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On the fundamental 3-classes of knot group representations

Takefumi Nosaka[1] (iD)

Abstract

We discuss the fundamental (relative) 3-classes of knots (or hyperbolic links), and provide diagrammatic descriptions of the push-forwards with respect to link-group representations. The point is an observation of a bridge between the relative group homology and quandle homology from the viewpoints of Inoue–Kabaya map (Geom Dedicata 171(1):265–292, 2014). Furthermore, we give an algorithm to algebraically describe the fundamental 3-class of any hyperbolic knot.

Keywords Knot · Relative group homology · Hyperbolicity · Malnormal subgroups · Quandle

Contents

✉ Takefumi Nosaka
  nosaka@math.titech.ac.jp

[1] Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-Ku, Tokyo 152-8551, Japan

1 Introduction

In the study of an oriented compact 3-manifold M for which each boundary component is a torus, for example a link complement, the (relative) fundamental homology 3-class $[M, \partial M]$ in $H_3(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ has essential information. The 3-class is analyzed quantitatively by examining the following situation. We suppose a pair of groups $K \subset G$ and a homomorphism $f : \pi_1(M) \to G$, which sends every boundary-element in $\pi_1(M)$ to some element in K. Then, for any relative group 3-cocycle $\theta \in H^3(G, K; A)$ with local coefficients [see (10) for the explicit definition], we can consider the following pairing valued in the coinvariant $A_G = H_0(G; A)$:

$$\langle \theta, \ f_*[M, \ \partial M] \rangle \in A_G = A/\{a - g \cdot a\}_{a \in A, \ g \in G}. \tag{1}$$

These settings appear in topics in low dimensional topology. For example, the volumes and the Chern–Simons invariants of hyperbolic manifolds can be described as this pairing, where $G = SL_N(\mathbb{C})$ (see [12,23,32]). Furthermore, this pairing includes the triple cup products of the form $\theta = a \smile b \smile c$; see [22,27]. In addition, if G is of finite order, the pairing is called *the Dijkgraaf–Witten invariant* [10], as a toy model of TQFT.

However, the pairing defined in the general situation is often considered to be uncomputable. Actually, we come up against difficulties: First, it is troublesome to explicitly describe a (truncated) triangulation in M, which represents the 3-class $[M, \partial M]$. Moreover, the 3-class $f_*[M, \partial M]$ is not always unique, but depends on the choices of $2^{|\pi_0(\partial M)|}$ decorations (when several boundary components are present), as mentioned in [32, § 5]. Next, the boundary condition is important; when dealing with the condition, we mostly need long and verbose explanations, as in [10,18,21,23,32] (cf. Homotopy quantum field theory [30]). In addition, since the relative homology is defined from some projective resolution (see Sect. 3), it is essentially a critical problem to choose an appropriate resolution and to find a presentation of the 3-cocycle θ. Further, even if we can succeed in doing so on θ, such presentations are quite intricate.

Nevertheless, this paper develops a diagrammatic computation of the pairing, according to geometric structures of links. Precisely, let $L \subset S^3$ be an oriented link in the 3-sphere, and $M = E_L$ be the 3-manifold which is obtained from S^3 by removing an open tubular neighborhood of L, i.e., $E_L = S^3 \setminus \nu L$. When the fundamental group of E_L is malnormal (which is the case for a broad class of link, e.g. hyperbolic), we compute the pairing (Theorem 2.1). In the cable cases, additional conditions are needed; see Sect. 7, cf. cabling formula [16]. As seen in Sect. 5, the result is summarized to that, if we know the presentation of θ and the JSJ decomposition of L, we can compute the pairing from a diagram. Here, the point is that the construction needs no triangulation of E_L.

Let us roughly explain our approach to the theorem. As seen in [8,17,24,26], quandle theory [19] and homology [8] have advantages to some diagrammatic computation in knot theory. Thus, inspired by the works [3,17], we will construct a bridge between the quandle and group homologies using chain maps, in order to reduce the 3-class $[M, \partial M]$ to a quandle 3-class. However, for technical reasons that arise from the scissors congruence formulas of [9,23], the bridge between these ideas factors through Hochschild relative homology [15]. It is expressed as a zigzag sequence [Expression (5) of Sect. 3]. In Theorem 2.3, we show that the malnormal property is a suitable condition for obtaining a quasi-inverse in the zigzag. In Sect. 5, this case is computed based upon the work of [3,13,29], which studied malnormal property of knots. In summary, compositions of chain maps gives the computation of the pairing.

In applications, we obtain four advantages from the approaches as follows. First, the above composite gives an algorithm to describe algebraically the fundamental 3-class $[M, \partial M]$ for hyperbolic links; see Sect. 5.2. Next, our results emphasize topological advantages of the quandle cocycle invariant [8]. Especially, for malnormal pairs (G, K), we will give a method to produce many quandle cocycles, and obtain a simple formulation of computing the pairing (see Theorem 2.3). The third is a result of determining the third homology of the link quandle Q_L, where L is a knot or a hyperbolic link. This quandle Q_L which is defined in [19] is analogous to the fundamental group of $S^3 \backslash L$. It plays a key role in the proof of the main theorems; details appear in "Appendix A". The fourth one is that our theorem is a generalization and application of the work [17]. To be precise, while the paper [17] showed the same theorem for only $G = SL_2(\mathbb{C})$ and hyperbolic links, our theorem points out the generalization applicable to groups $K \subset G$ with malnormality.

This paper is organized as follows. Section 2 states the theorems. Section 3 introduces relative group homology, and Sect. 4 reviews the quandle homology [8] and Inoue–Kabaya chain map [17]. Section 5 explains the algorithm to describe $[M, \partial M]$, and gives an example from the figure eight knot. Section 6 proves the main theorem, and Sect. 7 discusses cable knots. "Appendix A" computes the third homology of the link quandles Q_L for some links.

2 Statements; the main results

This section states the main results. For this, we fix terminology throughout this paper.
Conventional notation and assumption throughout this paper.

- By a link, L, we mean a C^∞-embedding of solid tori into the 3-sphere S^3 or into the solid torus $D^2 \times S^1$. We suppose an orientation of L, and denote $\pi_0(L)$ by $\#L \in \mathbb{Z}$.
- For short, the fundamentals group $\pi_1(S^3 \backslash L)$ is abbreviated to π_L, and the complement, $S^3 \backslash L$, is often denoted E_L
- For each $\ell \leq \#L$, we fix a meridian-longitude pair $(\mathfrak{m}_\ell, \mathfrak{l}_\ell) \in \pi_L$, and denote by $\mathcal{P}_\ell$ the subgroup generated by $(\mathfrak{m}_\ell, \mathfrak{l}_\ell) \in \pi_L$, which is called *a peripheral group*.
- We fix a group G and subgroups K_ℓ with $\ell \leq \#L$, and suppose a homomorphism $f : \pi_1(S^3 \backslash L) \to G$ such that $f(\mathcal{P}_\ell) \subset K_\ell$.
- By A we mean a left $\mathbb{Z}[G]$-module. The coinvariant $A/\{a - g \cdot a\}_{a \in A, g \in G}$ is denoted by A_G.

In this situation, although it seems easy to define a pushforward of the 3-class $f_*([E_L, \partial E_L])$ in the relative group homology $H_3(G, K_1, \ldots, K_{\#L}; \mathbb{Z})$, it is known (see [32, §5] or [3, §10]) that a canonical definition of such pushforwards depends on the choice of "f-decorations". However, if E_L is decomposed as a union of complete hyperbolic 3-manifolds, the 3-class $f_*([E_L, \partial E_L])$ is known to be well-defined (see [32, §5]). Therefore, similarly to (1), we can consider the pairing between this 3-class and a 3-cocycle of G relative to $(K_\ell)_{\ell \leq \#L}$.

2.1 The first statement

We will set up some terminology and state Theorem 2.1. Let D denote a diagram of the link L. Its sets of arcs Arc_D and regions Reg_D are defined as usual. Given a map $\phi : \mathrm{Reg}_D \times \mathrm{Arc}_D \times \mathrm{Arc}_D \to A$, let us consider a weight sum of the form

Fig. 1 Positive and negative crossings with labeled regions and labeled arc

$$\Psi_\phi(D) := \sum_\tau \epsilon_\tau \cdot \phi(x_\tau, y_\tau, z_\tau) \in A \tag{2}$$

running over all the crossings τ of D, where x_τ, y_τ, and z_τ are the region and the arcs shown in Fig. 1, and $\epsilon_\tau \in \{\pm 1\}$ is the sign of τ. Then, the main statement is as follows:

Theorem 2.1 (See Sect. 6 for the proof) *Assume that L is either a prime knot which is not cable [1] or a hyperbolic link. Then, for any relative group 3-cocycle $\theta \in H^3(G; K_1, \ldots, K_{\#L}; A)$, there is a map*

$$\phi_\theta : \mathrm{Reg}_D \times \mathrm{Arc}_D \times \mathrm{Arc}_D \longrightarrow A,$$

for which the following equality holds in the coinvariant A_G:

$$\langle \theta, f_*[E_L, \partial E_L] \rangle = \Psi_{\phi_\theta}(D) \in A_G. \tag{3}$$

The right hand side gives a method for computing the pairing that does not depend upon a triangulation. Here, it is important to express ϕ_θ concretely; in Sects. 5 and 6, we give a concrete expression of ϕ_θ for such links, where the expression is based on the proof of the theorem. However, the presentation essentially depends on the link type of L, and is not always simple. In fact, even for the figure eight knot L, the map ϕ_θ forms a sum of many terms; see Sect. 5.2.

Section 7 discusses the cable cases, and includes a similar statement (Theorem 7.1). Here, we see that the statement essentially should be considered modulo some integers, and that the pairing has no more information than the homology of cyclic groups.

2.2 The second statement from malnormality and transfer

In contrast, we will consider some conditions to get the map ϕ_θ and the diagrammatic description in a concrete way. Here, the subgroups $K_1, \ldots, K_{\#L} \subset G$ are said to be *malnormal (in G)*, if they satisfy

(⋆) For any $(i, j) \in \{1, \ldots, \#L\}^2$ and any $g \in G$ with $g \notin K_j$, the intersection $g^{-1} K_i g \cap K_j$ equals $\{1_G\}$.

The papers [13,14] give such examples, as in Gromov hyperbolic groups. Furthermore, as in [1,31], the malnormality plays a key role in studying the virtual Haken conjecture and "the Malnormal Special Quotient Theorem".

We will give a relatively simple description of the pairing, by using quandle theory. For this, we now review a quandle and colorings. A *quandle* [19] is a set, X, with a binary operation $\lhd : X \times X \to X$ such that

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[1] A knot K is *cable*, if there is a solid torus V embedded in S^3 such that V contains K as the (p, q)-torus knot for some $p, q \in \mathbb{Z}$. Furthermore, a knot L is *prime*, if it cannot be written as the knot sum of two non-trivial knots. Incidentally, the assumption in the theorems is inspired by Theorem 3.10 of [13] and the JSJ decomposition; see Sect. 5.
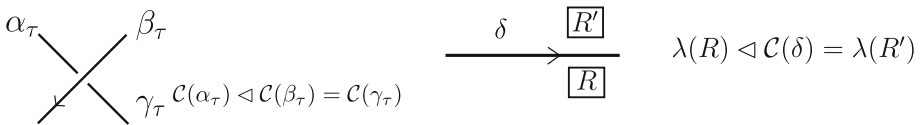
Fig. 2 The coloring conditions at each crossing τ and around each arcs

(I) The identity $a \triangleleft a = a$ holds for any $a \in X$.

(II) The map $(\bullet \triangleleft a) : X \to X$ that sends x to $x \triangleleft a$ is bijective, for any $a \in X$.

(III) The distributive identity $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ holds for any $a, b, c \in X$.

