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ORIGINAL PAPER

Vanishing zones and the topology of non-isolated singularities

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Abstract
We compare the topology of the link L_0 of non-isolated singularities defined by real analytic map-germs $(\mathbb{R}^m, 0) \xrightarrow{h} (\mathbb{R}^n, 0)$, $m > n$, with that of the boundary of a local non-critical level of h. We show that if the germ of h has an isolated critical value at $0 \in \mathbb{R}^n$ and admits a local Milnor-Lê fibration at 0, then there exists "a vanishing zone for h". This is an appropriate neighborhood of the set $L_0 \cap \Sigma$, where Σ denotes the critical set of h, such that away from it the topology of L_0 is fully determined by the boundary of the corresponding local Milnor fibre. We give conditions for the vanishing zone to be a fiber bundle over $L_0 \cap \Sigma$. A particular class of real singularities we envisage in this paper are those of the type $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with f, g holomorphic and satisfying certain conditions. We introduce for these a regularity criterium for having a local Milnor-Lê fibration, and we use this to produce an example of a real analytic singularity which does not have the Thom a_f-property and yet has a local Milnor-Lê fibration. Throughout this work we provide explicit examples of functions satisfying the hypothesis we need in each section.

Keywords Milnor-Lê fibration · Milnor fiber · Link · Vanishing zone · Lê's polyhedron · Non-isolated singularity

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1 Introduction

Given an analytic map-germ $(\mathbb{R}^m, 0) \xrightarrow{h} (\mathbb{R}^n, 0), m > n$, with an isolated critical value at 0, a fundamental problem is understanding the way how the non-critical levels $h^{-1}(t)$ degenerate to the special fiber $V := h^{-1}(0)$. For instance, when h is holomorphic with values in $\mathbb{C}$, the celebrated fibration theorem of Milnor [20] (refined by Lê Dũng Tráng in [12]) says that one has a locally trivial fibration:

$$h^{-1}\left(\mathbb{D}_\eta \backslash \{0\}\right) \cap \mathbb{B}_\epsilon \xrightarrow{h} \mathbb{D}_\eta \backslash \{0\},$$

where $\mathbb{B}_\epsilon$ denotes a sufficiently small ball around the origin and $\mathbb{D}_\eta$ is a sufficiently small disc around 0 in $\mathbb{C}$. The set $N(\epsilon, \eta) := h^{-1}(\mathbb{D}_\eta \backslash \{0\}) \cap \mathbb{B}_\epsilon$ is usually called a Milnor tube for h, and the fibers $F_t := h^{-1}(t) \cap \mathbb{B}_\epsilon$, $t \neq 0$, are now called the Milnor fibers of h. A lot of interesting work has been done studying how this degeneration process $F_t \rightsquigarrow F_0$, where $F_0 := V \cap \mathbb{B}_\epsilon$, takes place for holomorphic map-germs. For instance, a remarkable theorem of Lê Dũng Tráng [13] (see also [14]) says that when h further has an isolated critical point, inside each F_t one has a polyhedron P_t of middle dimension, which "collapses" as we approach the special fiber, and the complement $F_t \backslash P_t$ is diffeomorphic to $F_0 \backslash \{0\}$.

For real analytic map-germs the problem of studying Milnor fibrations is still in its childhood and it is an active field of current research (see for instance [15] and the survey paper [8]). We briefly discuss that topic below.

In this article we take an alternative viewpoint to the problem of studying how the non-critical levels degenerate to the special fiber. This springs from work by Siersma [32], Michel and Pichon [16–19], A. Nemethi and A. Szilard [21] and J. Fernándes de Bobadilla and A. Menegon [9]. For this we recall that a real analytic map-germ h as above has its *link*, which by definition is $L_0 := h^{-1}(0) \cap \mathbb{S}_\epsilon$, the intersection of V with a sufficiently small sphere. The link and its embedding in $\mathbb{S}_\epsilon$ determine fully the topology of V at 0 and its local embedding in the ambient space (cf. [20]).

The link is a real analytic variety, so L_0 is non-singular if h has an isolated critical point at 0. In that case L_0 is a smooth manifold, isotopic to the boundary L_t of the Milnor fiber F_t. Otherwise, when h has a non-isolated critical point on V, the variety L_0 may be singular: That is the setting we envisage in this paper.

Given an analytic map-germ h as above, consider a Milnor tube $N(\epsilon, \eta) := h^{-1}(\mathbb{D}_\eta \backslash \{0\}) \cap \mathbb{B}_\epsilon$, and let us assume this is a fiber bundle over $\mathbb{D}_\eta \backslash \{0\}$ with projection h (unlike in the complex setting, this hypothesis is not always satisfied). The fibers $F_t := h^{-1}(t) \cap \mathbb{B}_\epsilon$, $t \neq 0$, are compact manifolds with boundary L_t. While the family $\{F_t\}$ degenerates into the special fiber $F_0 := h^{-1}(t) \cap \mathbb{B}_\epsilon$, one also has the corresponding family of boundaries $\{L_t\}_{t \neq 0}$ degenerating to the link L_0, which may be singular. The purpose of this work is to study the topology of both L_t and L_0, endowing them with a good topological structure that may allow a further study of the degeneration process $\{L_t\}_{t \neq 0} \rightsquigarrow L_0$ for both, real and complex singularities.

This is interesting for two reasons. On the one hand the boundary of the Milnor fiber, being a smooth manifold, is in many ways easier to handle than the link; understanding the way how L_t degenerates into L_0 throws light into the topology of the link, and hence into that of V, just as the study of the vanishing cycles on the Milnor fiber throws light into the topology of the special fiber. On the other hand, we can argue conversely: Understanding the degeneration $L_t \rightsquigarrow L_0$ allows us to re-construct L_t out from L_0 itself, together with some additional information. For instance, this was the approach followed in [9,16,18,21] to show that in the case of holomorphic map-germs in 3 complex variables, the boundary L_t

is a Waldhausen manifold. This was extended in [9] to map germs of the form $f\bar{g}$, that we discuss below.

As noted by Nemethi and Szilard [21] in their interesting book, while the study of links of isolated complex hypersurface singularities has lead to remarkable discoveries and insights in manifolds theory, so too the boundaries of Milnor fibers of complex analytic map-germs with non-isolated singularities provide a rich source of interesting manifolds that may bring new insights into manifolds theory. And the case of singularities $f\bar{g}$ provides an even larger class of manifolds with a rich geometry and topology.

