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ORIGINAL PAPER

Total curvature and some characterizations of closed curves in CATk spaces

Areeyuth Sama-Ae[1] · **Aniruth Phon-on**[1]

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Abstract In this paper, we study the characterizations of a closed curve in a CAT(k) space that bounds a geodesic surface which is isometric to the disk bounded by a circle in the model space S_k with same perimeter.

Keywords CAT(k) · Total curvature · Closed curve

Mathematics Subject Classification (2000) 51K99

1 Introduction

In this paper, we study the characterizations of a closed curve γ in a CAT(k) space that bounds a geodesic surface which is isometric to the disk bounded by a circle γ' in the model space S_k with the same perimeter. These characterizations involve either the length or the total curvature of γ and all its subarcs and either the chord length or central angle. Here, properties of subarcs of γ are inherited from the same subarc of γ'. Specifically, the characterizations are:

(1) arclength of γ, including its subarcs, and chord length,
(2) arclength of γ, including its subarcs, and central angle,
(3) the total curvature of γ, including its subarcs, and chord length, and
(4) the total curvature of γ, including its subarcs, and central angle.

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✉ Areeyuth Sama-Ae
areeyuth.s@psu.ac.th

Aniruth Phon-on
aniruth.p@psu.ac.th

1 Department of Mathematics and Computer Sciences, Faculty of Science and Technology, Prince of Songkla University, Pattani Campus, Muang, Pattani 94000, Thailand

A curve γ in a CAT(k) space M is called a *spherical curve* if there is a point $q \in M$ and a positive real number r such that $d(x, q) = r$ for all $x \in \gamma$. The real number r is called the *radius* of γ. For example, a circle of radius $r > 0$ in the model space S_k is a closed spherical curve at a distance r from its center. In what follows, we let γ_{xy} be a spherical curve in a CAT(k) with endpoints x, y and let $\gamma'_{x'y'}$ be a subarc of a circle in the model space S_k with endpoints x', y'. Denote $\kappa_c(\gamma)$ and $\kappa(\gamma_{xy})$ the total curvature of a closed curve γ and a subarc γ_{xy}, respectively. We let $\ell(\gamma)$ stand for the length of a curve γ, while $\angle_q(x, y)$ is the angle at a point q between the geodesics from q to x and from q to y. We let $C(A)$ be the convex hull of a set A. Let M be with a metric d a CAT(k) space, and let d' be a metric on the model space S_k. The main result of this paper is presented as the following theorem:

Theorem 1.1 *Let γ be a closed spherical curve at a distance $r < \frac{\pi}{2\sqrt{k}}$ from q in a CAT(k) space M, and γ' be a circle of radius r centered at q' in the model space S_k. Suppose that one of the following statements holds:*

(1) $\ell(\gamma) = \ell(\gamma')$; $d(x, y) = d'(x', y')$ *whenever* $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ *for any subarc* γ_{xy} *of* γ *and any subarc* $\gamma'_{x'y'}$ *of* γ';
(2) $\ell(\gamma) = \ell(\gamma')$; $\angle_q(x, y) = \angle_{q'}(x', y')$ *whenever* $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ *for any subarc* γ_{xy} *of* γ *and any subarc* $\gamma'_{x'y'}$ *of* γ';
(3) $\kappa_c(\gamma) = \kappa_c(\gamma')$; $d(x, y) = d'(x', y')$ *whenever* $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ *for any subarc* γ_{xy} *of* γ *and any subarc* $\gamma'_{x'y'}$ *of* γ';
(4) $\kappa_c(\gamma) = \kappa_c(\gamma')$; $\angle_q(x, y) = \angle_{q'}(x', y')$ *whenever* $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ *for any subarc* γ_{xy} *of* γ *and any subarc* $\gamma'_{x'y'}$ *of* γ'.

Then $C(\gamma')$ is isometric to $C(\gamma)$, that is, the totally geodesic surface bounded by γ in M is isometric to the disk bounded by γ' in the model space S_k.