Let X be a quandle. An X-*coloring* of D is a map $\mathcal{C} : \mathrm{Arc}_D \to X$ such that $\mathcal{C}(\alpha_\tau) \triangleleft \mathcal{C}(\beta_\tau) = \mathcal{C}(\gamma_\tau)$ at each crossings τ of D illustrated as Fig. 2. Further, for $x_0 \in X$, a *shadow coloring* is a pair of an X-coloring $\mathcal{C}$ and a map $\lambda : \mathrm{Reg}_D \to X$ such that the unbounded exterior region is assigned by x_0, and if two regions R and R' are separated by an arc δ as shown in the right of Fig. 2, then $\lambda(R) \triangleleft \mathcal{C}(\delta) = \lambda(R')$. Here, notice that the assignment of every region is that of the unbounded region, by definition. Thus, from arbitrary $x_0 \in X$ and X-coloring, we obtain uniquely a shadow coloring such that the unbounded region is labeled by x_0.

This paper mainly deals with the following class of quandles.

Example 2.2 ([19]) This example is due to Joyce [19]. Under the above settings (f, G, K_ℓ), let X be the union of the left quotients $(K_\ell \backslash G)$, that is,

$$X = \sqcup_{\ell=1}^{\#L} (K_\ell \backslash G).$$

Let $k_\ell = f(\mathfrak{m}_\ell) \in K_\ell$. Assume that k_ℓ commutes with all the elements of the subgroup K_ℓ. Then, the union X is made into a quandle under the operation

$$[K_\ell x] \triangleleft [K_{\ell'} y] = [K_\ell^{-1} x y^{-1} k_{\ell'} y], \tag{4}$$

for any $x, y \in G$. In what follows, we will write the triple $(G, \mathcal{K})$ for this quandle.

Furthermore, as is known (see [26, Appendix] for the details), the homomorphism f admits uniquely an X-coloring $\mathcal{C}$ with $\mathcal{C}(\mathfrak{m}_\ell) = k_\ell$ via Wirtinger presentation, where $\mathcal{C}(\gamma)$ is defined to be $f(\gamma)^{-1} k_\ell f(\gamma)$ if γ lies in the ℓ-the component of L. Recall the above color x_0 in the unbounded region. Hence, by applying x_0 to k_1, we have the associated shadow coloring $\mathcal{S} : \mathrm{Reg}_D \times \mathrm{Col}_D \to X$.

Then, the main theorem in this subsection is stated as follows:

Theorem 2.3 *Suppose that k_ℓ commutes with any elements of K_ℓ, and that L is a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot, as in Theorem 2.1. Furthermore, assume one of the following two: (i) $(G, K_1, \ldots, K_{\#L})$ is malnormal. (ii) $K_1, \ldots, K_{\#L}$ are of finite order and and that each of the orders $|K_i|$ is invertible in the coefficient group A.*

Then, any relative group 3-cocycle of $(G, K_1, \ldots, K_{\#L})$ is represented as a map $\theta : X^4 \to A$ such that, the fundamental 3-class $\langle \theta, f_*[E_L, \partial E_L] \rangle$ is equal to the sum

$$\sum_\tau \epsilon_\tau \big(\theta(k_1, a_\tau, b_\tau, c_\tau) - \theta(k_1, a_\tau \triangleleft b_\tau, b_\tau, c_\tau) - \theta(k_1, a_\tau \triangleleft c_\tau, b_\tau \triangleleft c_\tau, c_\tau)$$

$$+ \theta(k_1, (a_\tau \triangleleft b_\tau) \triangleleft c_\tau, b_\tau \triangleleft c_\tau, c_\tau) \big).$$

running over the all crossings τ. Here, for the assignment (x_τ, y_τ, z_τ) around τ as in Fig. 1, we define $(a_\tau, b_\tau, c_\tau) \in X^3$ by setting $\big(\mathcal{S}(x_\tau), \mathcal{S}(y_\tau), \mathcal{S}(z_\tau) \big)$.

3 Relative group homology

First, we outline the proof of the theorems. The key step is to introduce three chain groups, and two chain maps (see Sects. 3, 4). These chain maps will be summarized to

$$C_*^R(X) \xrightarrow{\varphi_*} C_*^\Delta(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\mathrm{As}(X)]} \mathbb{Z} \xleftarrow{\alpha} C_*^{\mathrm{gr}}(G, \mathcal{K}; \mathbb{Z}). \qquad (5)$$

Roughly speaking, the right hand side denominates relative groups cocycles (see Sect. 3.1), and the left one can be diagrammatically described (see Sect. 4). Thus, if we construct a chain map from the middle term to the right hand side, we can obtain diagrammatic computations as in the theorems. However, as seen in Sect. 4, the existence of such a chain map depends on some properties of knot type. Thus, in the proof, we need careful verifications to deal with the chain maps.

To accomplish the outline in details we first introduce relative group homology in the family version; see Sect. 3.1. After that, we will give a key Proposition 3.7, and define the chain map α.

Throughout this section, we fix a group G and subgroups $K_1, \ldots, K_{\#L} \subset G$ as above. Furthermore, we denote the index set $\{1, \ldots, \#L\}$ by I, and denote $(K_1, \ldots, K_{\#L})$ by $\mathcal{K}$, for short.

3.1 Preliminaries; Two versions of group relative homology

The relative group homology is usually defined from a group pair $K \subset G$; see, e.g., [3, §3] or [32]. However, this paper generalizes the relative homology into the family version so as to deal with links.

Consider the union of the left quotients, $\sqcup_{i \in I}(K_i \backslash G)$, and set up the module of the form

$$C_n^{\mathrm{red}}(G, I) := \left\{ (a_1, \ldots, a_{\#L}) \in \mathbb{Z}[G^{n+1}]^{\#L} \;\middle|\; \sum_{i \in I} a_i = 0 \right\}. \qquad (6)$$

Then, letting $n = 0$, we canonically have a right G-module homomorphism

$$P_I : C_0^{\mathrm{red}}(G, I) \longrightarrow \mathbb{Z}[\sqcup_{i \in I}(K_i \backslash G)].$$

We define *the relative group homology* of $(G, \mathcal{K})$ to be the torsion $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathrm{Coker}(P_I), M)$, where M is a left $\mathbb{Z}[G]$-module. Precisely, taking the augmentation map $\varepsilon : \mathbb{Z}[\mathrm{Coker}(P_I)] \to \mathbb{Z}$ with a choice of a projective resolution

$$\mathcal{P}_* : \quad \cdots \xrightarrow{\partial_{n+1}} \mathcal{P}_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} \mathcal{P}_1 \xrightarrow{\partial_1} \mathrm{Coker}(P_I) \xrightarrow{\varepsilon} \mathbb{Z} \qquad (\text{exact}),$$

as right $\mathbb{Z}[G]$-modules, the relative homology is defined to be

$$H_n(G, \mathcal{K}; M) := H_n(\mathcal{P}_* \otimes_{\mathbb{Z}[G^n]} M, \partial_*).$$

Dually, we can define the cohomology as $\mathrm{Ext}_{\mathbb{Z}[G]}^*(\mathrm{Ker}(\varepsilon), M)$. For enough projectivity, we now cite an example of $\mathcal{P}_*$.

Example 3.1 (Mapping cone) For any set B, we define the map $\partial_n^\Delta : \mathbb{Z}[B^{n+1}] \to \mathbb{Z}[B^n]$ by setting

$$\partial_n^\Delta(x_0, \ldots, x_n) = \sum_{i: \, 0 \le i \le n} (-1)^i (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{Z}[B^n]. \qquad (7)$$

According to [32, §2], for $n > 1$, consider the following free G-module:

$$C_n^{\mathrm{gr}}(G, K_\ell) := \left((\mathbb{Z}[G^{n+1}] \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \oplus (\mathbb{Z}[(K_\ell)^n] \otimes_{\mathbb{Z}[K_\ell]} \mathbb{Z}) \right) \otimes_{\mathbb{Z}} \mathbb{Z}[G].$$

Furthermore, when $n = 1$, we define $C_1^{\mathrm{gr}}(G, \mathcal{K}) = C_1(G) = \mathbb{Z}[G^2]$. Then, we can easily see that the following assignments define a differential on these modules: $\partial_1 := \partial_1^\Delta$ and when $n > 1$,

$$\partial_n(\vec{g}, \vec{k}_\ell) := \left(\partial_n^\Delta(\vec{g}) + (-1)^n \iota_\ell(\vec{k}_\ell), \ \partial_{n-1}^\Delta(\vec{k}_\ell) \right)$$

for any $(\vec{g}, \vec{k}_\ell) \in G^{n+1} \times K_\ell^n$. Here ι_ℓ is the inclusion $K_\ell \hookrightarrow G$. From the definition in (6), it is sensible to consider a canonical inclusion

$$P_n^{(I)} : C_n^{\mathrm{red}}(G, I) \longrightarrow \bigoplus_{j : 1 \le j \le n} C_n^{\mathrm{gr}}(G, K_j). \tag{8}$$

Then, we denote the cokernel of $P_n^{(I)}$ by the pair $(C_*^{\mathrm{gr}}(G, \mathcal{K}), \partial_*)$. Thus, we define $H_n^{\mathrm{gr}}(G, \mathcal{K}; M)$ to be the homology of the complex $(C_*^{\mathrm{gr}}(G, \mathcal{K}) \otimes_{\mathbb{Z}[G]} M, \partial_*)$.

Proposition 3.2 *Then, the pair $(C_*^{\mathrm{gr}}(G, \mathcal{K}), \partial_*)$ gives a free resolution of* $\mathrm{Coker}(P_I)$.

Proof Since the statement with $|I| = 0, 1$ is known (see [32, Theorem 2.1]), we may assume $|I| > 1$. Then, we have two sequences with commutativity:

$$\cdots \longrightarrow C_n^{\mathrm{red}}(G, I) \xrightarrow{(\partial_n)^m} C_{n-1}^{\mathrm{red}}(G, I) \xrightarrow{(\partial_{n-1})^m} \cdots \longrightarrow C_0^{\mathrm{red}}(G, \emptyset, I) \xrightarrow{\varepsilon} \mathbb{Z} \qquad \text{(exact)}$$

$$\left\downarrow{P_n^{(I)}} \qquad\qquad \left\downarrow{P_{n-1}^{(I)}} \qquad\qquad\qquad\qquad\qquad \left\downarrow$$

$$\cdots \longrightarrow \oplus C_n^{\mathrm{gr}}(G, K_i) \longrightarrow \oplus C_{n-1}^{\mathrm{gr}}(G, K_j) \longrightarrow \cdots \longrightarrow \oplus \mathrm{Coker}(P_{[j]}) \xrightarrow{\varepsilon} \mathbb{Z} \qquad \text{(exact)}$$

Here, the exactness is ensured by the cases with $|I| \le 1$. The cokernel of the vertical map is exactly the complex $(C_*^{\mathrm{gr}}(G, \mathcal{K}), \partial_*)$ from definitions. Hence, by the five lemma, the cokernel gives a free resolution of $\mathrm{Coker}(P_I)$. □

Then, a standard discussion of mapping cones deduces the long exact sequence with $n \ge 2$:

$$\cdots \to H_{n+1}^{\mathrm{gr}}(G, \mathcal{K}; M) \xrightarrow{\delta_*} \oplus_j H_n^{\mathrm{gr}}(K_j; M) \xrightarrow{\oplus(\iota_j)_*} H_n^{\mathrm{gr}}(G; M) \longrightarrow H_n^{\mathrm{gr}}(G, \mathcal{K}; M) \to \cdots. \tag{9}$$

Example 3.3 As an example, let us describe 3-cocycles in the non-homogenous cochain $\mathrm{Coker}(P_I)$. Specifically, a 3-cocycle of G relative to $\mathcal{K}$ is represented by m maps $\theta_\ell : G^3 \to M$ and $\eta_\ell : (K_\ell)^2 \to M$ satisfying the two equations

$$g_1 \cdot \theta_\ell(g_2, g_3, g_4) - \theta_\ell(g_1 g_2, g_3, g_4) + \theta_\ell(g_1, g_2 g_3, g_4) - \theta_\ell(g_1, g_2, g_3 g_4)$$
$$+ \theta_\ell(g_1, g_2, g_3) = 0, \tag{10}$$
$$\theta_\ell(k_1, k_2, k_3) = k_1 \cdot \eta_\ell(k_2, k_3) - \eta_\ell(k_1 k_2, k_3) + \eta_\ell(k_1, k_2 k_3) - \eta_\ell(k_1, k_2), \tag{11}$$

for any $g_i \in G$ and $k_i \in K_\ell$, and $1 \le \ell \le \#L$.