Our first main result in this paper (statement (i) in Theorem 2.5) says that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is holomorphic with a critical point at 0, then there exists a *vanishing zone* for it. This means a compact regular neighborhood W of the set $L(\Sigma) := \Sigma \cap \mathbb{S}_\epsilon$ in $\mathbb{S}_\epsilon$, with smooth boundary, where Σ is the critical set of h and $\mathbb{S}_\epsilon$ is a Milnor sphere for f, such that for all t with $\|t\|$ sufficiently small, one has that $L_t \setminus \mathring{W}$ is diffeomorphic to $L_0 \setminus \mathring{W}$, where $\mathring{W}$ is the interior of W. Hence outside the vanishing zone W, the link L_0 is smooth and diffeomorphic to that part of the boundary of the Milnor fiber which lies outside W.

Then one must focus on what happens inside the vanishing zone W. Inspired by [9,16–19,21] we observe that under appropriate conditions, which are always satisfied if $n \geq 3$ and the critical set Σ of f has complex dimension 1, one indeed has a good control on what happens inside W. In fact, we prove that if the link $L(\Sigma)$ is smooth then W can be chosen to be a fiber bundle over $L(\Sigma)$ with fiber an $2(n-k)$-dimensional disk, where $k > 0$ is the dimension of Σ. And if we assume further that there exists a Whitney stratification of $V := f^{-1}(0)$ such that each connected component of $\Sigma \setminus \{0\}$ is a single stratum, then we prove that the portion of L_0 contained in W is a fiber bundle over $L(\Sigma)$ with fiber $V(f) \cap H$, where H is a $(n-k)$-dimensional complex slice transversal to $L(\Sigma)$. For $t \neq 0$, the portion of the boundary L_t contained in W is a fiber bundle over $L(\Sigma)$ with fiber the Milnor fiber of $V(f) \cap H$, that is, $L_t \cap H$.

In particular, if Σ has complex dimension 1, then $L(\Sigma)$ is a union of circles in the $(2n-1)$-sphere $\mathbb{S}_\epsilon$ and W is a union of products $\mathbb{S}^1 \times \mathbb{B}^{2n-2}$, since every circle in an oriented manifold has trivial normal bundle. Moreover, both W_0 and W_t are fiber bundles over the circle, with fiber given by the central and the Milnor fiber, respectively, of the restriction of f to a generic hyperplane section H.

As a corollary of Theorem 2.5, we get that if f is as above, then the boundary of the Milnor fiber is not homeomorphic to the link L_0. This applies, in particular, to any map-germ $\mathbb{C}^3 \to \mathbb{C}$.

The rest of this article, starting from Sect. 3, carries the previous discussion into the real analytic setting. Besides providing information about the topology of real singularities, which is a hard subject, this also gives, by comparison, a better understanding of the complex setting.

In Sect. 3 we look at real analytic singularities with a Milnor-Lê fibration in a tube. This implies that we have a family of manifolds (the Milnor fibers) degenerating into the special fiber, while their boundaries L_t degenerate into the corresponding link L_0, as in the complex setting. We give sufficient conditions that grant the existence of a vanishing zone for such map-germs.

In Sect. 4 we study map-germs of the type $f\bar{g}$ where f and g are holomorphic functions $\mathbb{C}^n \to \mathbb{C}$. These maps provide a specially interesting class of real analytic functions which share some properties of holomorphic map-germs and one can use complex geometry to study them. From the viewpoint of Milnor fibrations, these have been studied mainly by Pichon and Seade [25–29], though they appear also in previous work by A'Campo [1] and

Lee Rudolph [30]. These are a special class of "mixed singularities", studied by M. Oka and others (see [22,31]).

In Sect. 4.1 we discuss the existence of Milnor-Lê fibrations for real analytic map-germs of the form $f\bar{g}$, following [9,27]. We define, inspired by [9], a regularity condition that we call CT-regularity, that guarantees the existence of a Milnor-Lê fibration for $f\bar{g}$ (Proposition 4.2). This is very useful in practice, since it is easy to verify the condition in explicit examples. In particular we get an example where the map $f\bar{g}$ has a Milnor-Lê fibration in a tube but it does not have the Thom $a_{f\bar{g}}$-property, thus answering a question raised in [7, Remark 4.4] and in [3]. We notice that this is answered too in Oka's paper [23], as well as in the recent paper [24] by Parameswaran and Tibăr.

The discussion in Sects. 3 and 4.1 leads to Corollary 4.7: Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two holomorphic germs with no common irreducible component, such that $f\bar{g}$ has a Milnor-Lê fibration and $\Sigma\backslash\{0\}$ is smooth. Then there exists a vanishing zone W for $f\bar{g}$ which can be chosen to be a fiber bundle over $L(\Sigma)$ with fiber an 2-dimensional complex disk. If we further suppose that the Whitney stratification of V can be chosen so that $\Sigma\backslash\{0\}$ is a non-empty single stratum, then the intersection $L_0 \cap W$ is a bundle over $L(\Sigma)$ with fiber the singular variety defined by $f\bar{g}$ on a transversal slice, while $L_t \cap W$ is a bundle over $L(\Sigma)$ with fiber the corresponding Milnor fiber. We give examples of maps $f\bar{g}$ satisfying the hypotheses of Corollary 4.7.

As in the holomorphic case, we have as consequence (Theorem 4.11) that the boundary L_t of the Milnor fiber of $f\bar{g}$ is not homeomorphic to its link.

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2 The holomorphic case

In this section, we consider a holomorphic function-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. It is well-known that f has the Thom a_f-property (by [10]) and hence it has a Milnor-Lê fibration in a Milnor tube (cf. [12,27]):

$$f_| : f^{-1}\left(\mathbb{D}_\eta\backslash\{0\}\right) \cap \mathbb{B}_\epsilon \to \mathbb{D}_\eta\backslash\{0\}\,,$$

with $0 < \eta \ll \epsilon$. Set $V := f^{-1}(0)$ and let:

$$L_0 := V \cap \mathbb{S}_\epsilon$$

be the link of f. We denote by:

$$L(\Sigma) := \Sigma \cap \mathbb{S}_\epsilon$$

the link of the critical set Σ of f, where $\mathbb{S}_\epsilon$ is a Milnor sphere for f. By L_t we denote the boundary of a Milnor fiber F_t, so:

$$L_t := f^{-1}(t) \cap \mathbb{S}_\epsilon\,,$$

for $t \in \mathbb{D}_\eta\backslash\{0\}$. Then L_t is a smooth submanifold of $\mathbb{S}_\epsilon$ that degenerates to the link L_0 as $|t|$ goes to 0.