The curvature of a smooth curve parametrized by its arclength is the rate of change of direction of the tangent vector and measures the amount that the curve deviates from being straight. Pointwise curvature indicates how fast the direction changes at a point and total curvature depicts how much the change accumulates over the entire course of the curve. In the smooth case, the total curvature is the integral of pointwise scalar curvature in respect to arclength. In three-dimensional space, the total curvature of smooth curves was first studied in 1929 by Fenchel [12] who verified that any closed curve has total curvature greater than or equal to 2π; equality is realized if and only if the curve is a planar convex curve. In 1947, Borsuk [8] and Milnor [15] extended that fact to n-dimensional space. For the general case, the total curvature was introduced by Alexandrov [4] in 1946. An extensive development of this theory was given in [6], where the total curvature was defined by considering the total rotation of a sequence of geodesic polygons inscribed in the closed curve and arbitrarily close to it. The total curvature of a closed curve in a Riemannian manifold of non-positive curvature and in a hyperbolic space were studied in [9,13,20]. In a complete simply connected Riemannian manifold with negative sectional curvature, Tsukamoto [23] proved in 1974 that the total curvature of a smooth closed curve is also greater than 2π. The idea of the total curvature has been extended to a CAT(0) space by Alexander and Bishop [2] who showed that a lower bound for total curvature of a closed curve is 2π. If the total curvature of a closed curve is 2π then that curve is either a geodesic bigon or bounds a convex subset that is isometric to a convex set in the 2-dimensional Euclidean space. Furthermore, the idea for a CAT(k) space, with a real number k, was introduced by Maneesawarng and Lenbury [14].

Recently, Sama-Ae et al. proved in [17] that in the case of $k > 0$, the lower bound of the total curvature of any closed curve in a CAT(k) space with perimeter $s < \frac{2\pi}{\sqrt{k}}$ is greater than or equal to the total curvature of a circle in the model space S_k with the same circumference s. It was shown that the total curvature of any circle in the model space S_k with circumference $s < \frac{2\pi}{\sqrt{k}}$ is $2\pi\sqrt{1-\frac{ks^2}{4\pi^2}}$. It means that in the case of $k > 0$, the total curvature of a closed curve γ in a CAT(k) space with perimeter $s < \frac{2\pi}{\sqrt{k}}$ is greater than or equal to $2\pi\sqrt{1-\frac{ks^2}{4\pi^2}}$. In the case of a curve in a CAT(k) space for $k > 0$, the following corollary to Theorem 1.1 proves a conjecture of Sama-Ae et al. from [17]:

Corollary 1.2 *Let γ be a closed spherical curve at a distance $r < \frac{\pi}{2\sqrt{k}}$ from a point q with perimeter $s < \frac{2\pi}{\sqrt{k}}$ in a CAT(k) space M, for $k > 0$, and let γ' be a circle of radius r centered at a point q' in the model space S_k. If $\kappa(\gamma) = 2\pi\sqrt{1-\frac{ks^2}{4\pi^2}}$ and $d(x, y) = d'(x', y')$ (or $\angle_q(x, y) = \angle_{q'}(x', y')$) whenever $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ for any subarc γ_{xy} of γ and any subarc $\gamma'_{x'y'}$ of γ', then the totally geodesic surface bounded by γ in M is isometric to the disk bounded by γ' in the model space S_k.*

Sama-Ae and Maneesawarng [18] studied a comparison and rigidity theorem of spherical curves in a CAT(k) space. They compared the total curvature and the curvelength of spherical curves in a CAT(k) space with subarcs of circles of the same radii in the model space S_k. Their results are given in the following theorem:

