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ORIGINAL PAPER

On two inequality conjectures for the k-th Yau numbers of isolated hypersurface singularities

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Abstract

Let $(V, 0)$ be an isolated hypersurface singularity defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. In our previous work, we introduced a series of novel Lie algebras associated to $(V, 0)$, i.e., k-th Yau algebra $L^k(V)$, $k \geq 0$. It was defined to be the Lie algebra of derivations of the k-th moduli algebras $A^k(V) = \mathcal{O}_n/(f, m^k J(f))$, $k \geq 0$, where m is the maximal ideal of $\mathcal{O}_n$. I.e., $L^k(V) := \mathrm{Der}(A^k(V), A^k(V))$. The dimension of $L^k(V)$ was denoted by $\lambda^k(V)$. The number $\lambda^k(V)$, which was called k-th Yau number, is a subtle numerical analytic invariant of $(V, 0)$. Furthermore, we formulated two conjectures for these k-th Yau number invariants: a sharp upper estimate conjecture of $\lambda^k(V)$ for weighted homogeneous isolated hypersurface singularities (see Conjecture 1.2) and an inequality conjecture $\lambda^{(k+1)}(V) > \lambda^k(V)$, $k \geq 0$ (see Conjecture 1.1). In this article, we verify these two conjectures when k is small for large class of singularities.

Keywords Derivation · Lie algebra · Isolated singularity · Yau algebra

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1 Introduction

For any isolated hypersurface singularity $(V, 0) \subset (\mathbb{C}^n, 0)$ defined by the holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, one has the moduli algebra $A(V) := \mathcal{O}_n / \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right)$ which is finite dimensional. Its dimension $\tau(V)$ is called Tyurina number. The order of the lowest non-vanishing term in the power series expansion of f at 0 is called the multiplicity (denoted by $mult(f)$) of the singularity $(V, 0)$. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be weighted homogeneous if there exist positive rational numbers $w_1, \ldots, w_n$ (weights of $x_1, \ldots, x_n$) and d such that, $\sum a_i w_i = d$ for each monomial $\prod x_i^{a_i}$ appearing in f with nonzero coefficient. The number d is called weighted homogeneous degree (w-degree) of f with respect to weights w_j. The weight type of f is denoted as $(w_1, \ldots, w_n; d)$. By a beautiful result of Saito [19], we shall always assume without loss of generality that $2w_i \le d$ for all $1 \le i \le n$. Without loss of generality, we can assume that w-deg$f = 1$. and $w_i \le \frac{1}{2}$.

The well-known Mather–Yau theorem [17] stated that: Let V_1 and V_2 be two isolated hypersurface singularities and, $A(V_1)$ and $A(V_2)$ be the moduli algebras, then $(V_1, 0) \cong (V_2, 0) \iff A(V_1) \cong A(V_2)$. Motivated from the Mather–Yau theorem, the second author introduced the Lie algebra of derivations of moduli algebra $A(V)$, i.e., $L(V) = \mathrm{Der}(A(V), A(V))$. The finite dimensional Lie algebra $L(V)$ was called Yau algebra and its dimension $\lambda(V)$ was called Yau number in [26]. The Yau algerba plays an important role in singularity theory (cf. [11,21]). Yau and his collaborators have been systematically studying the Yau algebras of isolated hypersurface singularities begin from eighties [1,2,4–7,13,21–25,27,28]. In particular, Yau algebras of simple singularities and simple elliptic singularities were computed and a number of elaborate applications to deformation theory were presented in [1,21]. However, the Yau algbra can not characterize the simple singularties completely. In [8], it was shown that if X and Y are two simple singularities except the pair A_6 and D_5, then $L(X) \cong L(Y)$ as Lie algebras if and only if X and Y are analytically isomorphic. Therefore, a natural question is to find new Lie algebras which can be used to distinguish singularities (at least for the simple singularities) completely. In our previous work [14], we introduced the series of new k-th Yau algebra associated to isolated hypersurface singularities. We defined this new k-th Yau algebra as follows.

Recall that we have the following theorem.

Theorem 1.1 [10, Theorem 2.26] *Let $f, g \in m \subset \mathcal{O}_n$. The following are equivalent:*

(1) $(V(f), 0) \cong (V(g), 0)$.
(2) *For all $k \ge 0$, $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra.*
(3) *There is some $k \ge 0$ such that $\mathcal{O}_n/(f, m^k J(f)) \cong \mathcal{O}_n/(g, m^k J(g))$ as $\mathbb{C}$-algebra, where $J(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.*

In particular, if $k = 0$ and $k = 1$ above, and f, g define isolated singularities, then the claim of the equivalence of 1) and 3) is exactly the same as the Mather–Yau theorem.

Based on Theorem 1.1, it is natural for us to introduce the new series of k-th Yau algebras $L^k(V)$ which are defined to be the Lie algebra of derivations of the k-th moduli algebra $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), k \ge 0$, i.e., $L^k(V) = \mathrm{Der}(A^k(V), A^k(V))$. Its dimension is denoted as $\lambda^k(V)$. This number $\lambda^k(V)$ is a new numerical analytic invariant of a singularity. We call it k-th Yau number. In particular, $L^0(V)$ is exactly the Yau algebra, thus $L^0(V) = L(V), \lambda^0(V) = \lambda(V)$.

On the one hand, since $L(V)$ can not characterize the simple singularities completely, so there is a natural question: whether these simple singularities (or which classes of more general singularities) can be characterized completely by the Lie algebra $L^k(V), k \ge 1$? In

[16], we have proven that the simple singularities V can be characterized completely using the $L^1(V)$.

Theorem 1.2 [16] *If X and Y are two simple hypersurface singularities, then* $L^1(X) \cong L^1(Y)$ *as Lie algebras, if and only if X and Y are analytically isomorphic.*

We believe Theorem 1.2 is also true for $L^k(V)$, $k > 1$. Therefore the k-th Yau algebra $L^k(V)$, $k \geq 1$, is more subtle comparing to the Yau algebra $L(V)$ in some sense.