Following [9,16,21,32], we aim to study the topology of L_t and that of L_0.

2.1 The vanishing zone

We want to define and show the existence of a vanishing zone for f. For this, our starting point uses the idea of a *cellular tube* of a subvariety of a manifold M, as defined by Brasselet in [5].

Lemma 2.1 *There exists a compact regular neighborhood W of $L(\Sigma)$ in $\mathbb{S}_\epsilon$ such that its boundary ∂W is smooth and intersects L_0 transversally in $\mathbb{S}_\epsilon$, in the stratified sense. Moreover, if Σ is either a smooth submanifold or it has an isolated singularity, then W is a fiber bundle over $L(\Sigma)$ with fiber a $2(n-k)$-dimensional ball in $\mathbb{S}_\epsilon$, where k is the complex dimension of Σ.*

Proof Let us consider a Whitney stratification of $\mathbb{S}_\epsilon$ such that L_0 is a union of strata, and a triangulation (K) of $\mathbb{S}_\epsilon$ such that each stratum is a union of simplices. Let (K') be the barycentric decomposition of (K). Using (K') one constructs the associated cellular dual decomposition (D) of $\mathbb{S}_\epsilon$ as follows: Given a simplex σ in (K) of dimension s, its dual $d(\sigma)$ is the union of all simplices τ in (K') whose closure meets σ exactly at its barycenter $\hat{\sigma}$, *i.e.*,

$$\bar{\tau} \cap \sigma = \hat{\sigma}.$$

This is a cell of dimension $(2n - s - 1)$. Taking the union of all these dual cells we get the dual decomposition (D) of (K). By construction, each cell σ intersects its dual $d(\sigma)$ transversally. We let W be the union of cells in (D) which are dual of simplices in $L(\Sigma)$; it provides a cellular tube around $L(\Sigma)$ in $\mathbb{S}_\epsilon$, which means it satisfies the following properties:

(i) W is a compact neighborhood of $L(\Sigma)$ containing $L(\Sigma)$ in its interior, and ∂W is a retract of $W \backslash L(\Sigma)$;
(ii) W retracts to $L(\Sigma)$;
(iii) For every regular neighborhood U of $L(\Sigma)$ in $\mathbb{S}_\epsilon$, we may refine the triangulation of $\mathbb{S}_\epsilon$ if necessary, so that we can assume $W \subset U$.

Now consider the sets:

- $A := \{\sigma \in (K) \,|\, \sigma \text{ is a (1)-simplex of } L_0 \text{ whose closure intersects } \partial W\}$;
- $B := \{\sigma \in (K') \,|\, \sigma \text{ is an } (2n-2) \text{ simplex in } \partial W \text{ whose closure intersects } L_0\}$.

Then one can see that

$$B = \bigcup_{\sigma \in A} d(\sigma).$$

Thence ∂W intersects L_0 transversally. Moreover, if Σ is either a smooth manifold or an isolated singularity, it follows that $L(\Sigma)$ is a smooth manifold without boundary, and therefore (see for instance Sect. 1.1.2 of [6]) W is a bundle over $L(\Sigma)$ whose fibers are disks. Finally, by a theorem of Hirsch [11] we can choose W so that its boundary ∂W is a smooth manifold. □

Proposition 2.2 *For every t sufficiently close to 0 we have that $L_t \backslash \mathring{W}$ is diffeomorphic to $L_0 \backslash \mathring{W}$.*

Proof The first step is to show that for every t sufficiently close to 0, one has that L_t intersects ∂W transversally in $\mathbb{S}_\epsilon$. Let (p_t) be a sequence of points in $f^{-1}(\mathbb{D}_\eta \backslash \{0\}) \cap \partial W$, with $p_t \in L_t \cap \partial W$, that converges to $p_0 \in L_0 \cap \partial W$. Set $T := \lim_{t \to 0} T_{p_t} L_t$ and let $Reg\ L_0$ denote the regular set of L_0. Since f has the Thom a_f-property, it follows that $T_{p_0}(Reg\ L_0) \subset T$. And since $T_{p_0}(Reg\, L_0)$ intersects $T_{p_0}\partial W$ transversally, by the previous lemma, it follows that T and $T_{p_0}\partial W$ meet transversally. Consider d a metric in the corresponding Grassmannian. As ∂W is closed, the transversality condition is an open property, then we have that if $d(T, T_{p_t}L_t)$ and $d(T_{p_0}\partial W, T_{p_t}\partial W)$ are sufficiently small, then $T_{p_t}L_t$ intersects $T_{p_t}\partial W$ transversally in $\mathbb{S}_\epsilon$.

Now we set $M = \left(\mathbb{S}_\epsilon \cap f^{-1}(\mathbb{D}_\eta)\right) \setminus \mathring{W}$, where $\mathring{W}$ is the interior of W. Since $0 \in \mathbb{C}$ is the only critical value of f, by Ehresmann's fibration lemma we have that the restriction:

$$f_{|M} : \mathbb{S}_\epsilon \cap f^{-1}\left(\mathbb{D}_\eta\right) \setminus \mathring{W} \to \mathbb{D}_\eta$$

is a fiber bundle if one has the following conditions:

(1) For all $p \in M$, $D(f_{|M\setminus\partial M})_p : T_p(\mathbb{S}_\epsilon) \to T_{f(p)}\mathbb{R}^2$ is a surjection;
(2) For all $p \in \partial M$, $D(f_{|\partial M})_p : T_p(\partial M) \to T_{f(p)}\mathbb{R}^2$ is a surjection.

These conditions are equivalent to the following:

(1') The fibers $f^{-1}(t)$ are transversal to $\mathbb{S}_\epsilon$ in $\mathbb{R}^2$ at any point $p \in M$;
(2') The fibers $f^{-1}(t) \cap \mathbb{S}_\epsilon$ of the restriction of f to the sphere $\mathbb{S}_\epsilon$ are transversal to ∂M in $\mathbb{R}^2$ at any point $p \in \partial M$.

But $(1')$ follows from the fact that f has a Milnor-Lê fibration and $(2')$ follows from the fact that L_t intersects ∂W transversally. □

Now we may define:

Definition 2.3 A vanishing zone for f is a regular neighborhood W of $L(\Sigma)$ in $\mathbb{S}_\epsilon$ with smooth boundary such that, for every t sufficiently near 0, the smooth manifold $L_t\setminus\mathring{W}$ is diffeomorphic to $L_0\setminus\mathring{W}$.

