Geometriae Dedicata (2020) 204:365–375
https://doi.org/10.1007/s10711-019-00460-2

ORIGINAL PAPER

Integral geometric inequalities and valuations

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Received: 20 June 2018 / Accepted: 23 May 2019 / Published online: 8 June 2019

Abstract
General integral geometric invariants for convex bodies are introduced and two integral geometric inequalities for them are established. The equality cases for the inequalities are kinematic formulas, which are characterizations of integral geometric valuations. Those characterizations are analogues of Hadwiger's characterization theorem.

Keywords Integral geometry · Valuation · Hadwiger's Characterization theorem · Kinematic formula · Integral geometric inequalities

2010 Mathematics Subject Classification 52A22 · 52A20

1 Introduction

The *principal kinematic formula* is one of cornerstones in the classical integral geometry, which is now associated with the names of Blaschke, Santaló and Chern. It deals with integral mean values for distinguished geometric functionals with respect to the invariant measure on the group of proper rigid motions in the Euclidean space $\mathbb{R}^n$. When restricted to convex bodies, that is, compact convex sets with nonempty interiors, it involves the intrinsic volumes V_j $(j \in \{0, \ldots, n\})$. To be more specific, let G_n denote the motion group of $\mathbb{R}^n$, and let dg be the invariant measure of G_n whose restriction to the rotation group is the invariant probability measure and the restriction to the translation group is the Lebegue measure. Then the principal kinematic formula says that (see [16])

$$\int_{G_n} V_j(K \cap gM)dg = \sum_{k=j}^{n} c_{k,j} V_k(K) V_{n-k+j}(M), \tag{1.1}$$

The author was supported in part by the Fundamental Research Funds for the Central Universities (No. XDJK2017C056), Ph. D. Research Fundation of Southwest University (No. SWU115053), the Natural Science Foundation Project of CQ CSTC (No. cstc2016jcyjA0465) and NSFC (Nos. 11401486 and 11561012).

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where K, M are in $\mathcal{K}$, the class of convex bodies in $\mathbb{R}^n$, and the constant

$$c_{k,j} = \frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n-k+j+1}{2}\right)\Gamma\left(\frac{j+1}{2}\right)}$$

is expressed in terms of specific values of the Gamma function.

Related to the kinematic formula is the *Crofton formula*, which involves an integration over $\tilde{G}_{k,n}$, the affine Grassmann manifold of k-dimensional planes in $\mathbb{R}^n$. Then for $K \in \mathcal{K}$ (see [16])

$$\int_{\tilde{G}_{k,n}} V_j(K \cap \xi_k) d\xi_k = c_{k,j} V_{n-k+j}(K), \tag{1.2}$$

where $k \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, k\}$, $d\xi_k$ of $\tilde{G}_{k,n}$ is normalized so that ω_{n-k}, the volume of the $(n-k)$-dimensional unit ball, is the measure of the set of k-dimensional planes hitting the unit ball B in $\mathbb{R}^n$.

For the principal kinematic formula and the Crofton formula, there are different versions, such as differential-geometric versions [15], versions for support and curvature measures [17].

The following important formula is closely related to (1.2), which is involved in the integrals of square of volumes of higher dimensional sections of convex bodies, and was shown by Blaschke and Varga [3] in $\mathbb{R}^3$ and by Zhang [21] in $\mathbb{R}^n$

$$\int_{\tilde{G}_{k,n}} V_k(K \cap \xi_k)^2 d\xi_k = \frac{\omega_n}{k+1} \int_{\tilde{G}_{1,n}} V_1(K \cap \xi_1)^{k+1} d\xi_1, \tag{1.3}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, ξ_1 denotes a random line and $V_k(K \cap \xi_k)$ denotes the k-dimensional volume of the intersection $K \cap \xi_k$.