Furthermore, since derivations of moduli algebras are analogs of vector fields on smooth manifolds, such direction of research is in the spirit of the classical theorem of Pursell and Shanks stating that the Lie algebra of smooth vectors fields on a smooth manifold determines the diffeomorphism type of the manifold [18]. It is interesting to investigate the structure of k-th Yau algebras and find out, for which classes of singularities those Lie algebras determine the analytic or topological structures of singularities by analogy with the mentioned result of Pursell and Shanks. In fact, Theorem 1.2 yields a similar for the simple singularities.

On the other hand, it is well known that finite dimensional Lie algebras are semi-direct product of the semi-simple Lie algebras and solvable Lie algebras. Brieskorn gave the connection between simple Lie algebras and simple singularities. Simple Lie algebras have been well understood, but not the solvable (nilpotent) Lie algebras. It is extremely important to establish connections between singularities and solvable (nilpotent) Lie algebras. In fact $L^k(V)$ are finite dimensional solvable (nilpotent) Lie algebras naturally from isolated hypersurface singularities. These objects $L^k(V)$ help us to understand the solvable (nilpotent) Lie algebras from the geometric point of view. Moreover, it is known that the classification of nilpotent Lie algebras in higher dimensions (> 7) remains to be a vast open area. There are one-parameter families of non-isomorphic nilpotent Lie algebras (but no two-parameter families) in dimension seven. Dimension seven is the watershed of the existence of such families. It is well-known that no such family exists in dimension less than seven, while it is hard to construct one-parameter family in dimension greater than seven. However, such examples are hard to construct [20].

Recall that Griffiths has studied the Torelli problem when a family of complex projective hypersurfaces in $\mathbb{C}P^n$ is given and his school asks whether the period map is injective on that family, i.e., whether the family of complex hypersurfaces can be distinguished by means of their Hodge structures. A complex projective hypersurface in $\mathbb{C}P^n$ can be viewed as a complex hypersurface with isolated singularity in $\mathbb{C}^{n+1}$. Let $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ be a complex hypersurface with isolated singularity at the origin. Seeley and Yau [21] investigated the family of isolated complex hypersurface singularities using Yau algebras $L(V)$ and obtained two strong Torelli-type theorems for simple elliptic singularities $\tilde{E}_7$ and $\tilde{E}_8$. We obtained the following similar result for $L^k(V)$.

Theorem 1.3 [16] $L^2(\tilde{E}_6)$, $L^1(\tilde{E}_7)$, $L^2(\tilde{E}_7)$, $L^1(\tilde{E}_8)$ *and* $L^2(\tilde{E}_8)$ *are non-trivial one-parameter families. Thus the weak Torelli-type theorems hold for simple elliptic singularities* $\tilde{E}_6$, $\tilde{E}_7$ *and* $\tilde{E}_8$.

The second author [23] showed that the Yau algebra $L(V)$ of the family $\tilde{E}_6$ is constant, i.e., it does not depend on the parameter. In a recent paper [12], we have shown that the first Yau algebra $L^1(V)$ is also constant,. However, the strong Torelli-type theorem holds for $\tilde{E}_6$ using k-th Yau algebra $L^k(V)$, $k \geq 2$.

Theorem 1.4 [12] *Let* $\{V_t\}$ *represent a family of simple elliptic singularities* $\tilde{E}_6$. *If* $k \geq 2$, *then* L_t^k *and* L_s^k *are isomorphic as Lie algebras if and only if* V_t *is biholomorphic to* V_s.

As a corollary of Theorems 1.3 and 1.4, we can obtain many non-trivial one-parameter families of solvable (nilpotent) Lie algebras in dimension greater than seven.

Since the k-th Yau algebras $L^k(V)$ are more subtle invariants than the Yau algebras, we believe that these new Lie algebras $L^k(V)$ and numerical invariants $\lambda^k(V)$ will also play an important role in the study of singularities.

In this paper, we investigate the new analytic invariants $\lambda^k(V)$. A natural question arises: are there any numerical relations between the invariants $\lambda^k(V), k \geq 0$? We proposed the following conjecture:

Conjecture 1.1 [15] *With the above notations, let $(V, 0)$ be an isolated hypersurface singularity defined by $f \in \mathcal{O}_n,\ n \geq 2$, and $mult(f) \geq 3$. Then*

$$\lambda^{(k+1)}(V) > \lambda^k(V), k \geq 0.$$

The Conjecture 1.1 has already been verified for binomial singularities (see Definition 2.4) when $k = 0, 1$, and trinomial singularities (see Definition 2.4) when $k = 0$ by the authors in [15]. In this paper we shall prove this conjecture for trinomial singularities when $k = 1$ (see Theorem A).

It is also interesting to bound the k-th Yau number of weighted homogeneous singularities with a number that depends on weight type. In Yau and Zuo [28] firstly proposed the sharp upper estimate conjecture that bound the Yau number. They also proved that this conjecture holds in the case of binomial isolated hypersurface singularities. Furthermore, in [13], this conjecture was verified for trinomial singularities. We proposed the following sharp upper estimate conjecture which is a generalization of the conjecture in [28].

Conjecture 1.2 [14] *Assume that $\lambda^k(\{x_1^{a_1} + \cdots + x_n^{a_n} = 0\}) = h_k(a_1, \ldots, a_n)$, $(k \geq 0)$. Let $(V, 0) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n : f(x_1, x_2, \ldots, x_n) = 0\}$, $(n \geq 2)$ be an isolated singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, \ldots, x_n)$ of weight type $(w_1, w_2, \ldots, w_n; 1)$. Then $\lambda^k(V) \leq h_k(1/w_1, \ldots, 1/w_n)$.*

The Conjecture 1.2 tells, when fixing a weight type, the Brieskorn singularity $x_1^{a_1} + \cdots + x_n^{a_n}$ has maximal k-th Yau number. It has already been verified for binomial singularities and trinomial singualrities when $k = 0$ in [13,28] respectively and when $k = 1$ [14]. In this paper we shall prove this conjecture for binomial singularities and trinomial singualrities when $k = 2$ (see Theorems B, C and D). We obtain the following main results.