We know from Lemma 2.1 that if Σ has at most an isolated singularity then W can be chosen to be a fiber bundle over $L(\Sigma)$. In this case, we look for conditions which guarantee that the intersections $W_t := L_t \cap W$, for $t \in \mathbb{D}_\eta$, are also fiber bundles over $L(\Sigma)$. We have:

Proposition 2.4 *If each connected component of $\Sigma\setminus\{0\}$ is a single stratum of a Whitney stratification of V, then both W_0 and W_t are fiber bundles over $L(\Sigma)$.*

Proof Since each connected component of $\Sigma\setminus\{0\}$ is a single stratum of a Whitney stratification of V, one has that each connected component of $L(\Sigma)$ is a single stratum of the induced Whitney stratification of W_0. Hence each connected component of W_0 can be seen as a Whitney equisingular one-parameter family of isolated singularities. □

Thus we arrive to:

Theorem 2.5 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ with critical value at 0. Then:*

(i) *There exists a compact vanishing zone W for f.*
(ii) *If the critical set Σ of f is either smooth or an isolated singularity, then W can be chosen to be a fiber bundle over $L(\Sigma)$ with fiber a $2(n-k)$-dimensional ball, where k is the dimension of Σ.*
(iii) *If the Whitney stratification of V can be chosen so that each connected component of $\Sigma\setminus\{0\}$ is a single stratum, then the intersection $W_t := L_t \cap W$ is a fiber bundle over $L(\Sigma)$, for any $t \in \mathbb{D}_\eta$.*

Following [32], we say that $f : \mathbb{C}^n \to \mathbb{C}$ is a 1-singularity if the critical locus Σ of f has complex dimension one.

Example 2.6 If f is a 1-singularity then f clearly satisfies all the hypotheses of Theorem 2.5, since Σ is a union of strata and the only smaller stratum contained in Σ has dimension zero. In particular, if $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is reduced (that is, if the ideal (f) is radical) then it satisfies the hypotheses of Theorem 2.5. We remark that Theorem 2.5 for 1-singularities was already proved by Siersma in [32].

Example 2.7 Consider $f : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ given by $f(x, y, z, w) = w \cdot g(x, y, z)$, where $g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is a holomorphic function-germ with no critical point. Then $V = V(w) \cup V(g)$, which is (locally) analytically equivalent to $(\mathbb{C}^3 \times \{0\}) \cup (\mathbb{B}^4 \times \mathbb{C})$, where $\mathbb{B}^4$ is a 4-dimensional ball in $\mathbb{C}^3$. Its critical set is given by $\Sigma = V(w) \cap V(g)$, which is analytically equivalent to $\mathbb{B}^4 \times \{0\}$. It is easy to see that V can be given a Whitney stratification such that Σ is a single stratum.

We also have:

Theorem 2.8 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function-germ, with $n \geq 3$, such that the critical set Σ of f has complex dimension $(n-2)$ and such that each connected component of $\Sigma \backslash \{0\}$ is a non-empty single stratum of a Whitney stratification of V. Then the boundary of the Milnor fiber of f is not homeomorphic to the link of f.*

Proof Suppose that L_t is homeomorphic to L_0. By the Excision Theorem, we have that the homology group $H_*(W_t, \partial W_t)$ is isomorphic to $H_*(L_t, L_t \backslash W_t)$, for any t small. Since $L_t \backslash W_t$ is homeomorphic to $L_0 \backslash W_0$, it follows that $H_*(W_t, \partial W_t)$ is isomorphic to $H_*(W_0, \partial W_0)$. Note that W_0 is a topological manifold, since we are supposing that L_0 is homeomorphic to L_t. Then we can apply the Lefschetz Duality to get that the cohomology group $H^*(W_t)$ is isomorphic to $H^*(W_0)$. Since both W_t and W_0 are fiber bundles over $L(\Sigma)$ with fiber $f_s^{-1}(t)$ and $f_s^{-1}(0)$, respectively, we have that $H^*(f_s^{-1}(t))$ is isomorphic to $H^*(f_s^{-1}(0))$. But since $f_s^{-1}(t)$ deformation retracts to $P_{t,s}$ and $f_s^{-1}(0)$ deformation retracts to $\{s\}$, it follows that $P_{t,s}$ is a point, for any $s \in L(\Sigma)$. This means that f_s is regular, that is, $L(\Sigma)$ is the empty set, a contradiction. □

In particular, we have:

Corollary 2.9 *The boundary of the Milnor fiber of a reduced holomorphic function-germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is homeomorphic to its link if and only if f has isolated critical point.*

3 The real analytic case

We now look at real analytic map-germs :

$$h : \left(\mathbb{R}^m, 0\right) \to \left(\mathbb{R}^n, 0\right),$$

with $m > n > 0$. In this section we show that under certain stringent conditions, there exists a vanishing zone for such germs, and we study its topology. The definition of a vanishing zone in this setting will be similar to that of (Definition 2.3).

Given a Milnor tube $N(\epsilon, \eta) := h^{-1}(\mathbb{D}_\eta^*) \cap \mathbb{B}_\epsilon$, where $\mathbb{B}_\epsilon \subset \mathbb{R}^m$ is a Milnor ball for h and $\mathbb{D}_\eta^*$ is a punctured ball in $\mathbb{R}^n$ around 0, with $\epsilon \gg \eta > 0$, one has the restriction:

$$h_| : (h)^{-1}\left(\mathbb{D}_\eta^*\right) \cap \mathbb{B}_\epsilon \to \mathbb{D}_\eta^* . \quad (1)$$

If h has an isolated critical point or if h is holomorphic with values in $\mathbb{C}$, it follows from Milnor's work (together with [10,12]) that this is a fiber bundle (also called a locally trivial fibration in the literature). In the general case, this may not be a fiber bundle. Following [27], when this is a fiber bundle over its image, we call it the Milnor-Lê fibration of h.

Now suppose that h as above has an isolated critical value at $0 \in \mathbb{R}^n$ and that its zero locus $V(h) := h^{-1}(0)$ has dimension $m - n > 0$. It follows easily from Ehresmann's fibration theorem for manifolds with boundary that h has a Milnor-Lê fibration if and only if for every $t \in \mathbb{R}^n$ sufficiently close to 0, the smooth manifold $h^{-1}(t)$ intersects the boundary of the Milnor sphere $\mathbb{S}_\epsilon$ transversally (see for instance [7]).