Theorem 1.3 [18] *Let M be a $CAT(k)$ space, $q \in M$, $0 < r < \frac{\pi}{2\sqrt{k}}$ ($= \infty$ if $k \le 0$), and let γ_{xy} be a spherical curve at distance r from q with endpoints x, y and $\ell(\gamma_{xy}) < \frac{\pi}{\sqrt{k}}$. Let $\gamma'_{x'y'}$ be a subarc of a circle of radius r centered at q' in the model space S_k with endpoints x', y'.*

(1) *If $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ then $\kappa(\gamma_{xy}) \ge \kappa(\gamma'_{x'y'})$.*
(2) *If $d(x, y) = d'(x', y')$ then $\ell(\gamma_{xy}) \ge \ell(\gamma'_{x'y'})$, and hence $\kappa(\gamma_{xy}) \ge \kappa(\gamma'_{x'y'})$. In addition, if $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ or $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$, then $C(\{q'\} \cup \gamma'_{x'y'})$ is isometric to $C(\{q\} \cup \gamma_{xy})$.*
(3) *If $\angle_q(x, y) = \angle_{q'}(x', y')$ then $\ell(\gamma_{xy}) \ge \ell(\gamma'_{x'y'})$, and $\kappa(\gamma_{xy}) \ge \kappa(\gamma'_{x'y'})$. In addition, if $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ or $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ then $C(\{q'\}\cup\gamma'_{x'y'})$ is isometric to $C(\{q\}\cup\gamma_{xy})$.*

From Theorem 1.3, we have the following conclusions.

Remark 1.4 In a $CAT(k)$ space, the following statements hold.

(1) If $\ell(\gamma_{xy}) > \ell(\gamma'_{x'y'})$, then $\kappa(\gamma_{xy}) > \kappa(\gamma'_{x'y'})$. That is, if $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$, then $\ell(\gamma_{xy}) \le \ell(\gamma'_{x'y'})$.
(2) If $d(x, y) > d'(x', y')$, then $\ell(\gamma_{xy}) > \ell(\gamma'_{x'y'})$, and hence $\kappa(\gamma_{xy}) > \kappa(\gamma'_{x'y'})$. On the other hand, if $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ or $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$, then $d(x, y) \le d'(x', y')$.
(3) If $\angle_q(x, y) > \angle_{q'}(x', y')$, then $\ell(\gamma_{xy}) > \ell(\gamma'_{x'y'})$ and $\kappa(\gamma_{xy}) > \kappa(\gamma'_{x'y'})$. On the other hand, if $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ or $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$, then $\angle_q(x, y) \le \angle_{q'}(x', y')$.

So, in this paper, we shall use the facts in Remark 1.4 to study characterizations of a closed curve γ in a CAT(k) space M in both cases of $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ (or $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$) and $d(x, y) = d'(x', y')$ (or $\angle_q(x, y) = \angle_{q'}(x', y')$).

In Sect. 2, we give the definition of CAT(k) spaces and the meaning of the total curvature of closed curves in a CAT(k) space by considering closed polygonal curves. These are the curves that can be expressed as a concatenation of finitely many minimizing geodesics (distance-realizing curves). In Sect. 3, we deduce the characterizations of a closed spherical curve in a CAT(k) space.