Theorem A *Let $(V, 0)$ be a fewnomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, x_3)$ (see Proposition 2.2) with $mult(f) \geq 3$, then*

$$\lambda^2(V) > \lambda^1(V).$$

Theorem B *Let $(V, 0)$ be a binomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2)$ (see Corollary 2.1) with weight type $(w_1, w_2; 1)$ and $mult(f) \geq 3$, then*

$$\lambda^2(V) \leq h_2\left(\frac{1}{w_1}, \frac{1}{w_2}\right) = \begin{cases} \frac{2}{w_1 w_2} - 3\left(\frac{1}{w_1} + \frac{1}{w_2}\right) + 17; & w_1 \leq \frac{1}{5}, \quad w_2 \leq \frac{1}{5} \\ \frac{3}{w_2} + 5; & w_1 = \frac{1}{3}, \quad w_2 \leq \frac{1}{4} \\ 13; & w_1 = \frac{1}{3}, \quad w_2 = \frac{1}{3} \\ \frac{5}{w_2} + 4; & w_1 = \frac{1}{4}, \quad w_2 \leq \frac{1}{5} \\ 23; & w_1 = \frac{1}{4}, \quad w_2 = \frac{1}{4}. \end{cases}$$

Remark 1.1 If $multf(x_1, x_2)) = 2$, then $f(x_1, x_2)$ is contact equivalent [10] to $x_1^2+x_2^b$ where $b \geq 2$. Thus our Conjecture 1.2 is obviously true for this case. Therefore in Theorem B, we only need to consider $mult(f) \geq 3$.

Theorem C *Let $(V, 0)$ be a fewnomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, x_3)$ (see Proposition 2.2) with weight type $(w_1, w_2, w_3; 1)$ and mult$(f) \geq 3$, then*

$$\lambda^2(V) \leq h_2\left(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}\right) = \frac{3}{w_1 w_2 w_3} + 5\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) - 4\left(\frac{1}{w_1 w_2} + \frac{1}{w_1 w_3} + \frac{1}{w_2 w_3}\right) + 34, \quad w_1 \leq \frac{1}{3}, \quad w_2 \leq \frac{1}{3}, \quad w_3 \leq \frac{1}{3}.$$

Theorem D *Let $(V, 0)$ be a fewnomial singularity defined by the weighted homogeneous polynomial $f(x_1, x_2, x_3)$ (see Proposition 2.2) with weight type $(w_1, w_2, w_3; 1)$ and mult$(f) = 2$ (see Remark 1.2), then*

$$\lambda^2(V) \leq h_2\left(\frac{1}{w_1}, \frac{1}{w_2}, 2\right) = \begin{cases} \frac{2}{w_1 w_2} - 3\left(\frac{1}{w_1} + \frac{1}{w_2}\right) + 36; & w_1 \leq \frac{1}{5}, \quad w_2 \leq \frac{1}{5} \\ \frac{3}{w_2} + 22; & w_1 = \frac{1}{3}, \quad w_2 \leq \frac{1}{4} \\ 30; & w_1 = \frac{1}{3}, \quad w_2 = \frac{1}{3} \\ \frac{5}{w_2} + 23; & w_1 = \frac{1}{4}, \quad w_2 \leq \frac{1}{5} \\ 42; & w_1 = \frac{1}{4}, \quad w_2 = \frac{1}{4}. \end{cases}$$

Remark 1.2 If $f(x_1, x_2, x_3)$ is fewnomial and $mult(f(x_1, x_2, x_3)) = 2$, then $f(x_1, x_2, x_3)$ is contact equivalent to one of the following cases:

Case 1 (A) $x_1^{a_1} + x_2^{a_2} + x_3^2$, $a_1, a_2 \geq 3$; (B) $x_1^{a_1}x_2 + x_2^{a_2} + x_3^2$, $a_1 \geq 2, a_2 \geq 3$; (C) $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^2$, $a_1, a_2 \geq 2$;
Case 2 (A) $x_1^{a_1} + x_2^2 + x_3^2$, $a_1 \geq 2$.

The Conjecture 1.2 is true for case 2. Thus in Theorem D, we only need to verify the Conjecture 1.2 for case 1.

2 Generalities on derivation lie algebras of isolated singularities

In this section, we shall briefly define the basic definitions and important results that are helpful to solve the problem. The following basic concepts and results will be used to compute the derivation Lie algebras of isolated hypersurface singularities.

Let A, B be associative algebras over $\mathbb{C}$. The subalgebra of endomorphisms of A generated by the identity element and left and right multiplications by elements of A is called multiplication algebra $M(A)$ of A. The centroid $C(A)$ is defined as the set of endomorphisms of A which commute with all elements of $M(A)$. Obviously, $C(A)$ is an unital subalgebra of End(A). The following statement is a particular case of a general result from Proposition 1.2 of [3]. Let $S = A \otimes B$ be a tensor product of finite dimensional associative algebras with units. Then

$$\text{Der}S \cong (\text{Der}A) \otimes C(B) + C(A) \otimes (\text{Der}B).$$

We will only use this result for commutative associative algebras with unit, in which case the centroid coincides with the algebra itself and one has following result for commutative associative algebras A, B:

Theorem 2.1 [3] *For commutative associative algebras A, B,*

$$DerS \cong (DerA) \otimes B + A \otimes (DerB). \tag{1}$$

We shall use this formula in the sequel.

Definition 2.1 Let J be an ideal in an analytic algebra S. Then $\mathrm{Der}_J S \subseteq \mathrm{Der}_{\mathbb{C}} S$ is Lie subalgebra of all $\sigma \in \mathrm{Der}_{\mathbb{C}} S$ for which $\sigma(J) \subset J$.

We shall use the following well-known result to compute the derivations.