Remark 3.1 Recall that the map-germ h has the Thom a_h-property at 0 if there is a Whitney stratification of a neighborhood of 0 for which V is a union of strata and such that every limit of spaces tangent to the fibers on a sequence of regular points of h that converge to some $y \in V$, contains the space tangent at y to the corresponding stratum. It is clear that if h has the Thom a_h-property then the fibers of h sufficiently close to the special fiber V are transversal to the spheres, hence h has a Milnor-Lê fibration in the tube (see [27]).

Now suppose further that h has a Milnor-Lê fibration in the tube.

Set $F_t := h^{-1}(t) \cap \mathbb{B}_\epsilon$, $L_t := h^{-1}(t) \cap \mathbb{S}_\epsilon$ and $L(\Sigma) := \Sigma \cap \mathbb{S}_\epsilon$, as before, where Σ is the critical set of h. Also let k be the dimension of Σ.

One can easily check that with these conditions, the proofs of Lemma 2.1 and Propositions 2.2 and 2.4 go through with essentially no modification and we arrive to the following.

Proposition-Definition 3.2 *Assume that the analytic map-germ h has an isolated critical value, its zero locus $V(h) := h^{-1}(0)$ has dimension $m - n > 0$ and h has a local Milnor-Lê fibration. Then there exists a neighborhood W of $L(\Sigma)$ in $\mathbb{S}_\epsilon$ such that its boundary ∂W is smooth, it intersects L_0 transversally, and $L_t \setminus \mathring{W}$ is diffeomorphic to $L_0 \setminus \mathring{W}$. We call W a vanishing zone for h.*

We also have that if Σ is either smooth or it has an isolated singularity, then W can be chosen to be a fiber bundle over $L(\Sigma)$ with fiber an $(m - k)$-dimensional disk. And if the Whitney stratification of V can be chosen so that $\Sigma \setminus \{0\}$ is a non-empty single stratum, then the intersections $W_t := L_t \cap W$, for $t \in \mathbb{D}_\eta$, are fiber bundles over $L(\Sigma)$.

Yet, we notice that asking h to satisfy Thom's a_h-property is rather stringent, so we drop it from in the statement below.

Theorem 3.3 *Let $h : (\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$, with $m \geq n$, be a real analytic map-germ with an isolated critical value such that $V = V(h)$ has dimension $m - n$ and such that, for every Milnor sphere $\mathbb{S}_\epsilon$, h has an associated Milnor-Lê fibration in a Milnor tube. Then:*

(i) *If $\Sigma \setminus \{0\}$ is a non-empty smooth submanifold of V, then there exists a compact vanishing zone W for h and this can be chosen to be a fiber bundle over $L(\Sigma)$ with fiber an $(m - k)$-dimensional disk.*

(ii) *If there exists a Whitney stratification of a neighborhood of 0 in $\mathbb{R}^m$ for which V is union of strata and each connected component of $\Sigma \setminus \{0\}$ is one single stratum, then the intersection $W_t := L_t \cap W$ is a fiber bundle over $L(\Sigma)$, for any $t \in \mathbb{D}_\eta$.*

The proof mimics that of Theorem 2.5, so we leave the details to the reader.

We now give a couple of examples that illustrate Theorem 3.3 when $n \geq 3$.

Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^k, 0)$, with $2 < k < m$, be a real analytic map-germ with an isolated critical point, and let $g : (\mathbb{R}^k, 0) \to (\mathbb{R}^n, 0)$, with $1 < n < k$, be a real analytic map-germ with an isolated critical value. Then the composition $h := g \circ f$ is a real analytic map-germ $(\mathbb{R}^m, 0) \to (\mathbb{R}^n, 0)$ with an isolated critical value. In fact, for any $x \in \mathbb{R}^m \setminus V(h)$ one has that $f(x) \in \mathbb{R}^k \setminus V(g)$, and hence both $Df_x : T_x\mathbb{R}^m \to T_{f(x)}\mathbb{R}^k$ and $Dg_{f(x)} : T_{f(x)}\mathbb{R}^k \to T_{h(x)}\mathbb{R}^n$ are surjective. Thus $Dh_x : T_x\mathbb{R}^m \to T_{h(x)}\mathbb{R}^n$ is surjective.

Now suppose further that $V(g) = \{0\}$. Then if $0 \in \mathbb{R}^k$ is in fact a critical point of g, it follows that the critical set of h is given by $\Sigma_h = V(h) = V(f)$. Moreover, we claim that h has a Milnor-Lê fibration in a Milnor tube. In fact, let $\epsilon > 0$ be a Milnor radius for both f and h. Since f has an isolated critical point, it is well-known that there exists a positive real number $\eta_1 > 0$ sufficiently small such that $f^{-1}(s)$ intersects the sphere $\mathbb{S}^{m-1}_\epsilon$ transversally, for any $s \in \mathbb{D}^k_{\eta_1}$. On the other hand, since $V(g) = \{0\}$, we can take $\eta_2 > 0$ sufficiently small such that $g^{-1}(t)$ is contained in $\mathbb{D}^k_{\eta_1}$ for any $t \in \mathbb{D}^n_{\eta_2}$. But since $h^{-1}(t) = f^{-1}\big(g^{-1}(t)\big)$ it follows that $h^{-1}(t)$ intersects the sphere $\mathbb{S}^{m-1}_\epsilon$ transversally, for any $t \in \mathbb{D}^n_{\eta_2}$.

So using Kuiper's examples in [20] we have:

Example 3.4 Consider the map $F : \mathbb{H}^2 \to \mathbb{H}$ given by $F(q_1, q_2) := q_1\bar{q}_2$, where $\mathbb{H}$ denotes the set of the quaternions and $\bar{q}_2$ denotes the conjugate of the quaternion number q_2. It can be seen as a real analytic map $f : \mathbb{R}^8 \to \mathbb{R}^4$ with an isolated critical point at the origin. On the other hand, consider the map $G : \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}$ given by $G(z_1, z_2) := (z_1\bar{z}_2, \|z_2\|^2 - \|z_1\|^2)$, where $\bar{z}_2$ denotes the complex conjugate of z_2. It can be seen as a real analytic map $g : \mathbb{R}^4 \to \mathbb{R}^3$ with an isolated critical point at the origin. Moreover, we have that $V(g) = \{0\}$.

Hence the composition $h := g \circ f$ is a real analytic map-germ $(\mathbb{R}^8, 0) \to (\mathbb{R}^3, 0)$ with an isolated critical value and with a Milnor-Lê fibration in a Milnor tube. Moreover, $\Sigma_h = V(h) = V(f)$.