The right-hand side of (1.3) leads us to the frequently studied random chords of convex bodies, which are known as *chord power integrals* with the definition

$$I_p(K) = \frac{2\alpha_{n-1}}{n} \int_{\tilde{G}_{1,n}} V_1(K \cap \xi_1)^p d\xi_1, \quad p \geq 0. \tag{1.4}$$

The chord power integrals are fundamental geometric invariants. They are generations of the surface area $S(K)$ and the volume $V(K)$ of convex body K (see [21]). The inequalities of chord power integrals are important topics in convex geometry which imply the relationship among some important geometric invariants. Let K be a convex body of fixed volume in $\mathbb{R}^n$, then for $1 < p < n+1$, the ball maximizes the chord power integrals, that is,

$$I_p(K) \leq b_p V(K)^{(n+p-1)/n}, \tag{1.5}$$

while for $0 \leq p < 1$ or $p > n+1$, the ball minimizes the chord power integrals, that is,

$$I_p(K) \geq b_p V(K)^{(n+p-1)/n}, \tag{1.6}$$

where b_p is a sharp constant which can be computed as $I_p(B)/\omega_n^{(n+p-1)/n}$, where B is the unit ball. Each equality holds if and only if K is a ball.

The classical isoperimetric inequality involving the surface area and the volume of K in $\mathbb{R}^n$ is a special case $p = 0$ of (1.6). When p is the positive integer, then (1.5) and (1.6) are due to Ren [14]. Zhang [18] generalized the inequalities to any positive real number p.

Making the critical use of chord power integrals, Zhang [19] established the reverse Petty projection inequality, which is now known as the *Zhang projection inequality* (see [16]). In [21], Zhang established dual kinematic formulas for chord power integrals by using the dual quermassintegrals which generalized the famous Crofton-Hadwiger formula. The rapid developments of these integral formulas are motivated by their wide applications in stochastic geometry [17], geometric probabilities [15] and projection functions [17]. See [4,18,20] for more applications.

The theory of valuations on convex sets, with traditionally strong relations to integral geometry, has been an active and prominent part of mathematics (see [5,9]). A real function $\varphi : \mathcal{K} \to \mathbb{R}$ is called a valuation if

$$\varphi(K) + \varphi(L) = \varphi(K \cup L) + \varphi(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{K}$. Probably the most famous result on valuations is the following Hadwiger's characterization theorem.

Hadwiger's characterization theorem If the function $\varphi : \mathcal{K} \to \mathbb{R}$ is a valuation, continuous, and invariant under rigid motions, then

$$\varphi(K) = c_0 V_0(K) + c_1 V_1(K) + \cdots + c_n V_n(K) \tag{1.7}$$

with constants $c_0, \ldots, c_n$.

The first proof of Hadwiger's characterization theorem has been given in [7], then a brief one was presented by Klain [11]. Hadwiger's characterization theorem was the starting point for many results in the modern theory of valuations. For example, Alesker established a complete classification of continuous and merely translation invariant valuations, which laid the foundation for a new theory of algebraic integral geometry (see [1,2]). Ludwig and Reitzner [12] established an affine version of Hadwiger's characterization theorem, and they also established a complete classification of $SL(n)$ invariant valuations and asked for a centro-affine version in a landmark work [13], which was completely solved by Haberl and Parapatits [6]. Besides, Hadwiger's characterization theorem and its generalizations lead to effortless proofs of numerous results in integral geometry, including various kinematic formulas and the mean projection formulas for convex bodies (see [8,9,15–17]).

The intrinsic volumes are valuations which induce the following important integral geometric invariants mentioned above

$$\int_{\tilde{G}_{k,n}} V_j(K \cap \xi_k) d\xi_k \quad \text{and} \quad \int_{G_n} V_j(K \cap gM) dg. \tag{1.8}$$

They are special integral geometric valuations, which can be formulated by the linear combination of the intrinsic volumes from Hadwiger's characterization theorem.

Motivated by the fact that (1.8) are valuations, it is very interesting to know whether the following intrinsic volumes power integrals are valuations or not

$$\int_{\tilde{G}_{k,n}} V_j(K \cap \xi_k)^p d\xi_k \quad \text{and} \quad \int_{G_n} V_j(K \cap gM)^p dg, \quad p \geq 0. \tag{1.9}$$

The classical integral geometry mainly studies kinematic formulas and valuations, and they rarely involve inequality. However, (1.3), (1.4) and (1.5) yield the inequality

$$\int_{\tilde{G}_{k,n}} V_k(K \cap \xi_k)^2 d\xi_k \leq \tilde{b}_{k+1} V(K)^{(n+k)/n}, \tag{1.10}$$

where $\tilde{b}_{k+1} = b_{k+1}/(2k+2)$ is a constant, with equality if and only if K is a ball.