Theorem 2.2 [28] *Let J be an ideal in $R = \mathbb{C}\{x_1, \ldots, x_n\}$. Then there is a natural isomorphism of Lie algebras*

$$(Der_J R)/(J \cdot Der_{\mathbb{C}} R) \cong Der_{\mathbb{C}}(R/J).$$

Recall that a derivation of commutative associative algebra A is defined as a linear endomorphism D of A satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$. Thus for such an algebra A one can consider the Lie algebra of its derivations $\mathrm{Der}(A, A)$ with the bracket defined by the commutator of linear endomorphisms.

Definition 2.2 Let $f(x_1, \ldots, x_n)$ be a complex polynomial and $V = \{f = 0\}$ be a germ of an isolated hypersurface singularity at the origin in $\mathbb{C}^n$. Let $A^k(V) = \mathcal{O}_n/(f, m^k J(f))$, $1 \le k \le n$ be a moduli algebra. Then $Der(A^k(V), A^k(V))$ defined the derivation Lie algebras $L^k(V)$. The $\lambda^k(V)$ is the dimension of derivation Lie algebra $L^k(V)$.

It is noted that when $k = 0$, then derivation Lie algebra is called Yau algebra.

Definition 2.3 A polynomial $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ is called quasi-homogeneous (or weighted homogeneous) if there exist positive rational numbers $w_1, \ldots, w_n$ (called weights of indeterminates x_j) and d such that, for each monomial $\prod x_j^{k_j}$ appearing in f with non-zero coefficient, one has $\sum w_j k_j = d$. The number d is called the quasi-homogeneous degree (w-degree) of f with respect to weights w_j and is denoted $\deg f$. The collection $(w; d) = (w_1, \ldots, w_n; d)$ is called the quasi-homogeneity type (qh-type) of f.

Definition 2.4 An isolated hypersurface singularity in $\mathbb{C}^n$ is fewnomial if it can be defined by a n-nomial in n variables and it is a weighted homogeneous fewnomial isolated singularity if it can be defined by a weighted homogeneous fewnomial. 2-nomial (resp. 3-nomial) isolated hypersurface singularity is also called binomial (resp. trinomial) singularity.

Proposition 2.1 *Let f be a weighted homogeneous fewnomial isolated singularity with $mult(f) \ge 3$. Then f analytically equivalent to a linear combination of the following three series:*

Type A. $x_1^{a_1} + x_2^{a_2} + \cdots + x_{n-1}^{a_{n-1}} + x_n^{a_n}$, $n \ge 1$,
Type B. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n}$, $n \ge 2$,
Type C. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{n-1}^{a_{n-1}} x_n + x_n^{a_n} x_1$, $n \ge 2$.

Proposition 2.1 has an immediate corollary.

Corollary 2.1 *Each binomial isolated singularity is analytically equivalent to one from the three series: (A)* $x_1^{a_1} + x_2^{a_2}$, *(B)* $x_1^{a_1} x_2 + x_2^{a_2}$, *(C)* $x_1^{a_1} x_2 + x_2^{a_2} x_1$.

Wolfgang and Atsushi [9] give the following classification of weighted homogeneous fewnomial singularities.

Proposition 2.2 [9] *Let* $f(x_1, x_2, x_3)$ *be a weighted homogeneous fewnomial isolated singularity with mult*$(f) \geq 3$. *Then* f *is analytically equivalent to following five types:*

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$,
Type 5. $x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^{a_3}$.

In order to prove the Theorem A, we need to use the following propositions.

Proposition 2.3 [14] *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 1 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 3)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 1)$. *Then*

$$\lambda^1(V) = 3a_1a_2a_3 + 5(a_1 + a_2 + a_3) - 4(a_1a_2 + a_1a_3 + a_2a_3) + 6.$$

Proposition 2.4 [14] *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 2 which is defined by* $f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ *with weight type* $(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1)$. *Then*

$$\lambda^1(V) = \begin{cases} 4a_1a_3 - 2a_1 - 3a_3 + 11; & a_1 \geq 3, a_2 = 2, a_3 \geq 3 \\ 5a_3 + 7; & a_1 = 2, a_2 = 2, a_3 \geq 3 \\ 3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 & \\ -4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 5; & \textit{Otherwise}. \end{cases}$$

Proposition 2.5 [14] *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 3 which is defined by* $f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 2)$ *with weight type*

$$\left(\frac{1 - a_3 + a_2a_3}{1 + a_1a_2a_3}, \frac{1 - a_1 + a_1a_3}{1 + a_1a_2a_3}, \frac{1 - a_2 + a_1a_2}{1 + a_1a_2a_3}; 1\right).$$

Then

$$\lambda^1(V) = \begin{cases} 24; & a_1 = 2, a_2 = 2, a_3 = 2 \\ 3a_1a_2a_3 + 2(a_1 + a_2 + a_3) & \\ \quad -2(a_1a_2 + a_1a_3 + a_2a_3) + 11; & \textit{Otherwise}. \end{cases}$$

Proposition 2.6 [14] *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 4 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 2)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_2-1}{a_2a_3}; 1)$. *Then*

$$\lambda^1(V) = \begin{cases} 5a_1a_2 - a_1 - 7a_2 + 15; & a_1 \geq 3, a_2 \geq 3, a_3 = 3 \\ 3a_1a_2a_3 - 4a_1a_2 - 3a_1a_3 & \\ \quad -4a_2a_3 + 8a_1 + 5a_2 + 5a_3 - 1; & \textit{Otherwise}. \end{cases}$$

Proposition 2.7 [14] *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 5 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. *Then*

$$\lambda^1(V) = \begin{cases} 4a_2a_3 - 6a_2 + 12; & a_1 = 2, a_2 \geq 2, a_3 \geq 3 \\ 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 & \\ \quad -2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 6; & \textit{Otherwise}. \end{cases}$$