Remark 3.4 Here, we mention a topological meaning of the relative homology $H_n(G, \mathcal{K}; M)$. Let $K(K_j, 1)$ and $K(G, 1)$ be the Eilenberg–MacLane spaces of type $(K_j, 1)$ and of type $(G, 1)$, respectively. Let $(\iota_j)_* : K(K_j, 1) \to K(G, 1)$ be the map induced from the inclusions. Then, similarly to [32, §2 and §5], we can see that $H_n(G, \mathcal{K}; M)$ is isomorphic to the homology of the mapping cone of $\sqcup_j \iota_j : \sqcup_j K(K_j, 1) \to K(G, 1)$ with local coefficients M.

In another way, we will introduce the relative homology of $\sqcup_{i \in I}(K_i \backslash G)$, which is originally defined by Hochschild [15]. Let Y be the union $\sqcup_{i \in I}(K_i \backslash G)$, and let $C_n^{\mathrm{pre}}(Y)$ be the free $\mathbb{Z}$-module generated by $(n + 1)$-tuples $(y_0, y_1, \ldots, y_n) \in Y^{n+1}$. Consider the differential homomorphism defined by ∂_*^Δ as above. As is known (see [3,4,32]), the chain complex $(C_*^\Delta(Y), \ \partial_*^\Delta)$ is acyclic. From the natural action $Y \curvearrowleft G$, let us equip $C_n^{\mathrm{pre}}(Y)$ with the diagonal action. Furthermore, as a parallel to (6), from the definition of $C_n^{\mathrm{red}}(G, I)$, we can similarly consider a $\mathbb{Z}[G]$-homomorphism $Q_n : C_n^{\mathrm{red}}(G, I) \to C_n^{\mathrm{pre}}(Y)$.

Definition 3.5 We define the chain complex $(C_*^\Delta(Y), \ \partial_*^\Delta)$ to be the cokernel $\mathrm{Coker}(P_n)$, which is diagonally acted on by G.

Furthermore, $H_*^\Delta(Y; M)$ denotes the homology of the quotient complex $\mathrm{Coker}(P_n) \otimes_{\mathbb{Z}[G]} M$. Namely, $H_*^\Delta(Y; M) = H_*(\mathrm{Coker}(Q_n) \otimes_{\mathbb{Z}[G]} M)$.

This chain complex $(\mathrm{Coker}(Q_*), \ \partial_*^\Delta)$ is acyclic and is not always projective, even if $|I| = 1$. However, the projectivity of $\mathcal{P}_*$ admits, uniquely up to homotopy, a chain $\mathbb{Z}[G]$-map

$$\alpha : (\mathcal{P}_*, \partial_*) \longrightarrow (C_*^\Delta(Y), \partial_*^\Delta). \tag{12}$$

Example 3.6 When $\mathcal{P}_*$ is the complex $C_*^{\mathrm{gr}}(G, \mathcal{K})$ in Example 3.1, we will give an example of α. Consider the normalized complex of $\mathrm{Coker}(Q_n)$ subject to the submodule

$$\mathbb{Z}\langle (y_0, \ldots, y_n) \in Y^n \mid y_i = y_{i+1} \text{ for some } i \rangle,$$

and denote it by $C_*^{\mathrm{Nor}}(Y)$. As usual in normalization, we note $H_*^\Delta(Y; M) \cong H_*^{\mathrm{Nor}}(Y; M)$. Then, for $1 \le j \le \#L$, consider the correspondence

$$\alpha_j^{\mathrm{pre}} : G^{n+1} \times (K_j)^n \longrightarrow C_n^{\mathrm{pre}}(K_j \backslash G); \ (g_0, \ldots, g_n, k_0, \ldots, k_{n-1}) \longmapsto (K_j g_0, K_j g_1 \ldots, K_j g_n).$$

Here let us regard $\mathcal{P}_* = C_*^{\mathrm{gr}}(G, \mathcal{K})$ as the cokernel $\mathrm{Coker}(P_n^{(I)})$; see (8). Subject to the image of $C_n^{\mathrm{red}}(G, I)$, the direct sum of α_j^{pre} yields a chain map $C_*^{\mathrm{gr}}(G, \mathcal{K}) \to C_*^{\mathrm{Nor}}(Y)$.

3.2 A key proposition from malnormality, and some examples

Whereas this α is not always a quasi-isomorphism (see [3, §3.2] for counter-examples), we give a criterion which is a key in this paper.

Proposition 3.7 (A modification of [3, Proposition 3.23]) *The set* $Y = \sqcup_{i \le \#L}(K_i \backslash G)$ *is assumed to be of infinite order. Furthermore, the subgroups* $K_1, \ldots, K_{\#L} \subset G$ *are malnormal.*

Then, the chain map α *induces an isomorphism* $H_*^\Delta(Y; M) \cong H_*(G, \mathcal{K}; M)$ *for any coefficient* M.

Proof The proof is essentially due to [3]. Consider the submodule

$$C_n^{\neq}(Y) := \mathbb{Z}\{ [(y_0, \ldots, y_n)] \in \mathrm{Coker}(Q_n) \mid \text{If } s \neq t, \text{ then } y_s \neq y_t. \}. \tag{13}$$

Since Y is of infinite order, this $C_n^{\neq}(Y)$ is an acyclic subcomplex of $\mathrm{Coker}(Q_n)$, and the injection is quasi-isomorphic; see [3, Proposition 3.20] for the details. Furthermore, we can easily check that, if $\sigma \cdot g = \sigma$ with $g \in G$ and $\sigma := (y_0, \ldots, y_n) \in C_n^{\neq}(Y)$, the malnormal assumption implies $g = 1_G \in G$. That is, the action is free; therefore, $C_n^{\neq}(Y)$ is a free $\mathbb{Z}[G]$-module. Since the above α factors through $C_n^{\neq}(Y)$, we have the conclusion. $\square$

We end this section by giving three examples satisfying the assumptions.

Example 3.8 (Hyperbolic 3-manifolds) Let N be a compact hyperbolic 3-manifold with torus boundary $\partial N = T_1 \sqcup \cdots \sqcup T_m$. Apply G to $\pi_1(N)$ and K_i to $\pi_1(T_i)$ with a choice of base point. As is well-known as "algebraic atoroidality" in hyperbolic geometry (see [2]), the boundary group $\pi_1(T_i)$ injects $\pi_1(N)$, and the malnormal condition holds. Since N is also a $K(G, 1)$-space by hyperbolicity, we thus have the isomorphisms

$$H_*^\Delta(Y; \mathbb{Z}) \cong H_*^{\mathrm{gr}}(G, \mathcal{K}; \mathbb{Z}) \cong H_*(N, \partial N; \mathbb{Z}).$$

Example 3.9 (Knots) Furthermore, given a non-trivial knot L in the 3-sphere S^3, we replace G by $\pi_1(S^3 \setminus L)$ and K_1 by a peripheral subgroup $\pi_1(\partial(S^3 \setminus L)) \cong \mathbb{Z}^2$, which is generated by a meridian-longitude pair $(\mathfrak{m}, \mathfrak{l})$. By the loop theorem of 3-manifolds, K_1 injects G. Furthermore, $S^3 \setminus L$ is basically known to be a $K(G, 1)$-space.

Moreover, we mention a theorem to detect the malnormality of the knot group.

Theorem 3.10 ([13,29]) *Let $K_1 \subset G$ be as above. The pair (K_1, G) is malnormal if and only if the knot L is none of the following three cases: torus knots, cable knots, and composite knots.*

In particular, in the case, the isomorphism $\alpha : H_*^\Delta(Y; \mathbb{Z}) \cong H_*(E_L, \partial E_L; \mathbb{Z})$ holds.

Example 3.11 (Link quandles) More generally, let us consider a link $L \subset S^3$ and the link group $G = \pi_L = \pi_1(S^3 \setminus L)$. Let K_ℓ with $1 \leq \ell \leq \#L$ be the abelian subgroup generated by a meridian-longitude pair $(\mathfrak{m}_\ell, \mathfrak{l}_\ell)$ with respect to the ℓth link component, that is, K_ℓ is a peripheral group generated by $(\mathfrak{m}_\ell, \mathfrak{l}_\ell)$. We denote $\sqcup_\ell K_\ell$ by $\partial \pi_L$ hereafter.

However, there are many links satisfying non-malnormality on $(\pi_L, \partial \pi_L)$, as in the Hopf link. More generally, malnormality for non-splittable links is completely characterized in [13, Corollary 4].

Incidentally, the union $\sqcup_j (K_j \setminus \pi_L)$ with the binary operation (4) is called *the link quandle* [19]. We denote the link quandle by Q_L, since we later use it in many times.

4 Review; quandle homology and Inoue–Kabaya map

Next, regarding the middle term in the zigzag sequence (5), this section reviews the quandle homology [7] and Inoue–Kabaya chain map [17]. As seen in [8,17,24,26], quandle theory is useful for reducing some 3-dimensional discussions to diagrammatic objects.

We briefly explain the rack and quandle (co)homology groups [7,8]. Let X be a quandle, and $C_n^R(X)$ be the free right $\mathbb{Z}$-module generated by X^n. Namely, $C_n^R(X) := \mathbb{Z}[X^n]$. Define a boundary $\partial_n^R : C_n^R(X) \to C_{n-1}^R(X)$ by

$$\partial_n^R(x_1, \ldots, x_n) = \sum_{1 \leq i \leq n} (-1)^i \big((x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n)$$
$$- (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \big).$$

Since $\partial_{n-1}^R \circ \partial_n^R = 0$ as usual, we can define the homology $H_n^R(X)$ and call it *the rack homology*. Furthermore, let $C_n^D(X)$ be the submodule of $C_n^R(X)$ generated by n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_{i+1}$ for some $i \in \{1, \ldots, n-1\}$. One can easily see that this $C_n^D(X)$ is a subcomplex of $C_n^R(X)$. Then, the *quandle homology*, $H_n^Q(X)$, is defined to be the homology of the quotient complex $C_n^R(X)/C_n^D(X)$. In general, it is not easy to compute these homology groups.