(1.10) is an important inequality with the integrals of square of the volume of intersections which inspires us to consider the inequality with the integrals of higher power of intrinsic volumes of intersections, such as the integrals of p-th power of intrinsic volumes in (1.9), or we consider the following more general integral geometric invariants. That is, suppose K, M are convex bodies in $\mathbb{R}^n$ and function $f : [0, \infty) \longrightarrow [0, \infty)$ is right continuous at 0, then we define $I_{k,j}(f; K)$ and $I_j(f; K, M)$ as

$$I_{k,j}(f; K) = \int_{\tilde{G}_{k,n}} f\left(V_j(K \cap \xi_k)\right) d\xi_k, \tag{1.11}$$

and

$$I_j(f; K, M) = \int_{G_n} f\left(V_j(K \cap gM)\right) dg, \tag{1.12}$$

where $k \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, k\}$.

It is impossible to obtain general kinematic formulas for $I_{k,j}(f; K)$ and $I_j(f; K, M)$. But there are probably integral geometric inequalities for them. Furthermore, a natural question to ask is

Problem *Are the general integral geometric invariants $I_{k,j}(f; K)$ and $I_j(f; K, M)$, respectively, are valuations? How are these general integral geometric invariants related to fundamental invariants V_j and kinematic formulas?*

Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex or concave and right continuous at 0, for $K, M \in \mathcal{K}$, we then prove that $I_{k,j}(f; K)$ and $I_j(f; K, M)$ are both valuations if and only if $f(x)$ is linear in an interval. It follows immediately that the chord power integrals $I_p(K)$ are valuations if and only if $p = 0, 1$.

Then the following theorems give some answers to the Problem, each of them contains integral geometric inequality, kinematic formula and integral geometric valuation characterization.

Theorem 1.1 *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, and let K be a convex body in $\mathbb{R}^n$ and $m_1 = \max\{V_j(K \cap \xi_k) : \xi_k \in \tilde{G}_{k,n}\}$. Then*

$$I_{k,j}(f; K) \geq c_{k,j} f'(0) V_{n-k+j}(K) + c_{k,0} f(0) V_{n-k}(K), \tag{1.13}$$

with equality if and only if $I_{k,j}(f; K)$ is a valuation, in this case, if and only if $f(x)$ is linear in the interval $[0, m_1]$.

Theorem 1.2 *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, and let K, M be convex bodies in $\mathbb{R}^n$ and $m_2 = \max\{V_j(K \cap gM) : g \in G_n\}$. Then*

$$I_j(f; K, M) \geq \sum_{k=j}^{n} c_{k,j} f'(0) V_k(K) V_{n-k+j}(M) + \sum_{k=0}^{n} c_{k,0} f(0) V_k(K) V_{n-k}(M), \tag{1.14}$$

with equality if and only if $I_j(f; K, M)$ is a valuation in the variable M, in this case, if and only if $f(x)$ is linear in the interval $[0, m_2]$.

It should be noticed that the classical kinematic formulas are the equality cases in Theorems 1.1 and 1.2. To be more specific, the Crofton formula (1.2) and its special case $j = 0$ characterize the integral geometric valuation $I_{k,j}(f; K)$, and the principal kinematic formula (1.1) and its special case $j = 0$ characterize the integral geometric valuation $I_j(f; K, M)$. Those characterizations are analogues of Hadwiger's characterization theorem with certain coefficients, which can not be induced directly by Hadwiger's characterization theorem.

If the function $f : [0, \infty) \rightarrow [0, \infty)$ is concave and right continuous at 0, then (1.13) and (1.14) hold with the inequalities sign reversed. However, for general function f in (1.11) and (1.12), under what conditions $I_{k,j}(f; K)$ and $I_j(f; K, M)$ are valuations and their specific forms of Hadwiger's characterization theorem are unknown.