3 Proof of main theorems

Proposition 3.1 *Let* $(V, 0)$ *be a weighted homogeneous fewnomial isolated singularity of type A which is defined by* $f = x_1^{a_1} + x_2^{a_2}$ $(a_1 \geq 2,\ a_2 \geq 2)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 17; & a_1 \geq 5, a_2 \geq 5 \\ 3a_2 + 5; & a_1 = 3, a_2 \geq 4 \\ 13; & a_1 = 3, a_2 = 3 \\ 5a_2 + 4; & a_1 = 4, a_2 \geq 5 \\ 23; & a_1 = 4, a_2 = 4 \\ a_2 + 5; & a_1 = 2, a_2 \geq 3 \\ 6; & a_1 = 2, a_2 = 2. \end{cases}$$

Proof It follows that the generalized moduli algebra

$$A^2(V) = \mathbb{C}\{x_1, x_2\}/(f, m^2 J(f)),$$

has dimension $a_1a_2 - (a_1 + a_2) + 6$ and has a monomial basis of the form:

$$\text{(1)} \quad \text{if } a_1 \geq 3, \left\{x_1^{i_1}x_2^{i_2}, 1 \leq i_1 \leq a_1 - 2; 0 \leq i_2 \leq a_2 - 2; x_1^{a_1-1}; x_1^{a_1-1}x_2; x_1x_2^{a_2-1}; x_2^{i_2}, 0 \leq i_2 \leq a_2\right\}, \tag{2}$$

$$\text{(2)} \quad \text{if } a_1 = 2, a_2 \geq 3, \left\{x_2^{i_2}, 0 \leq i_2 \leq a_2; x_1x_2; x_1\right\}, \tag{3}$$

$$\text{(3)} \quad \text{if } a_1 = 2, a_2 = 2, \{1; x_1; x_1x_2; x_2; x_2^2\}, \tag{4}$$

$$\text{(4)} \quad \text{if } a_1 = 1, a_2 \geq 1, \{1; x_2\}, \tag{5}$$

with the following relations:

$$x_1^{a_1+1} = 0, \tag{6}$$
$$x_1^{a_1-1}x_2^2 = 0, \tag{7}$$
$$x_1^{a_1}x_2 = 0, \tag{8}$$
$$x_1^2x_2^{a_2-1} = 0, \tag{9}$$
$$x_2^{a_2+1} = 0, \tag{10}$$
$$x_1x_2^{a_2} = 0. \tag{11}$$

In order to compute a derivation D of $A^2(V)$ it suffices to indicate its values on the generators x_1, x_2 which can be written in terms of the basis (2), (3), (4) or (5). Without loss of generality, we write

$$Dx_j = \sum_{i_1=1}^{a_1-2}\sum_{i_2=0}^{a_2-2} c^j_{i_1,i_2} x_1^{i_1} x_2^{i_2} + c^j_{a_1-1,0} x_1^{a_1-1} + c^j_{a_1-1,1} x_1^{a_1-1} x_2 10$$

$$+ c^j_{1,a_2-1} x_1 x_2^{a_2-1} + \sum_{i_2=0}^{a_2} c^j_{0,i_2} x_2^{i_2}, \ \ j = 1, 2.$$

Using the relations (6)–(11) one easily finds the necessary and sufficient conditions defining a derivation of $A^2(V)$ as follows:

$$c^1_{0,0} = c^1_{0,1} = \cdots = c^1_{0,a_2-4} = 0; \tag{12}$$

$$c^2_{0,0} = c^2_{1,0} = \cdots = c^2_{a_1-4,0} = 0; \tag{13}$$

$$a_1 c^1_{1,0} = a_2 c^2_{0,1}. \tag{14}$$

Using (12)–(14) we obtain the following description of the Lie algebras in question. The following derivations form a basis of $\mathrm{Der}A^2(V)$:

$x_1^{i_1} x_2^{i_2}\partial_1, 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2; x_1^{a_1-1}x_2\partial_1; x_1x_2^{a_2-1}\partial_1; x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2;$
$x_1^{i_1}\partial_1, 2 \le i_1 \le a_1-1; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1^{i_1}\partial_2, a_1-3 \le i_1 \le a_1-1;$
$x_1^{i_1}x_2^{i_2}\partial_2, 1 \le i_1 \le a_1-2, 1 \le i_2 \le a_2-2; x_1^{a_1-1}x_2\partial_2;$
$x_1x_2^{a_2-1}\partial_2; x_1\partial_1 + \frac{a_1}{a_2}x_2\partial_2.$

Therefore we have the following formula

$$\lambda^2(V) = 2a_1a_2 - 3(a_1+a_2) + 17.$$

In case of $a_1 = 3, a_2 \ge 4$, we have following derivations form a basis of $\mathrm{Der}A^2(V)$:

$$x_1x_2^{i_2}\partial_1, 1 \le i_2 \le a_2-1; x_1^2x_2\partial_1; x_1^2\partial_1; x_2^{i_2}\partial_1, a_2-2 \le i_2 \le a_2;$$
$$x_1x_2^{i_2}\partial_2, 1 \le i_2 \le a_2-1; x_1^2x_2\partial_2; x_1^2\partial_2; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1\partial_1 + \frac{3}{a_2}x_2\partial_2.$$

Therefore we have the following formula

$$\lambda^2(V) = 3a_2 + 5.$$

In case of $a_1 = 3, a_2 = 3$, we have following derivations form a basis of $\mathrm{Der}A^2(V)$:

$$x_2^2\partial_1; x_2^3\partial_1; x_1x_2\partial_1; x_1x_2^2\partial_1; x_1^2\partial_1; x_1^2x_2\partial_1; x_2^2\partial_2; x_2^3\partial_2; x_1x_2\partial_2; x_1x_2^2\partial_2; x_1^2\partial_2; x_1^2x_2\partial_2; x_1\partial_1 + x_2\partial_2.$$

