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ORIGINAL PAPER

On open flat sets in spaces with bipolar comparison

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Abstract
We show that if a Riemannian manifold satisfies (3,3)-bipolar comparisons and has an open flat subset then it is flat. The same holds for a version of MTW where the perpendicularity is dropped. In particular we get that the (3,3)-bipolar comparison is strictly stronger than the Alexandrov comparison.

Keywords Metric geometry · Optimal transport · Differential geometry · Rigidity · Comparison geometry

1 Introduction

We say that a metric space X satisfies the (k, l)-*bipolar comparison* if for any $a_0, a_1, \dots, a_k$; $b_0, b_1, \dots, b_l \in X$ there are points $\hat{a}_0, \hat{a}_1, \dots, \hat{a}_k, \hat{b}_0, \hat{b}_1, \dots, \hat{b}_l$ in the Hilbert space $\mathbb{H}$ such that

$$|\hat{a}_0 - \hat{b}_0|_{\mathbb{H}} = |a_0 - b_0|_X, \quad |\hat{a}_i - \hat{a}_0|_{\mathbb{H}} = |a_i - a_0|_X, \quad |\hat{b}_i - \hat{b}_0|_{\mathbb{H}} = |b_i - b_0|_X$$

for any i, j and

$$|\hat{x} - \hat{y}|_{\mathbb{H}} \geqslant |x - y|_X$$

for any $x, y \in \{a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_l\}$.

This definition was introduced in [5]. The class of compact length metric spaces satisfying $(k, 0)$-bipolar comparison with $k \geqslant 2$ coincide with the class of Alexandrov spaces with nonnegative curvature, (for $k = 2$ it is just one of the equivalent definitions, for arbitrary k see [1], [3]). In general (k, l)-bipolar comparisons (with k or $l \geqslant 2$) for length metric spaces are stronger conditions than nonnegative curvature condition and they describe some new interesting classes of spaces. In particular, we prove in [5] that for Riemannian manifolds (4, 1)-bipolar comparison is equivalent to the conditions related to the continuity of optimal transport. Also in [5] we together with coauthors describe classes of Riemannian

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manifolds satisfying (k, l)–bipolar comparisons for almost all k, l excepting (2, 3) and (3, 3)-bipolar comparisons. In particular it was not known if (3, 3)-bipolar comparison differs from Alexandrov's comparison. In this note the affirmative answer is obtained as a corollary of some rigidity result for spaces with (3, 3)-bipolar comparison. To formulate exact statements we need some definitions and notations.

Let M be a Riemannian manifold and $p \in M$. The subset of tangent vectors $v \in \mathrm{T}_p$ such that there is a minimizing geodesic $[p\,q]$ in the direction of v with length $|v|$ will be denoted as $\overline{\mathrm{TIL}}_p$. The interior of $\overline{\mathrm{TIL}}_p$ is denoted by TIL_p; it is called *tangent injectivity locus* at p. If at TIL_p is convex for any $p \in M$, then M is called CTIL.

Riemannian manifold M satisfies MTW if the following holds. For any point $p \in M$, any $W \in \mathrm{TIL}_p$ and tangent vectors $X, Y \in \mathrm{T}_p$, such that $X \perp Y$ we have

$$\frac{\partial^4}{\partial^2 s\, \partial^2 t} \left|\exp_p(s \cdot X) - \exp_p(W + t \cdot Y)\right|^2_M \leqslant 0 \tag{1}$$

at $t = s = 0$.

This definition was introduced by Xi-Nan Ma, Neil Trudinger and Xu-Jia Wang in [7], Cedric Villani studied a synthetic version of this definition ([9]). If the same inequality holds without the assumption $X \perp Y$ Riemannian manifold M satisfies $\mathrm{MTW}^{\not\perp}$ [2].

MTW and CTIL are necessary condition for TCP (transport continuity property). In [2], Alessio Figalli, Ludovic Rifford and Cédric Villani showed that a strict version of CTIL and MTW provide a sufficient condition for TCP. A compact Riemannian manifold M is called TCP if for any two regular measures with density functions bounded away from zero and infinity the generalized solution of Monge–Ampère equation provided by optimal transport is a genuine (continuous) solution.