2 Preliminaries

As a rule, let B be the unit ball in $\mathbb{R}^n$ and its boundary is denoted by S^{n-1}. We write ω_n and α_{n-1} for the volume of B and the surface area of S^{n-1} in $\mathbb{R}^n$, respectively, with the representation

$$\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)},$$

where $\Gamma(\cdot)$ is the Gamma function, and $\alpha_{n-1} = n\omega_n$.

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interiors. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}$. For $K \in \mathcal{K}$, we denote by $V(K)$ the volume and by $S(K)$ the surface area of K. If K is a Borel subset of $\mathbb{R}^n$ and it is contained in an k-dimensional affine subspace of $\mathbb{R}^n$ but in no affine subspace of lower dimensional, the $V_k(K)$ will denote the k-dimensional Lebesgue measure of K. In this paper, for $K, L \in \mathcal{K}$, we always assume that $K \cup L \in \mathcal{K}$.

If K, L are compact convex sets in $\mathbb{R}^n$ and $\lambda \geq 0$, the *Minkowski sum* $K + \lambda L$ is defined by

$$K + \lambda L = \{x + \lambda y : x \in K \text{ and } y \in L\}.$$

For $\varepsilon \geq 0$, the volume $V(K + \varepsilon B)$ is given by *Steiner formula*

$$V(K + \varepsilon B) = \sum_{j=0}^{n} \omega_j V_{n-j}(K)\varepsilon^j, \quad \varepsilon \geq 0.$$

For $j \in \{0, \dots, n\}$, the coefficients $V_j(K)$ depend only on K and are called *j-th intrinsic volumes* of K. In particular, $V_0(K)$ is the *Euler characteristic* (that is, $V_0(K) = 1$ for $K \neq \emptyset$ and $V_0(\emptyset) = 0$), and

$$V_{n-1}(K) = \frac{1}{2}S(K), \quad V_n(K) = V(K),$$

and $V_j(K)$ is homogeneous of degree j, that is, for $\alpha > 0$,

$$V_j(\alpha K) = \alpha^j V_j(K).$$

The intrinsic volumes have the monotope property, for $K \subseteq L$, then

$$V_j(K) \leq V_j(L). \tag{2.1}$$

We need the following kinematic formulas, the special cases $j = 0$ of (1.1) and (1.2), respectively, (see [15,16])

$$\int_{G_n} \chi(K \cap gM)dg = \sum_{k=0}^{n} c_{k,0} V_k(K) V_{n-k}(M), \tag{2.2}$$

and

$$\int_{\tilde{G}_{k,n}} \chi(K \cap \xi_k)d\xi_k = c_{k,0} V_{n-k}(K). \tag{2.3}$$

The formula (2.2) is called the *fundamental kinematic formula*. Its differential-geometric version is due to Chern (see [15]) and its dual form is estabished by Zhang [21].

The investigation of convex sets is closely tied up with convex functions. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *convex* if

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda) f(x_1) + \lambda f(x_2),$$

for $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0, 1]$ and with equality if and only if all x are equal or $f(x)$ is linear in an interval including all the x. (see [10, p. 75]). A function f is *concave* if $-f$ is convex. Here we say a *linear function* if it is of the form

$$f(x) = ax + b,$$

where a and b are constants, and a is frequently referred to as the slope of the line.

3 Two auxiliary inequalities

In this section we prove two elemental inequalities that will be needed in the following sections. Throughout the paper, we always assume $f(0) \in \mathbb{R}$.

Lemma 3.1 *Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be convex and right continuous at 0, then for real numbers $a, b, c \geq 0$ and $c \leq \min\{a, b\}$, we have*

$$f(a + b - c) + f(c) \geq f(a) + f(b), \tag{3.1}$$

with equality if and only if $f(x)$ is linear in an interval including all the x or $a = b = c$.