In case of $a_1 = 4, a_2 \ge 5$, we have following derivations form a basis of $\mathrm{Der}A^2(V)$:

$$x_2^{i_2}\partial_1, a_2-3 \le i_2 \le a_2; x_1\partial_1 + \frac{4}{a_2}x_2\partial_2; x_1^{i_1}x_2\partial_1, 1 \le i_1 \le 2, 1 \le i_2 \le a_2-2; x_1x_2^{a_2-1}\partial_1; x_1^2\partial_1; x_1^3\partial_1;$$
$$x_1^3x_2\partial_1; x_2^{i_2}\partial_2, 2 \le i_2 \le a_2; x_1^{i_1}x_2^{i_2}\partial_2, 1 \le i_1 \le 2, 1 \le i_2 \le a_2-2; x_1x_2^{a_2-1}\partial_2; x_1^2\partial_2; x_1^3\partial_2; x_1^3x_2\partial_2.$$

Therefore we have the following formula

$$\lambda^2(V) = 5a_2 + 4.$$

In case of $a_1 = 4, a_2 = 4$, we have following derivations form a basis of $\mathrm{Der}A^2(V)$:

$x_2^2\partial_1; x_2^3\partial_1; x_2^4\partial_1; x_1\partial_1 + x_2\partial_2; x_1x_2\partial_1; x_1x_2^2\partial_1; x_1x_2^3\partial_1; x_1^2\partial_1; x_1^2x_2\partial_1; x_1^2x_2^2\partial_1; x_1^3$
$\partial_1; x_1^3x_2\partial_1; x_2^2\partial_2; x_2^3\partial_2;$
$x_2^4\partial_2; x_1x_2\partial_2; x_1x_2^{i_2}\partial_2; x_1x_2^3\partial_2; x_1^2\partial_2; x_1^2x_2\partial_2; x_1^2x_2^2\partial_2; x_1^3\partial_2; x_1^3x_2\partial_1.$

Therefore we have the following formula

$$\lambda^2(V) = 23.$$

In case of $a_1 = 2, a_2 \geq 3$, we have following derivations which form a basis of $L^2(V)$:

$$x_2^{i_2}\partial_1, a_2 - 1 \leq i_2 \leq a_2; x_1\partial_1 + \frac{2}{a_2}x_2\partial_2; x_1x_2\partial_1; x_2^{i_2}\partial_2, 2 \leq i_2 \leq a_2; x_1\partial_2; x_1x_2\partial_2.$$

Therefore we get following formula

$$\lambda^2(V) = a_2 + 5.$$

In case of $a_1 = 2, a_2 = 2$, we have following derivations which form a basis of $L^2(V)$:

$$x_2\partial_1 + x_1\partial_2; x_2^2\partial_1; x_1\partial_1 + x_2\partial_2; x_1x_2\partial_1; x_2^2\partial_2; x_1x_2\partial_2.$$

Therefore we get following formula

$$\lambda^2(V) = 6.$$

□

For the proofs of Propositions 3.2–3.11, we skip the details due to space constraints. Interested readers can find all the detailed proofs in a longer version of this paper at http://archive.ymsc.tsinghua.edu.cn/pacm_download/89/11687-HYZ2020.pdf.

Proposition 3.2 *Let $(V, 0)$ be a binomial isolated singularity of type B which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}$ $(a_1 \geq 1,\ a_2 \geq 2)$ with weight type $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}; 1)$. Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 20; & a_1 \geq 5, \quad a_2 \geq 5 \\ 5a_2 + 12; & a_1 = 4, \quad a_2 \geq 5 \\ 31; & a_1 = 4, \quad a_2 = 4 \\ 4a_1 + 7; & a_1 \geq 3, \quad a_2 = 3 \\ 2a_1 + 5; & a_1 \geq 2, \quad a_2 = 2 \\ a_2 + 11; & a_1 = 2, \quad a_2 \geq 4 \\ 13; & a_1 = 2, \quad a_2 = 3 \\ 6; & a_1 = 1, \quad a_2 \geq 2. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 5, a_2 \geq 5$, then $2a_1a_2 - 2a_1 - 3a_2 + 20 \leq \frac{2a_1a_2^2}{a_2-1} - 3(\frac{a_1a_2}{a_2-1} + a_2) + 17$.

Proposition 3.3 *Let $(V, 0)$ be a binomial isolated singularity of type C which is defined by $f = x_1^{a_1}x_2 + x_2^{a_2}x_1$ $(a_1 \geq 1,\ a_2 \geq 1)$ with weight type $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}; 1)$. Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2(a_1 + a_2) + 21; & a_1 \geq 4, \quad a_2 \geq 4 \\ 4a_2 + 13; & a_1 = 3, \quad a_2 \geq 4 \\ 2a_2 + 10; & a_1 = 2, \quad a_2 \geq 3 \\ 23; & a_1 = 3, \quad a_2 = 3 \\ 13; & a_1 = 2, \quad a_2 = 2 \\ 6; & a_1 = 1, \quad a_2 \geq 1. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 5, a_2 \geq 5$, *then* $2a_1a_2 - 2(a_1+a_2) + 21 \leq \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 17$.

Proposition 3.4 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 1 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 3)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}; 1)$. *Then*

$$\lambda^2(V) = 3a_1a_2a_3 + 5(a_1 + a_2 + a_3) - 4(a_1a_2 + a_1a_3 + a_2a_3) + 34.$$

Proposition 3.5 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 2 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ *with weight type* $(\frac{1-a_3+a_2a_3}{a_1a_2a_3}, \frac{a_3-1}{a_2a_3}, \frac{1}{a_3}; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 3a_1a_2a_3 + 2a_1 + 2a_2 + 6a_3 & \\ \quad -4a_2a_3 - 2a_1a_2 - 2a_1a_3 + 37; & a_1 \geq 3, \quad a_2 \geq 3, \quad a_3 \geq 3 \\ 4a_2 + 42; & a_1 = 2, \quad a_2 \geq 3, \quad a_3 = 3 \\ 2a_2a_3 - 2a_2 + 2a_3 + 38; & a_1 = 2, \quad a_2 \geq 3, \quad a_3 \geq 4 \\ 4a_1a_3 - 3a_3 - 2a_1 + 42; & a_1 \geq 3, \quad a_2 = 2, \quad a_3 \geq 4 \\ 5a_3 + 35; & a_1 = 2, \quad a_2 = 2, \quad a_3 \geq 4 \\ 46; & a_1 = 2, \quad a_2 = 2, \quad a_3 = 3 \\ 10a_1 + 32; & a_1 \geq 3, \quad a_2 = 2, \quad a_3 = 3. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, *then* $3a_1a_2a_3 - 2a_1a_2 - 2a_1a_3 - 4a_2a_3 + 2a_1 + 2a_2 + 6a_3 + 37 \leq 3\frac{a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)} - 4(\frac{a_1a_2^2a_3^3}{(1-a_3+a_2a_3)(a_3-1)} + \frac{a_1a_2a_3^2}{1-a_3+a_2a_3} + \frac{a_2a_3^2}{a_3-1}) + 5(\frac{a_1a_2a_3}{1-a_3+a_2a_3} + \frac{a_2a_3}{a_3-1} + a_3) + 34$.