Let us denote by $\mathcal{M}_{(k,l)}$ the class of smooth complete Riemannian manifolds satisfying (k, l)–bipolar comparison and by $\mathcal{M}_{\geqslant 0}$ the class of complete Riemannian manifolds with nonnegative sectional curvature.

It was mentioned above, that

$$\mathcal{M}_{\geqslant 0} = \mathcal{M}_{(k,0)}$$

for $k \geqslant 2$ and it is obvious from definition, that

$$\mathcal{M}_{(k',l')} \subset \mathcal{M}_{(k,l)}$$

if $k' \geqslant k$ and $l' \geqslant l$. It is proven in [5] that

$$\mathcal{M}_{\geqslant 0} = \mathcal{M}_{(2,2)} = \mathcal{M}_{(3,1)}$$

and

$$\mathcal{M}_{(4,1)} = \mathcal{M}_{(k,l)}$$

for $k \geqslant 4$ and $l \geqslant 1$. The most interesting fact proven in [5] is that

$$\mathcal{M}_{(4,1)} = \mathcal{M}_{CTIL} \cap \mathcal{M}_{MTW^{\not\perp}},$$

where $\mathcal{M}_{CTIL}, \mathcal{M}_{MTW^{\not\perp}}$ are classes of smooth Riemannian manifolds satisfying CTIL and $\mathrm{MTW}^{\not\perp}$ correspondingly. In particular this implies that $\mathcal{M}_{(4,1)} \neq \mathcal{M}_{\geqslant 0}$.

In this paper we prove the following two results.

Theorem 1.1 *Let M be a complete Riemannian manifold that satisfies (3,3)-bipolar comparison and contains a nonempty open flat subset. Then M is flat.*

Theorem 1.2 *Let M be a complete Riemannian manifold that satisfies MTW$^{\not\perp}$ and contains a nonempty open flat subset. Then M is flat.*

Corollary 1.3 *We have that $\mathcal{M}_{(3,3)} \neq \mathcal{M}_{\geqslant 0}$.*

Theorem 1.1 follows from Proposition 2.2 and Theorem 1.2 follows from Proposition 2.3, proved in the next section.

As a related result we would like to mention a rigidity result for manifolds with nonnegative sectional curvature with flat open subsets by Dmitri Panov and Anton Petrunin [8].

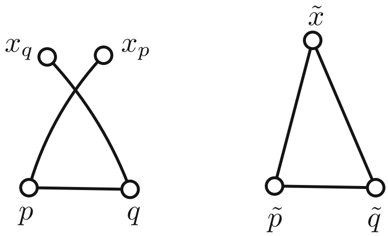
2 Proofs

For points a, b, c in a manifold we denote by $\measuredangle[a\,{}^{b}_{c}]$ the angle at a of the triangle $[abc]$.

Key lemma 2.1 *Let M be a complete Riemannian manifold that satisfies (3,3)-bipolar comparison. Assume that for the points x_p, p, q, x_q in M there is a triangle $[\tilde{p}\tilde{q}\tilde{x}]$ in the Euclidean plane $\mathbb{E}^2$ such that*

$$|x_p - p|_M = |\tilde{x} - \tilde{p}|_{\mathbb{E}^2}, \quad |p - q|_M = |\tilde{p} - \tilde{q}|_{\mathbb{E}^2}, \quad |q - x_q|_M = |\tilde{q} - \tilde{x}|_{\mathbb{E}^2}$$

and moreover a neighborhood $N \subset \mathbb{E}^2$ of the base $[\tilde{p}\tilde{q}]$ admits a globally isometric embedding ι into M such that $\iota([\tilde{p}\tilde{x}] \cap N) \subset [px_p]$ and $\iota([\tilde{q}\tilde{x}] \cap N) \subset [qx_q]$. Then $x_p = x_q$ and the triangle $[pqx_p]$ can be filled by a flat geodesic triangle.