The example above is not so good to illustrate Theorem 3.3 since $\Sigma_h = V(h)$. However, in the same vein we have:

Example 3.5 Consider the map $F : \mathbb{O}^2 \to \mathbb{O}$ given by $F(o_1, o_2) := o_1\bar{o}_2$, where $\mathbb{O}$ denotes the set of the octonions and $\bar{o}_2$ denotes the conjugate of the octonion number o_2. This can be regarded as a real analytic map $f : \mathbb{R}^{16} \to \mathbb{R}^8$ with an isolated critical point at the origin. On the other hand, consider the map $G : \mathbb{H}^2 \to \mathbb{H} \times \mathbb{R}$ given by $G(q_1, q_2) := (q_1\bar{q}_2, \|q_2\|^2 - \|q_1\|^2)$, where $\bar{q}_2$ denotes the conjugate of the quaternion number q_2. This can be seen as a real analytic map $g : \mathbb{R}^8 \to \mathbb{R}^5$ with an isolated critical point at the origin. Moreover, we have that $V(g) = \{0\}$.

By the discussion above, the composition $h := g \circ f$ is a real analytic map-germ $(\mathbb{R}^{16}, 0) \to (\mathbb{R}^5, 0)$ with an isolated critical value and with a Milnor-Lê fibration in a Milnor tube, and such that $\Sigma_h = V(h)$. If we remove the last coordinate function of g we obtain a real analytic map-germ $\tilde{g} : (\mathbb{R}^8, 0) \to (\mathbb{R}^4, 0)$ with an isolated critical point, as well as a real analytic map-germ $\tilde{h} = \tilde{g} \circ f : (\mathbb{R}^{16}, 0) \to (\mathbb{R}^4, 0)$ with an isolated critical value and a Milnor-Lê fibration in a Milnor tube (see Theorem 1.1 of [15]). Moreover, we have that $\Sigma_{\tilde{h}} = V(f) = V(h) = \Sigma_h$ and that $V(h) \subsetneq V(\tilde{h})$. Hence $\Sigma_{\tilde{h}} \subsetneq V(\tilde{h})$.

Since $\Sigma_{\tilde{h}} = V(f)$ and since f has an isolated critical point, it follows that $\Sigma_{\tilde{h}}$ is a non-empty smooth submanifold of $V(\tilde{h})$. So all the hypothesis in (i) of Theorem 3.3 are satisfied.

It is easy to see that there exists a Whitney stratification of a neighborhood of 0 in $\mathbb{R}^{16}$ for which $V(\tilde{h})$ is union of strata and $\Sigma_{\tilde{h}} \setminus \{0\}$ is a single stratum. In fact, we have that $V(\tilde{h}) = \{f_1 = \cdots = f_4 = 0\} \cup \{f_5 = \cdots = f_8 = 0\}$ and that $\Sigma_{\tilde{h}} = \{f_1 = \cdots = f_8 = 0\}$. Write $A := \{f_1 = \cdots = f_4 = 0\}$, $B := \{f_5 = \cdots = f_8 = 0\}$ and $C := \{f_1 = \cdots = f_8 = 0\}$.

Note that A, B and C are smooth outside the origin, since f has an isolated critical point. So the pairs $(A\backslash\{0\}, C\backslash\{0\})$ and $(B\backslash\{0\}, C\backslash\{0\})$ satisfy the Whitney conditions.

So f illustrates Theorem 3.3 when $n = 4$.

Note that if we now remove a coordinate function of $\tilde{h}$ we obtain a real analytic map-germ $(\mathbb{R}^{16}, 0) \to (\mathbb{R}^3, 0)$ with the same properties, which illustrates Theorem 3.3 when $n = 3$.

4 Real analytic map-germs of the type $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$

Now consider the case of real analytic map-germs of the form:

$$h := f\bar{g} : \left(\mathbb{C}^n, 0\right) \to (\mathbb{C}, 0)\,,$$

where $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic function-germs with no common irreducible components and such that $f\bar{g}$ has an isolated critical value. We give in this section a sufficient condition, weaker than the Thom a_h-property, that grants the existence of a Milnor-Lê fibration. Later we give some examples of map-germs of the type $f\bar{g}$ that satisfy the hypotheses of Theorem 3.3.

4.1 Existence of Milnor-Lê fibrations for $f\bar{g}$

If $n = 2$ and $f\bar{g}$ is as above, Pichon and Seade proved in [27] that the germ $h := f\bar{g}$ has the Thom a_h-property and therefore it has a Milnor-Lê fibration in a Milnor tube.

When $n > 2$ it is not true that every such map $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has the Thom a_h-property (see [29] for a counter-example). So we need extra conditions for $f\bar{g}$ to have a Milnor-Lê fibration.

If $n = 3$, Fernández de Bobadilla and Menegon proved in [9] that $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ has a Milnor-Lê fibration in the tube if and only if $f\bar{g}$ has an isolated critical value at $0 \in \mathbb{C}$ and for each irreducible component Σ_i of the intersection $V(f) \cap V(g)$, the restriction of $f\bar{g}$ to a generic hyperplane section H_i transversal to Σ_i has isolated critical values.

We want to extend that result from [9] for $n \geq 3$. For this, let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two holomorphic function-germs with no common irreducible component, such that $h := f\bar{g}$ has an isolated critical value.

Set $V := V(fg) = (fg)^{-1}(0)$ and consider a Whitney stratification V_α adapted to V. Recall that $\mathbb{S}_\epsilon$ is a Milnor sphere for h if every sphere centered at 0 and of radius less than or equal to ϵ meets transversally each stratum in V.

Notice that given a sphere $\mathbb{S}_\epsilon$ centered at 0 and a point $x \in \mathbb{S}_\epsilon$, there is a unique complex hyperplane $T_x^C(\mathbb{S}_\epsilon)$ tangent to $\mathbb{S}_\epsilon$ at x. This complex plane has real codimension 1 in the tangent space $T_x(\mathbb{S}_\epsilon)$ and the union of all these planes determines the canonical contact structure on the sphere.

We have:

Definition 4.1 The map-germ $h = f\bar{g}$ is CT-regular at 0 if for every Milnor sphere $\mathbb{S}_\epsilon$ and for every x in the critical set Σ of $V \cap \mathbb{S}_\epsilon$ one has that the restriction of $f\bar{g}$ to the contact plane $T_x^C(\mathbb{S}_\epsilon)$ has an isolated critical value at 0.

In the notation "CT-regular", the C stands for contact, since this is a condition that depends on the contact planes, while T stands for tube, because this condition concerns the behavior of $f\bar{g}$ in a "tube" around V, since it only cares about critical values near 0.