2 Definitions and preliminaries

A metric space M is a CAT(k) space for $k \leq 0$ if each pair of points of M is joined by a geodesic segment and the distance between any two points of any geodesic triangle $\triangle(x, y, z)$ in M is no greater than that between the corresponding points of the model triangle $\triangle(x', y', z')$ with the same sidelengths in the 2-dimensional space S_k of constant curvature k. That is, S_0 is the Euclidean plane and S_k for $k < 0$ is the hyperbolic plane with curvature k. Similarly, a metric space M is a CAT(k) space for $k > 0$ if each pair of points of M with distance less than $\frac{\pi}{\sqrt{k}}$ is joined by a geodesic segment, and the distance between any two points of any geodesic triangle $\triangle(x, y, z)$ of perimeter less than $\frac{2\pi}{\sqrt{k}}$ in M is no greater than that between the corresponding points of the model triangle $\triangle(x', y', z')$ with the same sidelengths in the 2-dimensional Euclidean sphere S_k of radius $\frac{1}{\sqrt{k}}$. We call S_k the *model space* of M. It is clear that a CAT(k_2) space is a CAT(k_1) for $k_1 > k_2$ because triangles with given sidelengths in the model space S_{k_1} are fatter than those in the model space S_{k_2}. A CAT(0) space is a generalization of a Hadamard manifold. It is known that the classical hyperbolic space, a complete simply connected Riemannian manifold having nonpositive sectional curvature, Euclidean buildings and the complex Hilbert ball with a hyperbolic metric are examples of CAT(0) spaces. Further discussion may be found in [5,7,10,11], and more properties of spaces of constant curvature can be found in [1]. In a CAT(k) space, the *angle* $\angle_w(\delta, \tau)$ between two curves δ and τ having a common starting point w is the limit, if it exists,

$$\lim_{u,v\to w} \arccos \frac{x^2 + y^2 - z^2}{2xy},$$

where u (resp., v) is a point on δ (resp., τ), and x, y and z are the lengths of minimizing geodesics between the pairs (v, w), (w, u) and (u, v), respectively.

As the total curvature will involve angles between two geodesics, the class of metric spaces we consider here is one where angles between geodesics starting from a common point always exist. The angle $\angle_x(y, z)$ at x of a triangle $\triangle(x, y, z)$ is defined to be the angle between the geodesics $[x, y]$ and $[x, z]$.

A *closed curve* in a CAT(k) space M is a continuous map of an oriented circle in the 2-dimensional Euclidean space. A *chain* V on a closed curve γ is a set of points corresponding to finitely many parameter values in order. The points in V are called the vertices of the chain. If γ consists of geodesic segments joining adjacent pairs in V, then γ and V form a *closed geodesic polygon* with vertex chain in V. A closed geodesic polygon δ is *inscribed* in a closed curve γ if its chain of vertices has an oriented reparametrization as a chain on γ. If δ is a closed geodesic polygon with a chain $\{\delta(s_0), \delta(s_1), \ldots, \delta(s_n) = \delta(s_0)\}$ inscribed in a closed curve γ, we define the *modulus* of δ associated with the closed curve γ, denoted by $\mu_\gamma(\delta)$, as $\mu_\gamma(\delta) = \max\{\mathrm{diam}(\gamma|_{[s_k, s_{k+1}]})\ ;\ 0 \leq k \leq n-1\}$, where for each $0 \leq k \leq n-1$, $\gamma|_{[s_k, s_{k+1}]}$ is the subarc with endpoints $\gamma(s_k)$ and $\gamma(s_{k+1})$ of γ. Let δ be a closed geodesic polygon inscribed in γ with consecutive vertices $x_1, x_2, \ldots, x_n = x_0$. The *total rotation* $\kappa_c^*(\delta)$ of δ is defined by rotations of δ, $\kappa_c^*(\delta) = \sum_{i=1}^{n}(\pi - \widehat{x_i})$, where $\widehat{x_i}$ is the angle at x_i. The *total curvature* $\kappa_c(\gamma)$ of a closed curve γ is defined by

$$\kappa_c(\gamma) = \lim_{\varepsilon \to 0} \sup_{\delta \in \Sigma_\varepsilon(\gamma)} \kappa_c^*(\delta),$$

where for each $\varepsilon > 0$, $\Sigma_\varepsilon(\gamma)$ is the set of all closed geodesic polygons δ inscribed in γ such that $\mu_\gamma(\delta) < \varepsilon$.

If γ itself is a closed geodesic polygon then $\kappa_c^*(\gamma) = \kappa_c(\gamma)$, see [17]. We note that our total curvature κ_c is based on the total curvature κ defined by Maneesawarng and Lenbury in [14]. For any closed geodesic polygon δ in a CAT(k) space, the total curvature $\kappa(\delta)$ of δ does not exceed $\kappa_c(\delta)$. Hence we have that $\kappa(\gamma) \leq \kappa_c(\gamma)$ for each closed curve γ in a CAT(k) space.