Proof Set $p_- = p$ and $q_- = q$.

Choose points $p_0, p_+ \in [p_-, x_p] \cap \iota(N)$ so that the points p_-, p_0, p_+, x_p appear in the same order on $[p_-, x_p]$. Analogously, choose points $q_0, q_+ \in [q_-, x_q] \cap N$ so that the points q_-, q_0, q_+, x_p appear in the same order on $[q_-, x_q]$. Denote by $\tilde{p}_-, \tilde{p}_0, \tilde{p}_+, \tilde{q}_-, \tilde{q}_0, \tilde{q}_+$ the corresponding points on the sides of triangle $[\tilde{p}\tilde{q}\tilde{x}]$; so $\tilde{p}_- = \tilde{p}$ and $\tilde{q}_- = \tilde{q}$.

Applying the comparison to $a_0 = p_0, a_1 = p_-, a_2 = p_+, a_3 = x_p$; $b_0 = q_0, b_1 = q_-, b_2 = q_+, b_3 = x_q$, we get a model configuration $\hat{p}_0, \hat{p}_-, \hat{p}_+, \hat{x}_p, \hat{q}_0, \hat{q}_-, \hat{q}_+, \hat{x}_q$ in the Hilbert space $\mathbb{H}$.

Note that from the comparison it follows that the quadruple $\hat{p}_-, \hat{p}_0, \hat{p}_+, \hat{x}_p$ lies on one line and the same holds for the quadruple $\hat{q}_-, \hat{q}_0, \hat{q}_+, \hat{x}_q$.

Since

$$|\hat{p}_0 - \hat{q}_+|_{\mathbb{H}} \geqslant |p_0 - q_+|_M = |\tilde{p}_0 - \tilde{q}_+|_{\mathbb{E}^2}, \quad |\hat{p}_0 - \hat{q}_0|_{\mathbb{H}} = |p_0 - q_0|_M = |\tilde{p}_0 - \tilde{q}_0|_{\mathbb{E}^2},$$
$$|\hat{q}_0 - \hat{q}_+|_{\mathbb{H}} = |q_0 - q_+|_M = |\tilde{q}_0 - \tilde{q}_+|_{\mathbb{E}^2},$$

we have $\measuredangle[\hat{q}_0\,{}^{\hat{p}_0}_{\hat{q}_+}] \geqslant \measuredangle[\tilde{q}_0\,{}^{\tilde{p}_0}_{\tilde{q}_+}]$. The same way we get that $\measuredangle[\hat{q}_0\,{}^{\hat{p}_0}_{\hat{q}_-}] \geqslant \measuredangle[\tilde{q}_0\,{}^{\tilde{p}_0}_{\tilde{q}_-}]$. Since the sum of adjacent angles is π, these two inequalities imply that

$$\measuredangle[\hat{q}_0\,{}^{\hat{p}_0}_{\hat{q}_\pm}] = \measuredangle[\tilde{q}_0\,{}^{\tilde{p}_0}_{\tilde{q}_\pm}].$$

The same way we get that

$$\measuredangle[\hat{p}_0\,{}^{\hat{q}_0}_{\hat{p}_\pm}] = \measuredangle[\tilde{p}_0\,{}^{\tilde{q}_0}_{\tilde{p}_\pm}].$$

From the angle equalities, we get that

$$|\hat{p}_- - \hat{q}_+|_{\mathbb{H}} \leqslant |\tilde{p}_- - \tilde{q}_+|_M \tag{1}$$

and the equality holds if the points $\hat{p}_-, \hat{q}_+$ lie in one plane and on the opposite sides from the line $\hat{p}_0\hat{q}_0$. By (3,3)-bipolar comparison the equality in 1 indeed holds.

It follows that configuration $\hat{p}_0, \hat{p}_-, \hat{p}_+, \hat{x}_p, \hat{q}_0, \hat{q}_-, \hat{q}_+, \hat{x}_q$ is isometric to the configuration $\tilde{p}_0, \tilde{p}_-, \tilde{p}_+, \tilde{x}, \tilde{q}_0, \tilde{q}_-, \tilde{q}_+, \tilde{x}$; in particular, $\hat{x}_q = \hat{x}_p$.

