (4.52) follows from (4.54) and the assumption (4.30). □

5 Convergence to a stationary solution

Let us go back to our original problem (1.1), which is a scalar parabolic differential equation defined on the cylinder $\Omega_T = \Omega \times [0, T]$ with initial value u_0. In view of a priori estimates, which we have estimated in the preceding sections, we know that

$$|D^2 u| \le C, \tag{5.1}$$

$$|Du| \le C, \tag{5.2}$$

and

$$|u| \le C. \tag{5.3}$$

Therefore,

$$F \text{ is uniformly elliptic.}$$

Moreover, since F is concave, we can apply the results of Chapter 14 in [7] to obtain uniform $C^{2,\alpha}$ estimates for u. Then standard Schauder estimates imply uniform bounds for u in C^k, $k \ge 0$. Therefore, a smooth solution of (1.1) exists for all $t \ge 0$.

Lemma 5.1 *If a solution of the flow Eq.* (1.1) *exists for all* $t \ge 0$. *Moreover, the initial surface satisfies* (1.15). *Then the solution converges uniformly to a solution of the Neumann boundary problem*

$$\begin{cases} F\left(\dfrac{\gamma^{ik}u^{\infty}_{kl}\gamma^{lj}}{w}\right) = \Phi(x, u^{\infty}) & \text{in } \Omega, \\ u^{\infty}_{\nu} = \varphi(x, u^{\infty}) & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Proof By integrating the flow equation with respect to t we get

$$u(x, t^*) - u(x, 0) = \int_0^{t^*} w(F - \Phi)dt. \tag{5.5}$$

In particular, by (5.3) we have

$$\int_0^{\infty} w(F - \Phi)dt < \infty \ \forall x \in \Omega. \tag{5.6}$$

Hence for any $x \in \Omega$ there existes a sequence $t_k \to \infty$ such that $F - \Phi \to 0$. On the other hand, by Lemmas 2.1 and 2.2 we know $u(x, \cdot)$ is monotone increasing and bounded. Therefore, $u(x, t)$ converges uniformly to u^{∞}. By virtue of our a priori estimates, we also know that u^{∞} is of class $C^{\infty}(\bar{\Omega})$. Moreover, it's easy to see that u^{∞} is a stationary solution of our problem, i.e., $f(\kappa[\Sigma^{\infty}]) = \Phi(x, u^{\infty})$ and $u^{\infty}_{\nu} = \phi(x, \infty)$. □

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