In addition, we will review the Inoue–Kabaya map whose codomain is the Hochschild complex in Definition 3.5. For this, we need some notation. A map $f : X \to X'$ between quandles is *a quandle homomorphism*, if $f(a \triangleleft b) = f(a) \triangleleft f(b)$ for any $a, b \in X$. Given a quandle X, the *associated group* $AS(X)$ is given via the presentation

$$As(X) := \langle\, e_x\ (x \in X) \mid e_{x \triangleleft y}^{-1} \cdot e_y^{-1} \cdot e_x \cdot e_y\ (x, y \in X)\, \rangle.$$

We call $As(X)$ *the associated group*. There is a right action of $AS(X)$ on X defined by $x \cdot e_y := x \triangleleft y$, where $x, y \in X$. Let $O(X)$ be the orbit set of $X \curvearrowleft As(X)$. With respect to $i \in O(X)$, we fix $x_i \in X$ in the orbit. As in Example 2.2, denoting $\mathrm{Stab}(x_i) \subset As(X)$ by K_i, we can consider the setting

$$X = Y = \sqcup_{i \in O(X)} (K_i \backslash G) \ \text{ with } \ G = As(X).$$

Furthermore, we set up the following set consisting of some maps:

$$J_n := \big\{\, \iota : \{2, 3, \ldots, n\} \longrightarrow \{0, 1\} \,\big\},$$

which is of order 2^{n-1}. Moreover, given a tuple $(x_1, \ldots, x_n) \in X^n$ and each $\iota \in J_n$, we define $x(\iota, i) \in X$ by the formula

$$x(\iota, i) := x_i \cdot \big(e_{x_{i+1}}^{\iota(i+1)} \cdots e_{x_n}^{\iota(n)}\big).$$

Then, with a choice of an element $p \in X$, we define a homomorphism

$$\varphi_n : C_n^R(X) \longrightarrow C_n^{\mathrm{pre}}(X) \otimes_{\mathbb{Z}[As(X)]} \mathbb{Z}$$

by setting

$$\varphi_n(x_1, \ldots, x_n) := \sum_{\iota \in J_n} (-1)^{\iota(2)+\iota(3)+\cdots+\iota(n)} \big(x(\iota, 1), \ldots, x(\iota, n), p\big).$$

Here are the descriptions of φ_* of lower degree:

$$\varphi_2(a, b) = (a, b, p) - (a \triangleleft b, b, p),$$
$$\varphi_3(a, b, c) = (a, b, c, p) - (a \triangleleft b, b, c, p) - (a \triangleleft c, b \triangleleft c, c, p) + ((a \triangleleft b) \triangleleft c, b \triangleleft c, c, p).$$

Then, it is shown [17, §4] that this φ_n is a chain map, i.e., $\partial_n^\Delta \circ \varphi_n = \varphi_{n-1} \circ \partial_n^R$, and that if $n \leq 3$, the image of the subcomplex $C_n^D(X)$ is nullhomotopic. Hence, the map φ_3 with $n = 3$ induces a homomorphism

$$(\varphi_3)_* : H_3^Q(X) \longrightarrow H_3^\Delta(X; \mathbb{Z}).$$

We refer the reader to several studies on the chain map; see [17,18,24–26].

Next, we review the quandle cocycle invariant. Given a shadow coloring $\mathcal{S}$ of a link diagram D, *the fundamental 3-class of $\mathcal{S}$*, denoted by $[\mathcal{S}]$, is defined to be the sum

$$[\mathcal{S}] := \sum_\tau \epsilon_\tau \big(\lambda(x_\tau),\ \mathcal{C}(y_\tau),\ \mathcal{C}(z_\tau)\big) \in C_3^Q(X)$$

running over all the crossings τ, where the triple $(x_\tau,\ y_\tau,\ z_\tau)$ are the three assignments around τ illustrated in Fig. 1, and $\epsilon_\tau \in \{\pm 1\}$ is the sign of τ. Then, we can easily see that $[\mathcal{S}]$ is a quandle 3-cycle in $C_3^Q(X)$; see [8]. If we have a quandle 3-cocycle $\phi : X^3 \to A$, the pairing $\langle \phi, [\mathcal{S}] \rangle \in A$ is called *the quandle cocycle invariant of $\mathcal{S}$*. Here, a map $\phi : X^3 \to A$ is a quandle 3-cocycle, if the followings hold by definition:

$$\phi(x, z, w) - \phi(x \triangleleft y, z, w) - \phi(x, y, w) + \phi(x \triangleleft z, y \triangleleft z, w)$$
$$= \phi(x \triangleleft w, y \triangleleft w, z \triangleleft w) - \phi(x, y, z),$$
$$\phi(x, x, y) = \phi(y, z, z) = 1, \quad \text{for any } x, y, z, w \in X.$$

For calculating the invariant $\langle \phi, [\mathcal{S}] \rangle$, it is important to find explicit formulas of quandle 3-cocycles ϕ, although it is difficult in general.

Example 4.1 Let X be the link quandle Q_L of a link; see Example 3.11. Consider the identity $\pi_L \to \pi_L$, which induces $\mathrm{id}_{Q_L} : Q_L \to Q_L$. Then, we obtain from Example 2.2 the Q_L-coloring $\mathcal{S}_{\mathrm{id}_{Q_L}}$, together with the associated 3-class $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$. This homology 3-class plays a key role later.

Example 4.2 In the hyperbolic case, Inoue and Kabaya obtained a 3-cocycle from the chain map φ_*, with a relation to the Chern–Simons invariant. Let X be the quotient set $\mathbb{C}^2 \backslash \{(0, 0)\} / \sim$ subject to the relation $(a, b) \sim (-a, -b)$. Equip X with a quandle operation

$$\begin{pmatrix} a & b \end{pmatrix} \triangleleft \begin{pmatrix} c & d \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} 1 + cd & d^2 \\ -c^2 & 1 - cd \end{pmatrix}.$$

One can easily verify that X is isomorphic to the triple (G, K, k_0) as in Example 2.2, where G is $PSL_2(\mathbb{C})$ and K is the unipotent subgroup of the form $\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{C} \right\}$, and $k_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Although this (G, K) is not malnormal, the paper [3, §4] showed that the chain map α in (12) is a quasi-isomorphism, which ensures a quasi-inverse β. Furthermore, Neumann [23] and Zickert [32] described the Chern–Simons 3-class as a relative group 3-cocycle

$$\mathrm{CS} \in C^3_{\mathrm{gr}}(PSL_2(\mathbb{C}), K; \ \mathbb{C}/\pi^2\mathbb{Z}), \tag{14}$$

together with a cocycle presentation (see [32] for the detail). As a consequence, we concretely get a quandle 3-cocycle $\varphi^* \circ \beta^*(\mathrm{CS})$.

Furthermore, let us consider a hyperbolic link L and explain (15) below. From the viewpoint of Example 2.2, the associated holonomy representation $\rho : \pi_L \to PSL_2(\mathbb{C})$ is regarded as a shadow X-coloring $\mathcal{S}_\rho$. Then, Inoue and Kabaya [17, Theorem 7.3] showed the equality

$$\langle \rho^*(\mathrm{CS}), [E_L, \partial E_L] \rangle = \langle (\beta \circ \varphi_3)^*(\mathrm{CS}), [\mathcal{S}_\rho] \rangle \in \mathbb{C}/\pi^2\mathbb{Z}. \tag{15}$$

Notice that, the right hand side is a quandle cocycle invariant, by definition. As a result, we can compute the Chern–Simons invariant without triangulation; see [17] for examples.

5 Algebraic representation of the fundamental homology 3-class

In this section, we give a method to algebraically represent the fundamental 3-class $[E_L, \partial E_L]$, where L is either a hyperbolic link or a prime non-cable knot.

To describe this, the following plays a key viewpoint (see Sect. 6 for the proof).

Theorem 5.1 *Assume that a prime knot L is neither a cable knot nor a torus knot, as in Theorem 3.10. Then, the Inoue–Kabaya chain map φ_3 induces an isomorphism $H^Q_3(Q_L) \to H^\Delta_3(Q_L; \mathbb{Z})$, which is identified with $\mathrm{id}_\mathbb{Z} : \mathbb{Z} \to \mathbb{Z}$.*

Theorem A.1 says that $H_3^Q(Q_L)$ is generated by some fundamental 3-class $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$. Hence, if we can explicitly formulate a quasi-inverse $\beta : C_*^\Delta(Q_L)_{\pi_L} \to C_*^{\mathrm{gr}}(\pi_L, \partial \pi_L; \mathbb{Z})$ in (17), then we obtain an algebraic presentation of the fundamental 3-class $[E_L, \partial E_L]$.

We will explain the reason why we focus on only hyperbolic links in Sect. 5.2. The key is the JSJ-decomposition of a knot (or the geometrization theorem). Precisely, as seen in [5, Theorem 4.18] or [2,13], there exist open sets $V_1 \subset V_2 \subset \cdots \subset V_n \subset S^3$ satisfying the followings:

1. The set V_i for any $i \le n$ is an open solid torus in S^3, and V_i contains the knot L.
2. For any $i \in \mathbb{Z}_{\ge 0}$, the difference $V_i - \overline{V}_{i-1}$ is homeomorphic to one of a composite knot or a hyperbolic knot or an (n_i, m_i)-torus knot in the solid torus for some $(n_i, m_i) \in \mathbb{Z}^2$. Here we denote the knot L by V_0.

As is known, the decomposition is unique in some sense. Here, remark (see [5, Corollary 4.19]) that L is a cable knot if and only if a difference $V_1 - \overline{V_0}$ is an (n, m)-torus knot in the solid torus; see Fig. 5.

Following the JSJ-decomposition, let us further examine the pairing (1). Denote the inclusion $V_i - V_{i-1} \subset S^3 - L$ by ι_i, and the torus-boundary $\overline{V_{i-1}} \cap \overline{V_i}$ by B_i. Then, given $f : \pi_L \to G$ and θ as before, we have $f \circ (\iota_i)_* : \pi_1(V_i - V_{i-1}) \to G$. Let $K_i \subset G$ be the image of $\pi_1(B_i)$ via $f \circ (\iota_i)_*$, where we appropriately choose a base point. Then, we can regard the pullback $\iota_i^* \circ f^*(\theta)$ as a relative group 3-cocycle of (G, K_i, K_{i+1}). Therefore, the excision axiom on ι_i's ensures the equality

$$\langle f^*(\theta), [E_L, \partial E_L] \rangle = \sum_{i\,:\,1 \le i \le n} \langle \iota_i^* \circ f^*(\theta), [\overline{V_i} - V_{i-1}, \partial(\overline{V_i} - V_{i-1})] \rangle. \quad (16)$$

To conclude, it is reasonable to deal with the fundamental 3-classes piecewise, according to the JSJ-decompositions of knots.

5.1 The fundamental relative 3-class of hyperbolic links

This subsection gives an explicit algorithm for describing the fundamental relative 3-class of hyperbolic links. Here, the description is done in truncated terms (Theorem 5.3).

We begin by reviewing the truncated complex, which is defined by Zickert [32, §3]. Fix a group G, and subgroups $K_1, \ldots, K_{\#L}$. For $n \ge 1$, consider the free abelian group $\mathbb{Z}[G^{n^2+n}]$, and denote the (ij)-th generator $g \in G$ by g_{ij} with $i \ne j$. Define $\overline{C}_n(G, \mathcal{K})$ by the submodule of $\mathbb{Z}[G^{n^2+n}]$ which is generated by g_{ij} satisfying

- for any $i \in \{0, \ldots, n\}$, there exists $m_i \in \{1, \ldots, \#L\}$ such that the classes of the n elements $g_{i0}, \ldots, \check{g}_{ii}, \ldots, g_{in}$ in the coset $K_{m_i} \backslash G$ are equal.

Then, right multiplication endows $\overline{C}_n(G, \mathcal{K})$ with a $\mathbb{Z}[G]$-module structure, and the usual simplicial boundary map gives rise to a boundary map ∂_* on $\overline{C}_n(G, \mathcal{K})$. The complex $(\overline{C}_*(G, \mathcal{K}), \partial_*)$ is called *the truncated complex* of $(G, \mathcal{K})$. As was similarly shown [32, Remark 3.2 and Proposition 3.7], we can easily verify that this complex is a free resolution of $\mathrm{Coker}(P_I)$ (Fig. 3).