One has:

Proposition 4.2 *Thom's a_h-property at* 0 *implies CT-regularity, and the latter implies that we have a Milnor-Lê fibration in a tube.*

Proof The first implication is immediate: We already know (see [7,27]) that Thom's a_h-property at 0 implies that every fiber $h^{-1}(t)$, $\|t\|$ sufficiently small, meets transversally the sphere $\mathbb{S}_\epsilon$. But in fact we know more: every limit of spaces tangent to the fibers of h contains the space tangent to the corresponding stratum of V. The result then follows because V is complex analytic, so the tangent bundle of each stratum is a complex vector bundle, and by definition of a Milnor sphere, it is traversal to the contact structure on the Milnor sphere.

For the second claim, that CT-regularity implies one has a Milnor-Lê fibration, we only need to observe that CT-regularity clearly implies that away from V, all the nearby fibers of h are transversal to the corresponding contact planes, and therefore they are transversal to the spheres. □

Notice that every holomorphic map-germ with values in $\mathbb{C}$ is CT-regular since it has the Thom property.

The following example shows that the reciprocal of Proposition 4.2 is false in general. This answers a question raised in [7, Remark 4.4] and in [3], which is whether or not having a local Milnor-Lê fibration is equivalent to having the (local) Thom a_h-property.

Example 4.3 Let $h = f\bar{g} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be given by $f(z_1, z_2) = z_1 z_2$ and $g(z_1, z_2) = z_2$. Parusinski showed that there is no stratification of fg satisfying Thom's a_h condition (see [29]). But one can show that $(f\bar{g})^{-1}(t)$ intersects the Milnor sphere transversally, for t sufficiently small, and hence $f\bar{g}$ has a Milnor-Lê fibration in the tube. In fact, for each $t \in \mathbb{C}\backslash\{0\}$ fixed, if we write a complex number in the form $z = |z|e^{i\theta_z}$, we have that $f\bar{g}(z_1, z_2) = z_1|z_2|^2 = t \Leftrightarrow |z_2|^2 = \dfrac{t}{z_1}$, which can occur if and only if $\theta_{z_1} = \theta_t$. One can see then that $(f\bar{g})^{-1}(t)$ is a fiber bundle over $\mathbb{R}^+$ with fiber $\mathbb{S}^1$, which intersects the "square Milnor ball" transversally. In order to show that this map is CT-regular we need some considerations (see Example 4.6 below).

Remark 4.4 Given a germ $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical value, for every Milnor sphere $\mathbb{S}_\epsilon$ one has the classical Milnor map:

$$\psi := \frac{f\bar{g}}{|f\bar{g}|} : \mathbb{S}_\epsilon\backslash\big(V \cap \mathbb{S}_\epsilon\big) \to \mathbb{S}^1,$$

which may or may not be a fiber bundle. In [7] (also in [4]) is proved that this is a fiber bundle if and only if the map $f\bar{g}$ satisfies a certain regularity condition, called d-regularity. Hence, if the germ $f\bar{g}$ is d-regular and CT-regular, then one has two local fibrations associated to the germ $f\bar{g}$. One is the Milnor fibration ψ on the sphere, as above; the other is the Milnor-Lê fibration on a tube, as in (1). The main result of [7] (see [8, Theorem 3.14]) implies that, just as in the holomorphic case, these two fibrations are equivalent (a fact already known from [2] for quasi-homogeneous germs).

Remark 4.5 In order to verify the CT-regularity, it is convenient to consider the polydisc

$$\Delta_\epsilon := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1| \le \epsilon_1, \ldots, |z_n| \le \epsilon_n\}.$$

Then $\partial\Delta_\epsilon$ is the union of the faces $\{|z_i| = \epsilon_i\}$ for $i = 1, \ldots, n$. We can suppose that Σ intersects Δ_ϵ in an union of open faces $\{|z_i| = \epsilon\}$. Then the CT-regularity is equivalent to the condition that $f\bar{g}$ intersects each hyperplane $H_{i,\theta} := \{z_i = \epsilon e^{i\theta}\}$ transversally, that is, that the restriction of $f\bar{g}$ to $H_{i,\theta}$ has an isolated critical value. This is very helpful in practice because it means we only need to check finitely many conditions: one for each face of a polydisc.

Example 4.6 The map-germ $h = \bar{z_2}(z_1 z_2)$ of Example 4.3 is CT-regular. In fact, the restriction of h to a hyperplane section $\{x = x_0\}$ has an isolated critical value at 0 and the restriction of h to a hyperplane section $\{y = y_0\}$ is regular.

In the sequel we give more explicit examples of map-germs $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ which are CT-regular.

4.2 The vanishing zone for $f\bar{g}$

Now suppose that $f\bar{g}$ has an isolated critical value and the Milnor-Lê fibration in a tube:

$$(f\bar{g})_| : \left(f\bar{g}\right)^{-1}\left(\mathbb{D}^*_\eta\right) \cap \mathbb{B}_\epsilon \to \mathbb{D}^*_\eta.$$

We consider the sets $V := (f\bar{g})^{-1}(0) = (fg)^{-1}(0)$, $F_t := (f\bar{g})^{-1}(t) \cap \mathbb{B}_\epsilon$ and $L_t := (f\bar{g})^{-1}(t) \cap \mathbb{S}_\epsilon$, as in the previous section.

An easy calculation shows that (if $f\bar{g}$ has an isolated critical value) one has that:

$$\Sigma\left(f\bar{g}\right) = \Sigma(f) \cup \Sigma(g) \cup \left(V(f) \cap V(g)\right) = \Sigma(fg).$$

We shall denote this set by Σ and set k to be its complex dimension. Also set $L(\Sigma) := \Sigma \cap \mathbb{S}_\epsilon$.

Notice that the critical set of $f\bar{g}$ contains the intersection $V(f) \cap V(g)$. Hence $f\bar{g}$ cannot have an isolated critical point if $n > 2$. In fact, if f, g have no common factor, then the singular set of $V = V(fg)$ has complex dimension $n - 2$.