Proposition 3.6 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 3 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 2)$ *with weight type*

$$\left(\frac{1 - a_3 + a_2a_3}{1 + a_1a_2a_3}, \frac{1 - a_1 + a_1a_3}{1 + a_1a_2a_3}, \frac{1 - a_2 + a_1a_2}{1 + a_1a_2a_3}; 1\right).$$

Then

$$\lambda^2(V) = \begin{cases} 3a_1a_2a_3 + 2(a_1 + a_2 + a_3) & \\ \quad -2(a_1a_2 + a_1a_3 + a_2a_3) + 43; & a_1 \geq 3, \quad a_2 \geq 3, \quad a_3 \geq 3 \\ 4a_2a_3 - 2(a_2 + a_3) + 46; & a_1 = 2, \quad a_2 \geq 3, \quad a_3 \geq 3. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 3, a_2 \geq 3, a_3 \geq 3$, *then* $3a_1a_2a_3 + 2(a_1 + a_2 + a_3) - 2(a_1a_2 + a_1a_3 + a_2a_3) + 43 \leq \frac{3(1+a_1a_2a_3)^3}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + 5(\frac{1+a_1a_2a_3}{1-a_3+a_2a_3} + \frac{1+a_1a_2a_3}{1-a_1+a_1a_3} + \frac{1+a_1a_2a_3}{1-a_2+a_1a_2}) - 4(\frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_1+a_1a_3)} + \frac{(1+a_1a_2a_3)^2}{(1-a_1+a_1a_3)(1-a_2+a_1a_2)} + \frac{(1+a_1a_2a_3)^2}{(1-a_3+a_2a_3)(1-a_2+a_1a_2)}) + 34$.

Proposition 3.7 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 4 which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$ $(a_1 \geq 3, a_2 \geq 3, a_3 \geq 2)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{a_2-1}{a_2a_3}; 1)$.

Then

$$\lambda^2(V) = \begin{cases} 7a_1a_3 - 6a_1 - 10a_3 + 48; & a_1 \geq 3, \quad a_2 = 3, \quad a_3 \geq 3 \\ 5a_2a_3 - 7a_2 - 4a_3 + 53; & a_1 = 3, \quad a_2 \geq 4, \quad a_3 \geq 3 \\ 3a_1a_2a_3 + 6a_1 + 5a_2 + 2a_3 & \\ -4a_2a_3 - 4a_1a_2 - 2a_1a_3 + 35; & a_1 \geq 4, \quad a_2 \geq 4, \quad a_3 \geq 3 \\ 11a_1a_2 - 3a_1 - 15a_2 + 41; & a_1 \geq 4, \quad a_2 \geq 4, \quad a_3 = 5 \\ 46; & a_1 = 3, \quad a_2 = 3, \quad a_3 = 2 \\ 2a_1a_2 + 2a_1 - 3a_2 + 36; & a_1 \geq 4, \quad a_2 \geq 4, \quad a_3 = 2 \\ 3a_2 + 40; & a_1 = 3, \quad a_2 \geq 4, \quad a_3 = 2 \\ 8a_1 + 26; & a_1 \geq 4, \quad a_2 = 3, \quad a_3 = 2 \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 4, a_2 \geq 4, a_3 \geq 3$, *then* $3a_1a_2a_3 + 6a_1 + 5a_2 + 2a_3 - 4a_2a_3 - 4a_1a_2 - 2a_1a_3 + 35 \leq \frac{3a_1a_2^2a_3}{a_2-1} + 5(a_1 + a_2 + \frac{a_2a_3}{a_2-1}) - 4(a_1a_2 + \frac{a_1a_2a_3}{a_2-1} + \frac{a_2^2a_3}{a_2-1}) + 34.$

Proposition 3.8 *Let* $(V, 0)$ *be a fewnomial surface isolated singularity of type 5 which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$ $(a_1 \geq 2, a_2 \geq 2, a_3 \geq 3)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{a_3}; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 & \\ \quad -2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 36; & a_1 \geq 4, \quad a_2 \geq 4, \quad a_3 \geq 3 \\ 7a_2a_3 - 10a_2 + 40; & a_1 = 3, \quad a_2 \geq 3, \quad a_3 \geq 3 \\ 4a_2a_3 - 6a_2 + 39; & a_1 = 2, \quad a_2 \geq 4, \quad a_3 \geq 4 \\ 8a_3 + 26; & A_1 = 2, \quad a_2 = 2, \quad a_3 \geq 4 \\ 46; & a_1 = 2, \quad a_2 = 2, \quad a_3 = 3 \\ 12a_3 + 21; & a_1 = 2, \quad a_2 = 3, \quad a_3 \geq 4 \\ 55; & a_1 = 2, \quad a_2 = 3, \quad a_3 = 3 \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 4, a_2 \geq 4, a_3 \geq 3$, *then* $3a_1a_2a_3 - 4a_1a_2 - 2a_2a_3 - 2a_1a_3 + 2a_1 + 2a_2 + 6a_3 + 36 \leq \frac{3(a_1a_2-1)^2}{(a_1-1)(a_2-1)}a_3 + 5(\frac{a_1a_2-1}{a_1-1} + \frac{a_1a_2-1}{a_2-1} + a_3) - 4(\frac{(a_1a_2-1)^2}{(a_1-1)(a_2-1)} + \frac{a_1a_2-1}{a_2-1}a_3 + \frac{a_1a_2-1}{a_1-1}a_3) + 34.$