By (3,3)-bipolar comparison $|x_p - x_q|_M \leqslant |\hat{x}_q - \hat{x}_p|_{\mathbb{H}}$; therefore $x_p = x_q$; so we can set further $x = x_p = x_q$.

Note that we also proved that the angles at p and q in the triangle $[pqx]$ coincide with their model angles; that is,

$$\measuredangle[p\,{}^{q}_{x}] = \measuredangle[\tilde{p}\,{}^{\tilde{q}}_{\tilde{x}}], \quad \measuredangle[q\,{}^{p}_{x}] = \measuredangle[\tilde{q}\,{}^{\tilde{p}}_{\tilde{x}}].$$

By the lemma on flat slices (see for example Lemma 2.1 in [4]), there is a global isometric embedding ι' of the solid model triangle $[\tilde{p}\tilde{q}\tilde{x}]$ to M which sends $[\tilde{p}\tilde{q}]$ to $[pq]$ and $[\tilde{p}\tilde{x}]$ to $[px]$. Note that ι' has to coincide with ι on N. It follows that ι' maps $[\tilde{q}\tilde{x}]$ to $[qx]$, which finishes the proof. □

Theorems 1.1 and 1.2 follow from the propositions below.

Proposition 2.2 *Let M be a complete Riemannian manifold that satisfies (3,3)–bipolar comparison. Then any point $x \in M$ admits a neighborhood $U \ni x$ such that if U contains a nonempty open flat subset, then U is flat.*

Proof Given a point p consider a convex neighborhood $U \ni p$ such that injectivity radius at any point of U exceeds the diameter of U; in particular any two points $p, q \in U$ are connected by unique minimizing geodesic $[pq]$ which lies in U. Denote by F an open flat subset in U; we can assume that F is convex. □

Note that by the key lemma we have the following:

Claim *For any $x \in U$ and any $p, q \in F$ the triangle $[pqx]$ admits a geodesic isometric filling by a flat triangle.*

Indeed, set $x_p = x$. Consider a plane triangle $[\tilde{p}\tilde{q}\tilde{x}]$ that has the same angle at $\tilde{p}$ and the same adjacent sides as the triangle $[pqx]$. Since F is flat and convex there is a flat open geodesic surface Σ containing $[pq]$ and a part of $[px]$ near p. Choose a direction at q that runs in Σ at the angle $\measuredangle[\tilde{q}\,{}^{\tilde{p}}_{\tilde{x}}]$ to $[qp]$. Consider the geodesic in this direction of the length $|\tilde{q}\tilde{x}|$. Since diameter of U exceeds the injectivity radius at q, this geodesic is minimizing. It remains to apply the key lemma.

From the claim, it follows that the sectional curvature $\sigma_x(X, Y)$ vanishes for any point $x \in U$ and any two velocity vectors $X, Y \in \mathrm{T}_x$ of minimizing geodesics from x to F. Since the set of such sectional directions is open, curvature vanish at x; hence the result.

Proposition 2.3 *Let M be a complete Riemannian manifold that satisfies $MTW^{\not\perp}$. Then any point $p \in M$ admits a neighborhood $U \ni p$ such that if U contains a nonempty open flat subset, then U is flat.*

Proof For a given $p \in M$ let us take a neighborhood $U \ni p$ as in the proof of the previous proposition. The same proof as (Thm 1.2 [5]) shows that U satisfies (4, 1)-bipolar comparison (CTIL condition is not necessary, because we stay away from the cut-locus). Again, same proof as (the Thm 1.2 [5]) shows that inside this neighborhood (4, 1)-bipolar comparison is equivalent to (4, 4)-bipolar comparison. Further note that (4, 4)-bipolar comparison implies (3, 3)-bipolar comparison. Now we can follow the same lines as in the proof of Proposition 2.2, because (3, 3)-bipolar comparison is used only locally in the proof. □

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