Proof For convenience, we always asume that $a \leq b$. From the assumption we have $c \leq a \leq b \leq a + b - c$. There exists $\lambda \in [0, 1]$ such that

$$a = \lambda c + (1-\lambda)b.$$

The previous formula can be written as

$$b = (1-\lambda)a + \lambda(a + b - c).$$

By the convexity of f we have

$$\begin{aligned} f(a) &= f(\lambda c + (1-\lambda)b) \\ &\leq \lambda f(c) + (1-\lambda) f(b) \\ &= \lambda f(c) - \lambda f(b) + f((1-\lambda)a + \lambda(a + b - c)) \\ &\leq \lambda f(c) - \lambda f(b) + (1-\lambda) f(a) + \lambda f(a + b - c). \end{aligned}$$

This yields the desired.

If $a = b = c$ or $f(x)$ is a linear function in an interval including all the x, it is easy to see that the equality in (3.1) holds. Conversely, if the equality in (3.1) holds, then we have $a = b = c$, or $f(x)$ is a linear function in an interval including all the x which follows from the equality condition of convex function. □

Lemma 3.2 *Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be convex and right continuous at 0, then*

$$f(x) \geq f'(0)\, x + f(0), \tag{3.2}$$

with inequality if and only if $f(x)$ is linear in an interval including all the x.

Proof Let $x, y \in (0, \infty)$ and $y \leq x$, by the convexity of f, we have

$$f(y) = f\left(\frac{x-y}{x}\,0 + \frac{y}{x}\,x\right) \leq \frac{x-y}{x} f(0) + \frac{y}{x} f(x), \tag{3.3}$$

hence

$$\frac{f(y) - f(0)}{y} \leq \frac{f(x) - f(0)}{x}.$$

Let $y \longrightarrow 0$ and then

$$f'(0) \leq \frac{f(x) - f(0)}{x},$$

for $x \in (0, \infty)$. That's the desired inequality.

With equality holds in (3.2) if and only if the equality holds in (3.3), then $f(x)$ must be a linear function in an interval including all the x which follows from the equality condition of convex function. □

4 Valuation and inequality for $I_{k,j}(f; K)$

Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, then a functional $I_{k,j}(f; \cdot) : \mathcal{K} \to \mathbb{R}$ is given by

$$I_{k,j}(f; K) = \int_{\tilde{G}_{k,n}} f(V_j(K \cap \xi_k))d\xi_k, \tag{4.1}$$

where $k \in \{0, \ldots, n\}$, $j \in \{0, \ldots, k\}$ and $K \in \mathcal{K}$. Recall that $V_k(K \cap \xi_k)$ and $V_0(K \cap \xi_k)$ are the k-dimensional volume and Euler characteristic of $K \cap \xi_k$, respectively.

Let

$$m_1 = \max\{V_j(K \cap \xi_k) : K \in \mathcal{K}, \xi_k \in \tilde{G}_{k,n}\}.$$

Then we have the following theorems.

Theorem 4.1 *Let $K \in \mathcal{K}$ and $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at* 0*, then $I_{k,j}(f; K)$ is a valuation if and only if $f(x)$ is linear on* $[0, m_1]$.

Proof Let $K, L \in \mathcal{K}$. Recall that $K \cup L \in \mathcal{K}$, then $K \cap \xi_k$, $L \cap \xi_k$, $(K \cap L) \cap \xi_k$, $(K \cup L) \cap \xi_k$ are convex bodies in ξ_k. By the fact that the intrinsic volumes are valuations in ξ_k, we have

$$V_j\left((K \cup L) \cap \xi_k\right) + V_j\left((K \cap L) \cap \xi_k\right) = V_j\left(K \cap \xi_k\right) + V_j\left(L \cap \xi_k\right). \tag{4.2}$$

Since $((K \cap L) \cap \xi_k) \subseteq (K \cap \xi_k)$ and $((K \cap L) \cap \xi_k) \subseteq (L \cap \xi_k)$, by (2.1) we get

$$V_j\left((K \cap L) \cap \xi_k\right) \leq \min\{V_j\left(K \cap \xi_k\right), V_j\left(L \cap \xi_k\right)\},$$

with equality if and only if $K \subseteq L$ or $L \subseteq K$.