Proposition 3.9 *Let* $(V, 0)$ *be a weighted homogeneous fewnomial isolated singularity which is defined by* $f = x_1^{a_1} + x_2^{a_2} + x_3^2$ $(a_1 \geq 3,\ a_2 \geq 3)$ *with weight type* $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{2}; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 3(a_1 + a_2) + 36; & a_1 \geq 5, \quad a_2 \geq 5 \\ 3a_2 + 22; & a_1 = 3, \quad a_2 \geq 4 \\ 30; & a_1 = 3, \quad a_2 = 3 \\ 5a_2 + 23; & a_1 = 4, \quad a_2 \geq 5 \\ 42; & a_1 = 4, \quad a_2 = 4. \end{cases}$$

Furthermore, when $a_1 \geq 2, a_2 = 2$ *then*

$$\lambda^2(V) = \begin{cases} a_1 + 19; & a_1 \geq 3, \\ 19; & a_1 = 2. \end{cases}$$

Proposition 3.10 *Let* $(V, 0)$ *be a weighted homogeneous fewnomial isolated singularity which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2} + x_3^2$ $(a_1 \geq 2,\ a_2 \geq 3)$ *with weight type* $(\frac{a_2-1}{a_1a_2}, \frac{1}{a_2}, \frac{1}{2}; 1)$.

Then

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2a_1 - 3a_2 + 39; & a_1 \geq 5, \quad a_2 \geq 5 \\ 5a_2 + 31; & a_1 = 4, \quad a_2 \geq 5 \\ 50; & a_1 = 4, \quad a_2 = 4 \\ 4a_1 + 24; & a_1 \geq 3, \quad a_2 = 3 \\ a_2 + 28; & a_1 = 2, \quad a_2 \geq 4 \\ 30; & a_1 = 2, \quad a_2 = 3. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 5, a_2 \geq 5$*, then* $2a_1a_2 - 2a_1 - 3a_2 + 39 \leq \frac{2a_1a_2^2}{a_2-1} - 3(\frac{a_1a_2}{a_2-1} + a_2) + 36$.

Proposition 3.11 *Let* $(V, 0)$ *be a weighted homogeneous fewnomial isolated singularity which is defined by* $f = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^2$ $(a_1 \geq 2, \quad a_2 \geq 2)$ *with weight type* $(\frac{a_2-1}{a_1a_2-1}, \frac{a_1-1}{a_1a_2-1}, \frac{1}{2}; 1)$. *Then*

$$\lambda^2(V) = \begin{cases} 2a_1a_2 - 2(a_1 + a_2) + 40; & a_1 \geq 4, \quad a_2 \geq 4 \\ 4a_2 + 32; & a_1 = 3, \quad a_2 \geq 4 \\ 2a_2 + 27; & a_1 = 2, \quad a_2 \geq 3 \\ 42; & a_1 = 3, \quad a_2 = 3 \\ 30; & a_1 = 2, \quad a_2 = 2. \end{cases}$$

Furthermore, we need to show that when $a_1 \geq 5, a_2 \geq 5$*, then* $2a_1a_2 - 2(a_1 + a_2) + 40 \leq \frac{2(a_1a_2-1)^2}{(a_1-1)(a_2-1)} - 3(\frac{a_1a_2-1}{a_2-1} + \frac{a_1a_2-1}{a_1-1}) + 36$.

4 Proof of Theorem A

Proof Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be classified into the following five types:

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}x_2$,
Type 5. $x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}$.

It is easy to see from Propositions 2.3, 2.4, 2.5, 2.6, 2.7, 3.4, 3.5, 3.6, 3.7 and 3.8, the conjecture $\lambda^{(k+1)}(V) > \lambda^k(V)$, $k = 1$ holds. Hence Theorem A is proved. □

5 Proof of Theorem B

Proof Let $f \in \mathbb{C}\{x_1, x_2\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be classified into the following three types:

Type A. $x_1^{a_1} + x_2^{a_2}$,
Type B. $x_1^{a_1}x_2 + x_2^{a_2}$,
Type C. $x_1^{a_1}x_2 + x_2^{a_2}x_1$.

It is easy to see from Propositions 3.1, 3.2 and 3.3, the conjecture $\lambda^2(V) \leq h_2(\frac{1}{w_1}, \frac{1}{w_2})$, holds. Hence Theorem B is proved. □

6 Proof of Theorem C

Proof Let $f \in \mathbb{C}\{x_1, x_2, x_3\}$ be a weighted homogeneous fewnomial isolated singularity. Then f can be classified into the following five types:

Type 1. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$,
Type 2. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3}$,
Type 3. $x_1^{a_1} x_2 + x_2^{a_2} x_3 + x_3^{a_3} x_1$,
Type 4. $x_1^{a_1} + x_2^{a_2} + x_3^{a_3} x_2$,
Type 5. $x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^{a_3}$.

It is easy to see from Propositions 3.4, 3.5, 3.6, 3.7 and 3.8, the conjecture $\lambda^2(V) \leq h_2(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3})$, holds. Hence Theorem C is proved. □

7 Proof of Theorem D

Proof It is easy to see from Propositions 3.9, 3.10 and 3.11, the following three cases:

(i) $x_1^{a_1} + x_2^{a_2} + x_3^2;\ a_1, a_2 \geq 3$,
(ii) $x_1^{a_1} x_2 + x_2^{a_2} + x_3^2;\ a_1 \geq 2, a_2 \geq 3$,
(iii) $x_1^{a_1} x_2 + x_2^{a_2} x_1 + x_3^2;\ a_1, a_2 \geq 2$,

satisfy the conjecture $\lambda^2(V) \leq h_2(\frac{1}{w_1}, \frac{1}{w_2}, 2)$. Hence Theorem D is proved. □

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