In addition, let us examine the case where $G_\mathbb{C}$ is $PSL_2(\mathbb{C})$ and every $K_{\mathbb{C},\ell}$ is conjugate to the unipotent subgroup such that $K_{\mathbb{C},s} \cap K_{\mathbb{C},t} = \{1_{G_\mathbb{C}}\}$ for $s \ne t$. Then, we have the quandle $X_\mathbb{C}$, from Example 4.2, as the union of the quandles $\sqcup_{\ell=1}^{\#L} K_{\mathbb{C},\ell} \backslash G_\mathbb{C} = \sqcup^{\#L}(\mathbb{C}^2 \backslash \{0,0\})/\sim$. We will describe a quasi-inverse β mentioned in Proposition 3.7. For this, consider the following subcomplex of $C_n^\Delta(X_\mathbb{C}; \mathbb{Z})$:
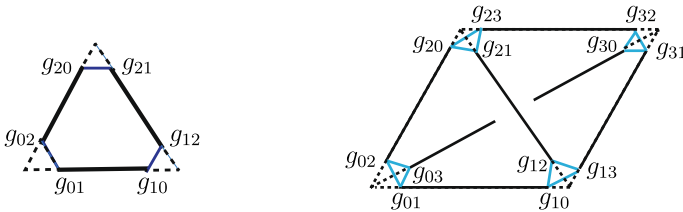
Fig. 3 A geometric description of generators of the truncated 2-, 3-simplexes with G-labels

$$C_n^{h\neq}(X_{\mathbb{C}}) = \mathbb{Z}\langle\, [(a_0, b_0), \ldots, (a_n, b_n)] \in C_n^{\Delta}(X_{\mathbb{C}}; \mathbb{Z}) \mid a_i b_j \neq a_j b_i \text{ for any } i,\ j \text{ with } i \neq j \,\rangle.$$

Then, this complex is known to be an acyclic $\mathbb{Z}[G_{\mathbb{C}}]$-free complex. Consider the correspondence

$$g_{ij} : (X_{\mathbb{C}})^{n+1} \longrightarrow G; \quad \big((a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)\big)$$
$$\longmapsto \begin{pmatrix} a_i & b_i \\ a_j/(a_i b_j - a_j b_i) & b_j/(a_i b_j - a_j b_i) \end{pmatrix}.$$

This gives rise to a homomorphism

$$\beta : C_n^{h\neq}(X_{\mathbb{C}}) \longrightarrow \overline{C}_n(G_{\mathbb{C}}, \mathcal{K}_{\mathbb{C}}).$$

Then, Zickert [32, §3] (see also [3, Corollary 9.6]) showed that this β is a chain map and a $\mathbb{Z}[G]$-homomorphism. To summarize, this β gives a quasi-inverse of the chain map α : $C_n^{\mathrm{gr}}(G_{\mathbb{C}}, \mathcal{K}) \to C_n(X_{\mathbb{C}})$.

We return to the discussion of a hyperbolic link L, and state Theorem 5.3 below. Fix a diagram D of L. Then, we have the holonomy representation $\rho : \pi_L \to PSL_2(\mathbb{C})$. As is well-known, ρ is injective. Thus, it is more reasonable to use matrices in $PSL_2(\mathbb{C})$, than to use (Wirtinger) group presentations of π_L. Here, we should mention the following lemma obtained from hyperbolicity.

Lemma 5.2 (see [32, §5] or [17, Lemma 7.2]) *Let* $\sigma \in C_3^{\Delta}(X_{\mathbb{C}}; \mathbb{Z})$ *be a 3-cycle which represents the fundamental 3-class* $\rho_*(E_L, \partial E_L)$ *of a hyperbolic link L. Then, this 3-cycle* σ *lies in the subcomplex* $C_3^{h\neq}(X_{\mathbb{C}})$ *in* (13).

In summary, we obtain the conclusion:

Theorem 5.3 *Let L be a hyperbolic link with the holonomy representation* $\rho : \pi_L \to PSL_2(\mathbb{C})$. *Let G be the image* $\rho(\pi_L)$, *and $\mathcal{K}$ be the subgroups* $\rho(\partial\pi_L)$. *Choose a diagram D, and take the quandle 3-class* $[\mathcal{S}_\rho]$; *see Sect. 4. Then, the following 3-cycle represents the fundamental 3-class in* $H_3(E_L, \partial E_L; \mathbb{Z}) \cong H_3^{\mathrm{gr}}(\pi_L, \partial\pi_L; \mathbb{Z}) \cong \mathbb{Z}$.

$$\mathrm{res}(\beta) \circ \varphi_3([\mathcal{S}_\rho]) \in \overline{C}_3(G, \mathcal{K}; \mathbb{Z}).$$

5.2 Example; the figure eight knot

As the simplest case, we let L be the figure eight knot as in Fig. 4. By the Wirtinger presentation, we have

$$\pi_1(S^3 \backslash L) \cong \langle g, h \mid h^{-1} g h = g^{-1} h^{-1} g h g^{-1} h g \rangle,$$

$$\mathcal{C}(\alpha_1) := (1, 0)$$
$$\mathcal{C}(\alpha_2) := (0, \omega)$$
$$\mathcal{C}(\alpha_3) := (1, \omega - 1)$$
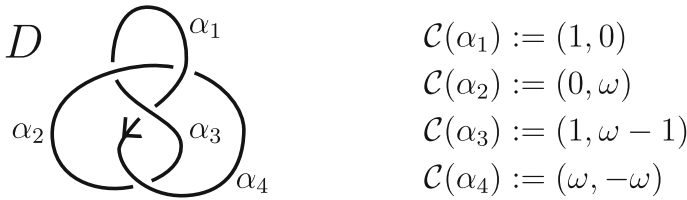$$\mathcal{C}(\alpha_4) := (\omega, -\omega)$$

Fig. 4 The holonomy representation of 4_1 as an X-coloring. Here $\omega = (-1 + \sqrt{-3})/2$

where g and h are meridians derived from the arcs α_1 and α_2, respectively. We denote the two classes in Q_L of g and $h \in \pi_L$ by a and b, respectively. Then, by definition, the fundamental 3-class $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$ is given by

$$-(b \lhd a, a, b) - (b \lhd a, b, a) + ((b \lhd a) \lhd b, a, a \lhd b) + (b, b, b \lhd a) \in C_3^Q(Q_L; \mathbb{Z}).$$

Notice that the first term is sent to zero by φ_*. Then, $\varphi_*[\mathcal{S}_{\mathrm{id}_{Q_L}}]$ is computed as

$$- (b \lhd a, a, b, p) + ((b \lhd a) \lhd a, a, b, p) + ((b \lhd a) \lhd b, a \lhd b, b, p)$$
$$- (((b \lhd a) \lhd a) \lhd b, a \lhd b, b, p)$$
$$- (b \lhd a, b, a, p) + ((b \lhd a) \lhd b, b, a, p) + ((b \lhd a) \lhd a, b \lhd a, a, p)$$
$$- (((b \lhd a) \lhd b) \lhd a, b \lhd a, a, p) + ((b \lhd a) \lhd b, a, a \lhd b, p)$$
$$- (((b \lhd a) \lhd b) \lhd a, a, a \lhd b, p)$$
$$- (((b \lhd a) \lhd b) \lhd (a \lhd b), a \lhd (a \lhd b), a \lhd b, p)$$
$$+ ((((b \lhd a) \lhd b) \lhd a) \lhd (a \lhd b), a \lhd (a \lhd b), a \lhd b, p).$$

Thus, following Theorem 5.3, we consider the well-known holonomy representation $\rho : \pi_L \to PSL_2(\mathbb{Z}[\omega])$, where ω is $(-1 + \sqrt{-3})/2$. This is represented by the $X_{\mathbb{C}}$-coloring $\mathcal{C}$ in Fig. 4. Accordingly, if we replace a by $(1, 0)$ and b by $(0, \omega)$, the 3-cycle $\varphi_*[\mathcal{S}_{\mathrm{id}_{Q_L}}]$ above is reduced to

$$- ((-\omega, \omega), (1, 0), (0, \omega), p) + ((-2\omega, \omega), (1, 0), (0, \omega), p)$$
$$+ ((-\omega, \omega + 1), (1, \omega - 1), (0, \omega), p)$$
$$- ((-2\omega, \omega + 2), (1, \omega - 1), (0, \omega), p)$$
$$- ((-\omega, \omega), (0, \omega), (1, 0), p) + ((-\omega, \omega + 1), (0, \omega), (1, 0), p)$$
$$+ ((-2\omega, \omega), (-\omega, \omega), (1, 0), p)$$
$$- ((-2\omega - 1, \omega + 1), (-\omega, \omega), (1, 0), p)$$
$$+ ((-\omega, 1 + \omega), (0.\omega), (1, \omega - 1), p) - ((-\omega, 2 + \omega), (0.\omega), (1, \omega - 1), p)$$
$$- ((-2\omega - 1, 4), (-\omega, 1 + \omega), (1, \omega - 1), p)$$
$$+ ((-2\omega - 2, 6 - \omega), (-\omega, 1 + \omega), (1, \omega - 1), p).$$

Hence, for example, if $p = (0, 1)$ and we apply the homomorphism β to this cycle, we can describe explicitly the fundamental 3-class by Theorem 5.3. However, the description forms long; we omit writing it.

6 Proofs of the main theorems

We will prove the theorems in Sect. 2. If L is the trivial knot, the theorems are obvious. Thus, we may assume that L is non-trivial in what follows. We begin by proving Theorem 2.1.

6.1 Proofs of Theorems 2.1 and 5.1

Proof of Theorems 2.1 Recall the complexes in Sects. 3 and 4. Since they are functorial by construction, we obtain a commutative diagram:

$$
\begin{array}{ccccc}
C_*^R(Q_L;\mathbb{Z}) & \xrightarrow{\varphi_*} & C_*^\Delta(Q_L;\mathbb{Z}) & \xleftarrow{\alpha} & C_*^{\mathrm{gr}}(\pi_L,\partial\pi_L;\mathbb{Z}) \\
{\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow & {\scriptstyle \beta} & {\scriptstyle f_*}\downarrow \\
C_*^R(X;\mathbb{Z}) & \xrightarrow{\varphi_*} & C_*^\Delta(X;\mathbb{Z}) & \xleftarrow{\alpha} & C_*^{\mathrm{gr}}(G,K;\mathbb{Z}).
\end{array}
$$

$$(17)$$

Since L is either a prime non-cable knot or a hyperbolic link by assumption, the right β comes from the quasi-isomorphism α in Examples 3.8 and 3.9.

We will explain (18) below. Recall the quandle 3-class $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$ from Example 4.1. Then, denoting by $[\pi_L,\partial\pi_L]$ a generator of $H_3^{\mathrm{gr}}(\pi_L,\partial\pi_L)\cong\mathbb{Z}$, the diagram (17) admits some $N_L\in\mathbb{Z}$ such that

$$
\beta\circ\varphi_3([\mathcal{S}_{\mathrm{id}_{Q_L}}]) = N_L[\pi_L,\partial\pi_L]\in H_3^\Delta(Q_L;\mathbb{Z})\cong\mathbb{Z}.
$$

Then, for every group relative 3-cocycle θ, setting $\phi_\theta=(\beta\circ\varphi_3)^*\circ f^*(\theta)$ yields the equalities

$$
\langle\phi_\theta,\ [\mathcal{S}_{\mathrm{id}_{Q_L}}]\rangle = \langle f^*(\theta),\ \beta\circ\varphi_3([\mathcal{S}_{\mathrm{id}_{Q_L}}])\rangle = N_L\langle f^*(\theta),\ [\pi_L,\partial\pi_L]\rangle \in A_G. \tag{18}
$$

Hence, it is enough to show $N_L=\pm 1$. Actually, if $N_L=\pm 1$, we canonically obtain a map $\phi_\theta:\mathrm{Reg}_D\times(\mathrm{Arc}_D)^2\to A$ from the definition of $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$, which justifies the desired equality (3).

If L is a hyperbolic link, and f is the associated holonomy $\pi_L\to PSL_2(\mathbb{C})$ with $\theta=\mathrm{CS}$, we immediately have $N_L=\pm 1$ by (15).