From Lemma 3.1 we have

$$\begin{aligned} &f\left(V_j\left(K \cap \xi_k\right) + V_j\left(L \cap \xi_k\right) - V_j\left((K \cap L) \cap \xi_k\right)\right) \\ &\quad + f\left(V_j\left((K \cap L) \cap \xi_k\right)\right) \geq f\left(V_j\left(K \cap \xi_k\right)\right) + f\left(V_j\left(L \cap \xi_k\right)\right). \end{aligned}$$

By (4.2), the previous formula can be rewritten as

$$\begin{aligned} &f\left(V_j\left((K \cup L) \cap \xi_k\right)\right) + f\left(V_j\left((K \cap L) \cap \xi_k\right)\right) \\ &\geq f\left(V_j\left(K \cap \xi_k\right)\right) + f\left(V_j\left(L \cap \xi_k\right)\right). \end{aligned}$$

Via the definition of $I_{k,j}(f; K)$ we have

$$I_{k,j}(f; K \cup L) + I_{k,j}(f; K \cap L) \geq I_{k,j}(f; K) + I_{k,j}(f; L).$$

From equality condition in Lemma 3.1 we have

$$I_{k,j}(f; K \cup L) + I_{k,j}(f; K \cap L) = I_{k,j}(f; K) + I_{k,j}(f; L),$$

if and only if $f(x)$ is a linear function on $[0, m_1]$. □

Let $f(t) = t^p$ $(p \geq 0)$ in (4.1), then

$$I_{k,j;p}(K) = \int_{\tilde{G}_{k,n}} V_j(K \cap \xi_k)^p d\xi_k, \quad 0 \leq p < \infty. \tag{4.3}$$

From Theorem 4.1, we get the following corollary immediately.

Corollary 4.2 *For $K \in \mathcal{K}$, then $I_{k,j;p}(K)$ is a valuation if and only if $p = 0, 1$.*

Finally, we get an analog of Hadwiger characterization theorem.

Theorem 4.3 *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, for $K \in \mathcal{K}$ we have*

$$I_{k,j}(f; K) \geq c_{k,j} f'(0) V_{n-k+j}(K) + c_{k,0} f(0) V_{n-k}(K), \tag{4.4}$$

with equality if and only if $I_{k,j}(f; K)$ is a valuation, in this case, if and only if $f(x)$ is linear in the interval $[0, m_1]$.

Proof From $f : [0, \infty) \longrightarrow [0, \infty)$ is convex and right continuous at 0, let $x = V_j(K \cap \xi_k)$ in Lemma 3.2 we then have

$$f(V_j(K \cap \xi_k)) \geq f'(0) V_j(K \cap \xi_k) + f(0). \tag{4.5}$$

This along with the definition (4.1), we have

$$\begin{aligned} I_{k,j}(f; K) &= \int_{\tilde{G}_{k,n}} f(V_j(K \cap \xi_k)) d\xi_k \\ &\geq f'(0) \int_{\tilde{G}_{k,n}} V_j(K \cap \xi_k) d\xi_k + f(0) \int_{\tilde{G}_{k,n}} \chi(K \cap \xi_k) d\xi_k. \end{aligned}$$

By (1.2) and (2.3) we thus get

$$I_{k,j}(f; K) \geq f'(0) c_{k,j} V_{n-k+j}(K) + f(0) c_{k,0} V_{n-k}(K).$$

The equality condition in (4.4) is equivalent to that in (4.5), that is, $f(x)$ is a linear function on $[0, m_1]$. From Theorem 4.1 we know that $I_{k,j}(f; K)$ must be a valuation. □

5 Valuation and inequality for $I_j(f; K, M)$

Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, then we fix a convex body $K \in \mathcal{K}$ and define a functional by

$$I_j(f; K, M) = \int_{G_n} f\left(V_j(K \cap gM)\right) dg, \tag{5.1}$$

for $j \in \{0, \ldots, n\}$ and $M \in \mathcal{K}$.