So Theorem 3.3 gives:

Corollary 4.7 *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two holomorphic function-germs with no common irreducible components. Suppose that the real analytic map-germ $f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has an isolated critical value at $0 \in \mathbb{C}$ and that it has a Milnor-Lê fibration in the tube. Then:*

(i) *If $\Sigma\backslash\{0\}$ is a non-empty smooth submanifold, then there exists a vanishing zone W for $f\bar{g}$ which can be chosen to be a fiber bundle over $L(\Sigma)$ with fiber a 2-dimensional complex disk.*

(ii) *If the Whitney stratification of V can be chosen so that each connected component of $\Sigma\backslash\{0\}$ is a non-empty single stratum, then the intersection $W_t := L_t \cap W$ is a fiber bundle over $L(\Sigma)$, for any $t \in \mathbb{D}_\eta$.*

If we further assume that $n = 3$ and that both f and g are reduced, the condition that a Whitney stratification of V can be chosen so that $\Sigma\backslash\{0\}$ is a non-empty single stratum is always satisfied, since Σ is a complex curve, as we have said before.

Bellow we give some examples of function-germs $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ with $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ non-constant, holomorphic and with no common irreducible components, such that $f\bar{g}$ has an isolated critical value and the Milnor-Lê fibration in the tube:

Example 4.8 $f\bar{g} : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $f(x, y, z) = x^2 + y^3 + xz^2$ and $g(x, y, z) = x$. It is an exercise to show that $\Sigma = \{x = y = 0\}$ and that $f\bar{g}$ has a Milnor-Lê fibration in the tube.

Example 4.9 $f\bar{g}:(\mathbb{C}^3,0)\to(\mathbb{C},0)$ given by $g(z_1,z_2,z_3)=z_i^k$ for some $i\in\{1,2,3\}$ and $k\geq 1$, and f a homogeneous polynomial of degree $m\neq k$ with $\frac{\partial f}{\partial z_l}=0$ for some $l\in\{1,2,3\}, l\neq i$.

Consider coordinates $z_1,\bar{z_1},z_2,\bar{z_2},z_3,\bar{z_3}$ for $\mathbb{C}^3$. Then:

$$f\bar{g}=(\Re f\bar{g},\Im f\bar{g})=\frac{1}{2}\left(f\bar{g}+\bar{f}g,\frac{1}{i}\left(f\bar{g}-\bar{f}g\right)\right),$$

where $\Re$ denotes the real part and $\Im$ denotes the imaginary part of $f\bar{g}$. A straightforward computation shows that the set of critical points of $f\bar{g}$, denoted by C, is defined by the following equations:

- $fg\left(\frac{\partial f}{\partial z_i}\frac{\partial g}{\partial z_j}-\frac{\partial f}{\partial z_j}\frac{\partial g}{\partial z_i}\right)=0,\quad$ for all $i\neq j$;
- $|f\frac{\partial g}{\partial z_i}|=|g\frac{\partial f}{\partial z_i}|,\quad$ for all $i\in\{1,2,3\}$;
- $|f|^2\frac{\partial g}{\partial z_i}\overline{\frac{\partial g}{\partial z_j}}=|g|^2\frac{\partial f}{\partial z_i}\overline{\frac{\partial f}{\partial z_j}},\quad$ for all $i\neq j$.

Since we are supposing that $g=z_i^k$, we have that

$$C=\{f=g=0\}\bigcup\left(\bigcap_{i\neq j}\left\{\frac{\partial f}{\partial z_j}=0\right\}\cap\left\{\left|f\frac{\partial g}{\partial z_i}\right|-\left|g\frac{\partial f}{\partial z_i}\right|=0\right\}\right).$$

Since f is a homogeneous polynomial of degree m, we have that

$$mf=\sum_{j=1}^{n}\frac{\partial f}{\partial z_j}z_j.$$

We claim that $f\bar{g}$ has an isolated critical value. In fact, suppose that there exists $x=(x_1,x_2,x_3)\in C$ with $f(x)\neq 0$. We have that $mf(x)=\frac{\partial f}{\partial z_i}(x)x_i\neq 0$ and then $|f(x)\frac{\partial g}{\partial z_i}(x)|=|g(x)\frac{\partial f}{\partial x_i}|$ implies that $m|x_i^{k-1}||f(x)|=|x_i^k||\frac{\partial f}{\partial z_i}(x)|=|f(x)\frac{\partial g}{\partial z_i}(x)|=k|x_i^{k-1}||f(x)|$, which implies that $m=k$, a contradiction.

Now we will show that $f\bar{g}$ is CT-regular. For each irreducible component of the intersection $V(f)\cap\{z_i=0\}$, take the transversal section $\{z_j=s\}$, for some small real number s, with $j\neq i$ and $j\neq l$. Then one can easily check that the restriction of $f\bar{g}$ to such transversal section has an isolated critical value.

Now we give an example of a map-germ $f\bar{g}$ defined in $\mathbb{C}^4$ that satisfies all the hypotheses of Corollary 4.7 above:

Example 4.10 Consider $f,g:(\mathbb{C}^4,0)\to(\mathbb{C},0)$ given by $f(x,y,z,w)=x^a+y^b+z$, with $a,b>1$, and $g(x,y,z,w)=w$. Then the real analytic map-germ $f\bar{g}:(\mathbb{C}^4,0)\to(\mathbb{C},0)$ given by

$$f\bar{g}(x,y,z,w)=\bar{w}\left(x^a+y^b+z\right)$$

has an isolated critical value and the critical set of $V(f\bar{g})$ is given by

$$\Sigma=V(f)\cap V(g)$$

and hence it is a complex 2-manifold. Since the restrictions $(f\bar{g})_{x_0}=\bar{w}(x_0^a+y^b+z)$ and $(f\bar{g})_{y_0}=\bar{w}(x^a+y_0^b+z)$ have isolated critical values, for any x_0 and y_0 fixed with $|x_0|=\epsilon_1$ and $|y_0|=\epsilon_2$, it follows that $f\bar{g}$ is CT-regular at 0, and hence it has a Milnor-Lê fibration in the tube. Look at Example 2.7 to see that V can be given a Whitney stratification such that Σ is a single stratum.

As a consequence, analogous to Theorem 2.8, we have:

Theorem 4.11 *Let $f, g : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be reduced holomorphic function-germs with no common irreducible component such that $f\bar{g}$ has an isolated critical value at 0 and a Milnor-Lê fibration in a tube. Then the boundary of the Milnor fiber of $f\bar{g}$ is not homeomorphic to its link.*

Notice that if Corollary 3.7 holds for $f\bar{g}$, it also holds for $g\bar{f}$. In fact, the critical locus of $f\bar{g}$ equals the critical locus of $g\bar{f}$ and $(f\bar{g})^{-1}(\bar{t}) = (g\bar{f})^{-1}(t)$. In particular, one can interchange the roles of f and g in the examples 3.8 to 3.10 above.

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