It remains to work in the case where L is a knot which is neither hyperbolic nor cable. Then, thanks to the JSJ-decomposition (16) above, there is a solid torus $V_1\subset S^3$, such that V_1 contains the link L and $V_1\backslash L$ is either hyperbolic or reducible.

In the former case, when $V_1\backslash L$ is hyperbolic, we now give a diagram (19) below. Regarding $V_1\backslash L$ as a hyperbolic link, L', in the 3-sphere, we denote by K_2 another subgroup of $\pi_1(S^3\backslash L')$ arising from $\partial(V_1)$. Then, the link quandle $Q_{L'}$ is bijective to $K_1\backslash\pi_{L'}\sqcup K_2\backslash\pi_{L'}$ by definition. We consider the homogenous quandle of the form $K_1\backslash\pi_L\sqcup K_2\backslash\pi_L$, and denote it by Q_W. Then, the inclusion $j:S^3\backslash L'\hookrightarrow S^3\backslash L$ defines a quandle map $j_*:Q_{L'}\to Q_W$, and we take a canonical injection $i:Q_L\hookrightarrow Q_W$. In summary, we have the commutative diagram on the third homology groups:

$$
\begin{array}{ccccc}
H_3^Q(Q_L;\mathbb{Z}) & \xrightarrow{i_*^Q} & H_3^Q(Q_W;\mathbb{Z}) & \xleftarrow{j_*^Q} & H_3^Q(Q_{L'};\mathbb{Z}) \\
{\scriptstyle \varphi_*}\downarrow & & {\scriptstyle \varphi_*}\downarrow & & {\scriptstyle \varphi_*}\downarrow \\
H_3^\Delta(Q_L;\mathbb{Z}) & \xrightarrow{i_*} & H_3^\Delta(Q_W;\mathbb{Z}) & \xleftarrow{j_*} & H_3^\Delta(Q_{L'};\mathbb{Z}).
\end{array}
\tag{19}
$$

Here, i_*^Q and j_*^Q are the induced maps on the third quandle homology from i and j, respectively. In the appendix (Lemma A.3), we later show the isomorphisms $H_3^Q(Q_W; \mathbb{Z}) \cong H_3^Q(Q_{L'}; \mathbb{Z}) \cong \mathbb{Z}^2$, such that the matrixes of i_*^Q and j_*^Q are given by $(1, 0)$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, respectively. Hence, since the right φ_* is surjective by the former discussion, the commutativity with (16) implies $N_L = \pm 1$ as desired.

Finally, we discuss the case where $V_1 \backslash L$ is reducible. For this, take a hyperbolic link L_0 in $V_1 \cong D^2 \times S^1$, and attach it to $V_1 \backslash L$. Then, the situation is reduced to the above case. Hence, the proof completes. $\qquad\square$

Proof of Theorem 5.1 Since L is neither composite nor cable, the above discussion readily implies the bijectivity of φ_3. $\qquad\square$

Remark 6.1 We mention the assumption of hyperbolicity. As a counter example, consider the Hopf link L. Since $\pi_1(S^3 \backslash L) \cong \mathbb{Z}^2$ and the boundary inclusions induce isomorphisms on π_1, the link quandle Q_L consists of two points. Hence, the chain map φ_* is zero by definition. However, the pairing $\langle \theta, \ f_*[E_L, \ \partial E_L] \rangle$ is not always trivial. In summary, it is seemingly hard to generalize the Theorem 2.1 in every link case.

6.2 Proof of Theorem 2.3; malnormality and transfer

Next, turning to malnormality and transfer, we will complete the proof of Theorem 2.3. For this, we shall mention a key proposition obtained from transfer.

Proposition 6.2 (cf. Transfer; see [4, §III.10]) *Let $K_1, \ldots, K_{\#L}$ be finite subgroups of G, and let Y be $\sqcup_i (K_i \backslash G)$ on which G has a right action. Assume that each order $|K_i|$ is invertible in the coefficient group A. Then, the chain map*

$$\alpha : (\mathcal{P}_* \otimes_{\mathbb{Z}} A, \partial_*) \longrightarrow (C_*^\Delta(Y) \otimes_{\mathbb{Z}} A, \partial_*)$$

with coefficients A is a quasi-isomorphism.

Proof According to the same discussion on the transfer; see [4, § III.9–10]. $\qquad\square$

Proof of Theorem 2.3 First examine the case (i), that is, each $K_i \subset \mathcal{K}$ is a malnormal subgroup in G. Then, Proposition 3.7 again ensures a quasi-inverse $\beta' : C_*^\Delta(X)_G \to C_*^{\text{gr}}(G, \mathcal{K})$, where $X = \sqcup_{i \in I} K_i \backslash G$ as in Example 2.2. Then, for any group 3-cocycle θ, we set $\phi_\theta = (\beta' \circ f_* \circ \varphi_3)^*(\theta) \in A$ as a quandle 3-cocycle, and the group 3-cocycle θ is represented by a map $X^4 \to A$.

We will show the equality below. Let L be a hyperbolic link or a prime non-cable knot, which ensures the 3-class $[\pi_L, \partial \pi_L]$ by the malnormal property of L. Compute the pairing $\langle \phi_\theta, [\mathcal{S}_f] \rangle$ as

$$\langle \phi_\theta, [\mathcal{S}_f] \rangle = \langle f^* \circ (\beta')^*(\theta), \ (\varphi_3)_*[\mathcal{S}_f] \rangle = \langle f^* \circ (\beta')^*(\theta), \ (\alpha)_*[\pi_L, \partial \pi_L] \rangle$$
$$= \langle \alpha^* \circ f^* \circ (\beta')^*(\theta), \ [\pi_L, \partial \pi_L] \rangle = \langle f^*(\theta), \ [\pi_L, \partial \pi_L] \rangle. \qquad (20)$$

Here, the second equality is obtained by (18) and Proposition 3.7, and the others are done by functoriality. From the definition of $[\mathcal{S}_f]$ and ϕ_θ, this equality (20) is equivalent to the desired statement.

Finally, we turn to the assumption (ii). By Proposition 6.2, we can similarly get a quasi-inverse β of α in the coefficient group A. Thus, the required equality is done by the same discussion as (20); hence, the desired statement also holds in (ii). $\qquad\square$

6.3 Examples of 3-cocycles

Finally, we will point out that, in some cases, such group 3-cocycles θ have much simpler expressions in terms of quandle cocycles. We end this section by giving two examples.

Example 6.3 (cf. [24]) First, we will observe some triple Massey products. This example is essentially due to [24, §4.2]. Regard the finite field $\mathbb{F}_q$ as the abelian group $(\mathbb{Z}/p)^m$, where $q = p^m$ and $p \neq 2$. Consider the (nilpotent) group on the set

$$G := \mathbb{Z}/2 \times \mathbb{F}_q \times (\mathbb{F}_q \wedge_{\mathbb{Z}} \mathbb{F}_q),$$

with operation

$$(n, a, \kappa) \cdot (m, b, \nu) = (n + m, \ (-1)^m a + b, \ \kappa + \nu + [(-1)^m a \otimes b]). \tag{21}$$

Letting the subgroup K be $\mathbb{Z}/2 \times \{0\} \times \{0\} \subset G$ and $x_0 \in K$ be $(1, 0, 0)$, we have the quandle of the form $X = \mathbb{F}_q \times (\mathbb{F}_q \wedge_{\mathbb{Z}} \mathbb{F}_q)$.

The cohomology $H^3(G; \mathbb{F}_q)$ is complicated. In fact, the cohomology has 3-cocycles θ_Γ, which are derived from triple Massey products (see [24, Proposition 4.8]). However, the author [24, Lemma 4.7] showed that the pullback $\varphi_3^* \theta_\Gamma : X^3 \to \mathbb{F}_q$ is formulated as

$$(\varphi_3^* \theta_\Gamma)\big((x, \alpha), \ (y, \beta), \ (z, \gamma)\big) = (x - y)^{q_1} (y - z)^{q_2 + q_3} z^{q_4}.$$

with some prime powers $q_1, q_2, q_3, q_4 \in \mathbb{Z}$. Since this formula is relatively simple, we can compute the relative fundamental 3-class in an easier way than the group-theoretic method; see [24, §5] for a computation.

Example 6.4 Next, we focus on the quandles given in the paper [24] and show Proposition 6.7 below. The result will be useful to show theorems in [27].

Let L be a prime knot or a hyperbolic link, and $\mathcal{O}_\ell$ be the set $\{g^{-1}\mathfrak{m}_\ell g\}_{g \in \pi_L}$. Fix, the inclusion from the link quandle Q_L into $\sqcup_{\ell=1}^{\#L} \mathcal{O}_\ell$ which sends $K_\ell g$ to $g^{-1}\mathfrak{m}_\ell g$. Given a right $\mathbb{Z}[\pi_L]$-module M, we have the semi-direct product $G := M \ltimes \pi_L$. Fix $b_1, b_2, \ldots, b_{\#L} \in M$. Then, for $\ell \leq \#L$, we fix $b_\ell \in M$, and consider the subgroup

$$K_\ell := \big\{ \big(b_\ell(1 - \mathfrak{m}_\ell^s \mathfrak{l}_\ell^t), \ \mathfrak{m}_\ell^s \mathfrak{l}_\ell^t\big) \in M \ltimes \pi_L \, \big| \, s, t \in \mathbb{Z}^2 \big\}.$$

There is a quandle $X = \sqcup_{i \leq \#L} K_\ell \backslash (M \ltimes \pi_L)$ that is in bijective correspondence with $M \times \sqcup_{\ell=1}^{\#L} \mathcal{O}_\ell$, and the quandle structure is equivalent to

$$(a, g) \lhd (b, h) = ((a - b)h + b, \ h^{-1}gh)$$

for $a, b \in M$ and $g, h \in \sqcup_{\ell=1}^{\#L} \mathcal{O}_\ell$. We notice two lemmas:

Lemma 6.5 *If the pair $(\pi_L, \partial \pi_L)$ is malnormal, so is the family $K_1, \ldots, K_{\#L} \subset M \rtimes \pi_L$.*

Lemma 6.6 *Take a π_L-invariant multi-linear map $\psi : M^n \to A$, where π_L trivially acts on A. Let X be the quandle on the union $\sqcup_{j=1}^{\#L} K_\ell \backslash G$. Then, the following map from the normalized complex (see Example 3.6) is an n-cocycle.*

$$\Psi : C_n^{\mathrm{Nor}}(X)_{\mathrm{As}(X)} \longrightarrow A; \quad ((a_0, g_0) \ldots, (a_n, g_n)) \longmapsto \psi(a_0 - a_1, a_1 - a_2, \ldots, a_{n-1} - a_n).$$

Next we consider the pullbacks of φ and α in the diagram (17) for the case $n = 3$. Then, as in [25, Theorem 5.2], we can see that the pullback of the IK map φ is formulated by

$$\varphi_3^*(\Psi)((a_1, z_1), (a_2, z_2), (a_3, z_3)) = \psi\big((a_1 - a_2)(1 - z_2), a_2 - a_3, \ a_3 - a_3 z_3^{-1}\big).$$
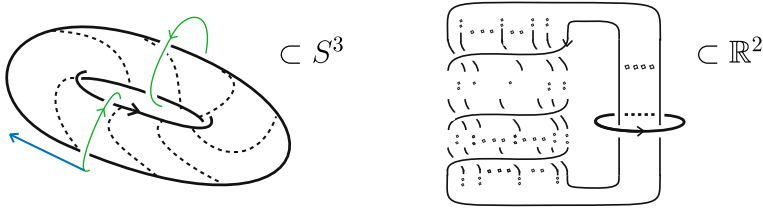
Fig. 5 The torus knot $S_{m,n}$ in the solid torus, and the link diagram on $\mathbb{R}^2$.