Let

$$m_2 = \max\{V_j(K \cap gM) : K, M \in \mathcal{K}, g \in G_n\}.$$

Theorem 5.1 *Let $K, M \in \mathcal{K}$ and $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, then $I_j(f; K, M)$ is a valuation in the variable M if and only if $f(x)$ is linear on $[0, m_2]$.*

Proof For convex bodies $K, L \in \mathcal{K}$ such that $K \cup L \in \mathcal{K}$, then we have [16, p. 140]

$$K + L = (K \cup L) + (K \cap L). \tag{5.2}$$

Then, for $K, M, N \in \mathcal{K}$ and $M \cup N \in \mathcal{K}$, (5.2) implies

$$\begin{aligned}(K \cap M) + (K \cap N) &= ((K \cap M) \cup (K \cap N)) + ((K \cap M) \cap (K \cap N)) \\ &= (K \cap (M \cup N)) + (K \cap (M \cap N)).\end{aligned}$$

Since the intrinsic volumes are valuations, then

$$V_j(K \cap M) + V_j(K \cap N) = V_j(K \cap (M \cup N)) + V_j(K \cap (M \cap N)).$$

This leads to

$$V_j(K \cap (M \cap N)) \leq \min\{V_j(K \cap M), V_j(K \cap N)\}.$$

By Lemma 3.1 again we get

$$\begin{aligned}&f\left(V_j(K \cap (M \cup N))\right) + f\left(V_j(K \cap (M \cap N))\right) \\ &\quad \geq f\left(V_j(K \cap M)\right) + f\left(V_j(K \cap N)\right).\end{aligned}$$

Integrating both sides over G_n, we have

$$\begin{aligned}&\int_{G_n} f\left(V_j(K \cap g(M \cup N))\right) dg + \int_{G_n} f\left(V_j(K \cap g(M \cap N))\right) dg \\ &\quad \geq \int_{G_n} f\left(V_j(K \cap gM)\right) dg + \int_{G_n} f\left(V_j(K \cap gN)\right) dg.\end{aligned}$$

That is

$$I_j(f; K, M \cup N) + I_j(f; K, M \cap N) \geq I_j(f; K, M) + I_j(f; K, N).$$

Therefore, by equality condition in Lemma 3.1 we get

$$I_j(f; K, M \cup N) + I_j(f; K, M \cap N) = I_j(f; K, M) + I_j(f; K, N),$$

only and if only $f(x)$ is linear on $[0, m_2]$. □

Let $f(t) = t^p$ $(p \geq 0)$ in (4.1), then

$$I_{j;p}(K; M) = \int_{G_n} V_j(K \cap gM)^p dg, \quad 0 \leq p < \infty, \tag{5.3}$$

From Theorem 5.1, we get the following corollary.

Corollary 5.2 *For $K, M \in \mathcal{K}$, then $I_{j;p}(K; M)$ be a valuation in the variable M if and only if $p = 0, 1$.*

Theorem 5.3 *Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at* 0*, for $K, M \in \mathcal{K}$ we have*

$$I_j(f; K, M) \geq \sum_{k=j}^{n} c_{k,j} f'(0) V_k(K) V_{n-k+j}(M) + \sum_{k=0}^{n} c_{k,0} f(0) V_k(K) V_{n-k}(M),$$

with equality if and only if $I_j(f; K, M)$ is a valuation in the variable M, in this case, if and only if $f(x)$ is linear in the interval $[0, m_2]$.

Proof Let $f : [0, \infty) \longrightarrow [0, \infty)$ be convex and right continuous at 0, and let $x = V_j(K \cap gM)$ in Lemma 3.2, we have

$$f(V_j(K \cap gM)) \geq f'(0) V_j(K \cap gM) + f(0).$$

By the definition (5.1), we have

$$\begin{aligned} I_j(f; K, M) &= \int_{G_n} f\left(V_j(K \cap gM)\right) dg \\ &\geq f'(0) \int_{G_n} V_j(K \cap gM) dg + f(0) \int_{G_n} \chi(K \cap gM) dg. \end{aligned}$$

The previous formula combines with (1.1) and (2.2), we have

$$I_j(f; K, M) \geq f'(0) \sum_{k=j}^{n} c_{k,j} V_k(K) V_{n-k+j}(M) + f(0) \sum_{k=0}^{n} c_{k,0} V_k(K) V_{n-k}(M).$$

The equality condition follows from Lemma 3.2 and Theorem 5.1. □

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