Moreover, consider another pullback by α, where we use the expression of α in Example 3.6. Notice that the projection $M \rtimes \pi_L \to M \times \mathcal{O}_\ell$ is represented by the map which takes (m, g) to $(y_\ell g + m, g^{-1} \mathfrak{m}_\ell g)$. Then, in terms of the non-homogenous complex in Example 3.3, the pullback, $\alpha^*(\Psi)$, is represented by

$$\theta_\ell : \big((a_1, g_1), (a_2, g_2), (a_3, g_3)\big) \longmapsto \psi\big((a_1 + y_\ell - y_\ell \cdot g_1)g_2g_3, (a_2 + y_\ell - y_\ell \cdot g_2) \cdot g_3, a_3 + b_\ell - b_\ell \cdot g_3\big), \tag{22}$$

and $\eta_\ell = 0$. By Theorems 2.1 and 7.1, we accordingly have the following:

Proposition 6.7 *As in Theorem* 2.1, *let L be either a hyperbolic link or a prime knot which is neither a cable knot nor a torus knot. Then, the pairing of the 3-cocycle $\alpha^*(\Psi)$ in* (22) *and $[Y_L, \partial Y_L]$ is equal to the quandle cocycle invariant $\langle \varphi_3^*(\Psi), [\mathcal{S}] \rangle$.*

Furthermore, if K is the (n, m)-torus knot, the same equality holds modulo nm.

7 Theorem for cable knots

While Theorems 2.1 and 2.3 assumed non-cabling knots, this section focuses on cable knots. To simplify the study, consider a solid torus $V \subset S^3$ such that $V \backslash L$ is the (m, n)-torus knot. By the formula (16) from the JSJ-decomposition, it is sensible to consider either the torus knot $T_{m,n}$ in S^3 or the solid one in V. We denote the latter by $S_{m,n}$; Regarding $S^3 \backslash (L \sqcup (S^3 \backslash V))$ as a link complement, the knot $S_{m,n}$ admits a link-diagram in $\mathbb{R}^2$; see Fig. 5.

As in the Theorems 2.1 and 2.3, we get similar results modulo some integer. Precisely,

Theorem 7.1 (cf. Theorems 2.1 and 2.3. See Sect. 7.2 for the proof) *Assume that L is either the torus knot $T_{m,n} \subset S^3$ or $S_{m,n} \subset V$. Let $N = n$ if L is $S_{m,n}$, and let $N = mn$ if L is $T_{m,n}$.*

Then, for any relative group 3-cocycle $\theta \in H^3(G, \mathcal{K}; A)$, there is a map $\phi_\theta : \mathrm{Reg}_D \times \mathrm{Arc}_D \times \mathrm{Arc}_D \to A$ for which the following equality holds in the coinvariant $A_G = H_0(G; A)$,:

$$\langle \theta, f_*[E_L, \partial E_L] \rangle = \langle \phi_\theta, [\mathcal{S}_f] \rangle \in A_G \quad \text{modulo the integer } N. \tag{23}$$

Furthermore, if the pair $(G, \mathcal{K})$ is malnormal, then there are a quandle 3-cocycle ϕ_θ and an X-coloring $\mathcal{S}_f$ such that $\langle \theta, f_*[E_L, \partial E_L] \rangle = \langle \phi_\theta, [\mathcal{S}_f] \rangle$ modulo the integer N.

In conclusion, we have a diagrammatic computation for cable knots, although the statement is considered modulo some integer. In addition, we later see that the discussion to show (23) is reduced to the homology of cyclic groups; hence, the pairing modulo N does not have more information than cyclic groups.

7.1 Observations on the torus knots

Before going to the proof, we show two lemmas, and observe an essential reason why the statement is considered modulo N.

Consider the (n, m)-torus knot L in the 3-sphere S^3. Let G be $\pi_1(S^3 \backslash L)$, and let K be the peripheral subgroup. Fix $(a, b, n, m) \in \mathbb{Z}^4$ with $an + bm = 1$. According to [14, § 2],

$$\pi_1(S^3 \backslash T_{n,m}) \cong \langle\, x, y, \mid x^n = y^m \,\rangle \supset \langle\, x^a y^b, (x^a y^b)^{-nm} x^n \,\rangle = K.$$

Then, Theorem 3.10 says that the pair (G, K) is not malnormal. However, regarding the center $Z = \langle x^n \rangle \subset G$, the quotients G/Z and K/Z are isomorphic to the free product $\mathbb{Z}/n * \mathbb{Z}/m$ and $\mathbb{Z}$, respectively. Then, it is known [14, §2] that the pair of the quotients $(G/Z, K/Z)$ is malnormal. We will show the following:

Lemma 7.2 *For $* \geq 2$, there are isomorphisms*

$$H_*^\Delta(K \backslash G; \mathbb{Z}) \cong H_*^\Delta((K/Z) \backslash (G/Z); \mathbb{Z}) \cong H_*^{\mathrm{gr}}(G/Z, K/Z; \mathbb{Z})$$

$$\cong \begin{cases} \mathbb{Z}, & \text{if } * = 2, \\ \mathbb{Z}/nm, & \text{if } * \text{ is odd, and } * \geq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Proof The first isomorphism is obtained by the set-theoretic equality $K \backslash G = (K/Z) \backslash (G/Z)$. The second isomorphism follows from the malnormality. We explain the last one: by a Mayer-Vietoris argument, the homology of $\mathbb{Z}/n * \mathbb{Z}/m$ is that of the pointed sum $L_n^\infty \vee L_m^\infty$, where L_m^∞ is the infinite dimensional lens space with fundamental group $\mathbb{Z}/m$. Hence, the long exact sequence (9) readily leads to the conclusion. □

In summary, since the proofs in this paper often employ the Hochschild homology $H_*^\Delta(K \backslash G)$, it is reasonable to consider the pairing modulo nm.

In addition, we similarly observe another knot $S_{n,m}$ in the solid torus, in details. Fix four integers $(n, m, a, b) \in \mathbb{Z}^4$ with $an + bm = 1$. Consider the following subspace

$$\Big\{ (z, w) \in \mathbb{C}^2 \ \Big| \ |z|^2 + |w|^2 = 1, \quad |z^n + w^m| < \frac{1}{n^m m^n}, \quad |z| < \frac{1}{3} \Big\}.$$

We can easily check that the space is homeomorphic to the (n, m)-torus knot $V \backslash T_{n,m}$. Since the space is regarded as a restriction of a Milnor fibration over S^1, it is an Eilenberg–MacLane space. Furthermore, as in the usual computation of $\pi_1(S^3 \backslash T_{n,m})$, set up the two subsets

$$U_1 := \big\{\, (z, w) \in V \backslash T_{n,m} \ \big| \ |z|^2 \leq 1/2 \,\big\}, \qquad U_2 := \big\{\, (z, w) \in V \backslash T_{n,m} \ \big| \ |z|^2 \geq 1/2 \,\big\}.$$

Since $U_1 \simeq S^1$ and $U_2 \simeq S^1 \times S^1$, a van-Kampen argument (see [6, §15]) can conclude

$$\pi_1(V \backslash T_{n,m}) \cong \langle\, x, z' \mid x^m z' = z' x^m \,\rangle.$$

Here, the longitude $\mathfrak{l}$ is represented by $\mathfrak{m}^{-nm} x^n$, as before. In summary, we can easily obtain the following:

Lemma 7.3 *(i) The center of $\pi_1(V \backslash T_{n,m})$ is generated by x^m and is isomorphic to $\mathbb{Z}$.*
(ii) The quotient group of $\pi_1(V \backslash T_{n,m})$ subject to the center is isomorphic to $\mathbb{Z}/m * \mathbb{Z}$.
(iii) The subgroup generated by the meridian $[\mathfrak{m}]$ is malnormal, and is isomorphic to $\mathbb{Z}$.

Since $S_{n,m}$ is embedded in S^3, we have the link quandle Q_V. As a result, we similarly have

$$H_k^\Delta(Q_V; \mathbb{Z}) \cong H_k^\Delta(K \backslash G; \mathbb{Z}) \cong H_k^{\mathrm{gr}}(\mathbb{Z}/m * \mathbb{Z}, \mathbb{Z}, \mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 2, \\ \mathbb{Z}/m, & \text{if } * \text{ is odd, and } * \geq 3. \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we mention that the inclusion $j : V\backslash T_{n,m} \hookrightarrow S^3\backslash T_{n,m}$ induces $H_3^\Delta(Q_V) \to H_3^\Delta(\mathcal{K}\backslash\mathcal{G})$ as an injection $\mathbb{Z}/m \to \mathbb{Z}/mn$.

7.2 Proof of Theorem 7.1

We will give the proof of Theorem 7.1, based on the discussion in Sect. 7.1.

Proof In this proof, we always deal with homology in torsion coefficients $\mathbb{Z}/N$. Since the latter part from malnormality can be proven in a similar way to Theorem 2.1, we will only show the former statement.

By the above computations of homologies, the chain map α modulo N yields an iso-morphism on $H_*(\bullet; \mathbb{Z}/N)$, which gives a quasi-inverse $\beta : C_*^\Delta(Q_V) \to C_*^{\mathrm{gr}}(\pi_V, \partial\pi_V)$. Furthermore, we suppose a quandle homomorphism $f : Q_{T_{m,n}} \to X$ with $X = H\backslash G$.

Similarly to (17), we have the following commutative diagram by functoriality.

$$
\begin{array}{ccccc}
H_3^R(Q_V; \mathbb{Z}/N) & \xrightarrow{\varphi_*} & H_3^\Delta(Q_V; \mathbb{Z}/N)_{\pi_V} \xleftarrow{\alpha} & H_3^{\mathrm{gr}}(\pi_V, \partial\pi_V; \mathbb{Z}/N) \\
\downarrow{j_*} & & \downarrow{j_*} \quad\quad \beta & \downarrow{j_*} \\
H_3^R(Q_{T_{m,n}}; \mathbb{Z}/N) & \xrightarrow{\varphi_*} & H_3^\Delta(Q_{T_{m,n}}; \mathbb{Z}/N)_{\pi_L} \xleftarrow{\alpha} & H_3^{\mathrm{gr}}(\pi_{T_{m,n}}, \partial\pi_{T_{m,n}}; \mathbb{Z}/N) \\
\downarrow{f_*} & & \downarrow{f_*} \quad\quad \beta & \downarrow{f_*} \\
H_3^R(X; \mathbb{Z}/N) & \xrightarrow{\varphi_*} & H_3^\Delta(X; \mathbb{Z}/N)_G \xleftarrow{\alpha} & H_*^{\mathrm{gr}}(G, K; \mathbb{Z}/N).
\end{array}
\tag{24}
$$

Here, by the above discussion, the middle j_* is reduced to the injection $\mathbb{Z}/m \to \mathbb{Z}/nm$. Hence, if we show that all the left φ_*'s are surjective, then the rest of the proof runs as in the proof of Theorems 2.1 and 2.3.

To show the surjectivity, we now set up appropriate X and f. Although there are many choices of such X and f for the proof, this paper relies on some results in [24] as follows: Take arbitrary prime p which divides m. Choose the minimal $k \in \mathbb{N}$ such that n is not relatively prime to $1 + p^k$. We set up the semi-direct product $G = (\mathbb{Z}/p)^k \rtimes \mathbb{Z}/(1 + p^k)$ and the subgroup $K = \mathbb{Z}/(1 + p^k)$. Then, we can easily construct a group homomorphism $f : \mathbb{Z}/m * \mathbb{Z}/n \to G$ that sends the subgroup $\mathbb{Z}$ to K, and induces the injection $f_* : H_3(\mathbb{Z}/m * \mathbb{Z}/n; \mathbb{Z}/p) \to H_3(G; \mathbb{Z}/p)$ on homology. Thus, it is enough for the surjectivity of the left φ_*'s to show that the left bottom map $\varphi_* : H_3^R(X; \mathbb{Z}/N) \to H_3^\Delta(X; \mathbb{Z}/N)_G$ is injective. However, by noticing that the binary operation on $X = (\mathbb{Z}/p)^k = \mathbb{F}_{p^k}$ is $x \lhd y = \omega(x - y) + y$ for some $\omega \in \mathbb{F}_{p^k}\backslash\{0, 1\}$ the injectivity of X is already shown in the previous paper [24, Lemmas 4.5–4.6], which studies the chain map φ_* for the quandle operation.

As a parallel discussion, when we choose any prime p which divides n, the same injectivity can be shown. To summarize, since such a p is arbitrary, we have shown the surjectivity of the left φ_*'s. Hence, we complete the proof.   □

Finally, we give a corollary:

Corollary 7.4 *Let L be the (m, n)-torus knot, and let E_L be the complement space. Let K be a malnormal subgroup of G. Then, for any relative 3-cocycle θ, the ℓ-torsion part of the pairing $\langle \theta, f_*[E_L, \partial E_L]\rangle$ is zero. Here ℓ is either the prime number coprime to nm or $\ell = 0$.*

In fact, as seen in the above proof, the fundamental 3-class of the torus knot must factor through $H_3^\Delta(Q_V; \mathbb{Z}) \cong \mathbb{Z}/mn$. As a result, for example, if θ is the Chern–Simon 3-class in (14), the free (volume) part of the pairing turns out to be zero.

A Appendix: The third quandle homology of some link quandles

The purpose of this appendix is to determine the third quandle homology of some link quandles (Theorems A.1, A.2). In what follows, we assume the terminology in Sect. 4, and we deal with only integral homology (so we often omit writing $\mathbb{Z}$)

For the purpose, we adopt an approach on the basis of [26, § 8]. Thus, we shall review rack spaces from a quandle X. Consider the orbit decomposition $X = \sqcup_{i \in O(X)} X_i$ from the action of $\mathrm{As}(X)$ on X. For each orbit $i \in O_X$, we fix $x_i \in X_i$. Let Y be either X_i or the single point with their discrete topology. Then, let us consider a disjoint union $\bigsqcup_{n \geq 0} (Y \times ([0, 1] \times X)^n)$, with the following two relations:

$$(y, t_1, x_1, \ldots, x_{j-1}, 0, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (y, t_1, x_1, \ldots t_{j-1}, x_{j-1}, t_{j+1}, x_{j+1}, \ldots, t_n, x_n),$$
$$(y, t_1, x_1, \ldots, t_{j-1}, x_{j-1}, 1, x_j, t_{j+1}, x_{j+1}, \ldots, t_n, x_n)$$
$$\sim (y \triangleleft x_j, t_1, x_1 \triangleleft x_j, \ldots, t_{j-1}, x_{j-1} \triangleleft x_j, t_{j+1}, x_{j+1}, \ldots, t_n, x_n).$$

Then, the *rack space* $B(X, Y)$ is defined to be the quotient space, which is path connected. When Y is a single point, we denote it by BX for short.

We will list some properties on the space from [11]. By observing the cellular complexes, the following isomorphisms are known:

$$H_n^R(X) \cong H_n(BX; \mathbb{Z}), \qquad H_{n+1}^R(X) \cong \bigoplus_{i \in O_X} H_n^R(B(X, X_i); \mathbb{Z}). \qquad (25)$$

Furthermore, concerning fundamental groups, we mention the following isomorphisms [11]:

$$\pi_1(BX) \cong \mathrm{As}(X), \qquad \pi_1(B(X, X_i)) \cong \mathrm{Stab}(x_i) \subset \mathrm{As}(X). \qquad (26)$$

It is shown [11, Proposition 5.2] that the action of $\pi_1(BX)$ on $\pi_*(BX)$ is trivial, and the projection $p : B(X, X_i) \to BX$ is a covering. Therefore, we have functorially the Postnikov tower written in

$$
\begin{array}{ccccccccc}
H_3(\mathrm{Stab}(x_i)) & \longrightarrow & \pi_2(B(X, X_i)) & \longrightarrow & H_2(B(X, X_i)) & \longrightarrow & H_2^{\mathrm{gr}}(\mathrm{Stab}(x_i)) & \longrightarrow & 0 \quad \text{(exact)} \\
\downarrow p_* & & \downarrow p_* \cong & & \downarrow p_* & & \downarrow p_* & & \\
H_3(\pi_1(BX)) & \longrightarrow & \pi_2(BX) & \longrightarrow & H_2(BX) & \longrightarrow & H_2^{\mathrm{gr}}(\pi_1(BX)) & \longrightarrow & 0 \quad \text{(exact).}
\end{array}
$$
$$(27)$$

Furthermore, we mention [20, Theorem 7], which claims the isomorphisms

$$H_2^R(X) \cong H_2^Q(X) \oplus \mathbb{Z}^{\oplus O(X)}, \qquad H_3^R(X) \cong H_3^Q(X) \oplus H_2^Q(X) \oplus \mathbb{Z}^{\oplus O(X) \times O(X)}. \quad (28)$$

Using the above results, we will show the following theorem:

Theorem A.1 *Let Q_L be the link quandle of a non-trivial knot L. Then,*

$$H_3^R(Q_L) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \qquad H_3^Q(Q_L) \cong \mathbb{Z}.$$

Furthermore, the quandle homology $H_3^Q(Q_L) \cong \mathbb{Z}$ is generated by the fundamental 3-class $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$ (recall Example 4.1 *for the definition).*

Proof Note $O(X) = 1$. Then, $\pi_1(B(X, X)) \cong \mathrm{Stab}(x_0)$ is the peripheral group $\cong \mathbb{Z}^2$. Hence, $H_*^{\mathrm{gr}}(\mathrm{Stab}(x_i)) \cong H_*(S^1 \times S^1; \mathbb{Z})$. Furthermore, $\pi_2(BX) \cong \mathbb{Z}^2$ is known [11]. Hence, the above sequence deduces $H_3^R(Q_L) \cong \mathbb{Z}^3$ as desired. Furthermore, since $H_2^R(Q_L) \cong H_1(B(Q_L, Q_L)) \cong H_1^{\mathrm{gr}}(\mathrm{Stab}(x_i)) \cong \mathbb{Z}^2$ (which is already known [11,28]), we have $H_2^Q(Q_L) \cong \mathbb{Z}$ by (26). Hence, we obviously obtain $H_3^Q(Q_L) \cong \mathbb{Z}$ from (28).

We complete the remaining proof on the generator. Consider a chain map $q : C_n^R(X) \to C_{n-1}^R(X)$ induced by $(x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$. We can easily check that the map on the homology level coincides with the above $p_* : H_n(BX) \to H_{n-1}(BX)$. Here note the fact [11,28] that $H_2^Q(Q_L) \cong \mathbb{Z}$ is generated by the 2-class $q_*[\mathcal{S}_{\mathrm{id}_{Q_L}}]$. Hence, $H_3^Q(Q_L)$ must be generated by the 3-class $[\mathcal{S}_{\mathrm{id}_{Q_L}}]$. □

Next, we will deal with the link case. As seen in Remark 6.1 or Seifert pieces, it is complicated to deal with $H_3^Q(Q_L)$ of every links. Let us assume a property:

The centralizer subgroup of each peripheral group $\mathcal{P}_i \subset \pi_L$ is equal to $\mathcal{P}_i$. (†).

For example, if L is hyperbolic or if the subgroups $\mathcal{P}_i$ are malnormal, the property holds.

Theorem A.2 *Let $L \subset S^3$ be a non-splitting link with the property* (†)*. Then,*

$$H_3^R(Q_L) \cong \mathbb{Z}^{\#L(\#L+2)}, \quad H_3^Q(Q_L) \cong \mathbb{Z}^{\#L}.$$

Proof First, we show $H_2^R(Q_L) \cong \mathbb{Z}^{2\#L}$. By (†), we can easily verify $\mathrm{Stab}(x_i) \cong \mathbb{Z}^2$ and $|O(Q_L)| = \#L$. Hence, it follows from (25) that $H_2^R(Q_L) \cong \oplus_{i \in O(X)} H_1^{\mathrm{gr}}(\mathrm{Stab}(x_i)) \cong \mathbb{Z}^{2\#L}$ as desired.

Next, we show $\pi_2(BX) \cong \mathbb{Z}^{\#L+1}$. Since L is non-splitting, $S^3 \setminus L$ is an Eilenberg-MacLane space of type $(\pi_L, 1)$; see, e.g., [2]. Accordingly, $H_*^{\mathrm{gr}}(\mathrm{As}(Q_L)) \cong H_*(S^3 \setminus L)$. Thus, $H_3^{\mathrm{gr}}(\mathrm{As}(Q_L)) \cong 0$ and $H_2^{\mathrm{gr}}(\mathrm{As}(Q_L)) \cong \mathbb{Z}^{\#L-1}$. By (27) and $H_2(BQ_L) \cong \mathbb{Z}^{2\#L}$, we obviously have $\pi_2(BQ_L) \cong \mathbb{Z}^{\#L+1}$.

Next, we will complete the proof. Since $\pi_2(B(X, X_i)) \cong \mathbb{Z}^{\#L+1}$ and $\mathrm{Stab}(x_i) \cong \mathbb{Z}^2$, we have $H_2(B(X, X_i)) \cong \mathbb{Z}^{\#L+2}$ by (27). Hence, we have $H_3^R(Q_L) \cong \mathbb{Z}^{\#L(\#L+2)}$ by (25) with $n = 2$. Furthermore, regarding $H_3^Q(Q_L)$, the proof is readily due to (28). □

Finally, we prove Lemma A.3 which is used in Sect. 6. In what follows, we use the notation in Sect. 6. More precisely, we should recall the associated quandles Q_L and $Q_{L'}$ of links L and L', respectively, and employ the homogenous quandle of the form $Q_W = (K_1 \backslash \pi_L) \sqcup (K_2 \backslash \pi_L)$, together with the injections $i_* : Q_L \to Q_W$ and $j_* : Q_{L'} \to Q_W$; see Sect. 6 for the details. Furthermore, we assume $L \subset L'$ and the hyperbolicity of $S^3 \backslash L \setminus (S^3 \backslash L')$.

Lemma A.3 *Assume that L' is a hyperbolic link. Then, $H_3^Q(Q_W) \cong \mathbb{Z}^2$. Furthermore, the induced maps on the third homology level*

$$i_*^Q : H_3^Q(Q_L) \longrightarrow H_3^Q(Q_W), \quad \text{and} \quad j_*^Q : H_3^Q(Q_{L'}) \longrightarrow H_3^Q(Q_W)$$

are given by $(1, 0)$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, respectively.

Proof By the form of Q_W, we can easily show $\mathrm{As}(Q_W) \cong \pi_1(S^3 \backslash L)$. Further, notice the amalgamation

$$\pi_L = \pi_1(S^3 \backslash V_1) *_{\pi_1(\partial(S^3 \backslash V_1))} \pi_1(V_1 \backslash L).$$

Since $V_1 \backslash L$ is hyperbolic, and $S^3 \backslash V_1$ is a knot, the centralizer of $\pi_1(\partial(S^3 \backslash V_1))$ in π_L is itself isomorphic to $\mathbb{Z}^2$. Thus, $\mathrm{Stab}(x_i) \cong \mathbb{Z}^2$ and $|O(X)| = 2$. Therefore, the isomorphism (25) readily implies $H_2^R(Q_W) \cong \mathbb{Z}^4$. Then, by the same discussion of Theorem A.2, we can show $H_3^Q(Q_W) \cong \mathbb{Z}^2$.

Furthermore, by the above proofs, we already know the basis of second quandle homologies, as the stabilizer subgroups. Thus, by functoriality from boundary inclusions, we can check the matrix presentations. $\qquad\square$

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