By the definition of the total curvature for a closed curve given above, if γ is a closed curve in a CAT(k) space for $k \leq 0$, then $\kappa_c(\gamma)$ and the total curvature of γ defined by Alexander and Bishop in [2] coincide; that is, $\kappa_c(\gamma)$ is the supremum of $\kappa_c(\delta)$ over all closed polygons δ inscribed in γ.

3 Results

In the first two theorems of this paper, we present characterizations of a closed curve in a CAT(k) space which has the same length and total curvature as a circle in the model space S_k. Throughout the paper, we set γ_{xy} to be a subarc of γ with the starting point x and the ending point y, and if it is not ambiguous we sometimes call these points x and y the endpoints of γ_{xy}.

Theorem 3.1 *Let γ be a closed spherical curve at a distance r from q in a CAT(k) space and let γ' be a circle of radius r centered at q' in the model space S_k. Additionally, if $k > 0$ we assume $r < \frac{\pi}{2\sqrt{k}}$. If $\ell(\gamma) = \ell(\gamma')$ then $\kappa_c(\gamma) \geq \kappa_c(\gamma')$. In addition, if $\kappa_c(\gamma) = \kappa_c(\gamma')$ then the angle between the opposite directions is π at any point.*

Proof Assume that $\ell(\gamma) = \ell(\gamma')$. Then, by Theorem 1.3, it is clear that $\kappa_c(\gamma) \geq \kappa_c(\gamma')$. Assume additionally that $\kappa_c(\gamma) = \kappa_c(\gamma')$, and let $q_1, q_2, \ldots, q_n, q_{n+1} = q_1$ be consecutive vertices of γ and $q_1', q_2', \ldots, q_n', q_{n+1}' = q_1'$ consecutive vertices of γ' such that $\ell(\gamma_{q_k q_{k+1}}) = \ell(\gamma'_{q_k' q_{k+1}'}) = \frac{\ell(\gamma)}{n}$, for $k = 1, 2, \ldots, n$. Therefore we have that $\kappa(\gamma_{q_k, q_{k+1}}) \geq \kappa(\gamma'_{q_k' q_{k+1}'})$ for all $k = 1, 2, \ldots, n$. Since all q_k' are points of γ', we obtain that $\sum_{k=1}^n (\pi - \widehat{q_k'}) = 0$. Consequently, $\kappa_c(\gamma') = \sum_{k=1}^n \kappa(\gamma'_{q_k' q_{k+1}'})$, and since

$$\begin{aligned}
\kappa_c(\gamma) &= \sum_{k=1}^n \kappa(\gamma_{q_k q_{k+1}}) + \sum_{k=1}^n (\pi - \widehat{q_k}) \\
&\geq \sum_{k=1}^n \kappa\left(\gamma'_{q_k' q_{k+1}'}\right) + \sum_{k=1}^n (\pi - \widehat{q_k}) \\
&\geq \sum_{k=1}^n \kappa\left(\gamma'_{q_k' q_{k+1}'}\right) \\
&= \kappa_c(\gamma'),
\end{aligned}$$

we have that $\sum_{k=1}^n \kappa(\gamma'_{q_k' q_{k+1}'}) + \sum_{k=1}^n (\pi - \widehat{q_k}) = \sum_{k=1}^n \kappa(\gamma'_{q_k' q_{k+1}'})$, that is, $\sum_{k=1}^n (\pi - \widehat{q_k}) = 0$. Hence the angle between the opposite directions is π at the point q_k. □

Theorem 3.2 *Let γ be a closed spherical curve at a distance r from q in a CAT(k) space and let γ' be a circle of radius r centered at q' in the model space S_k. Additionally, if $k > 0$ we assume $r < \frac{\pi}{2\sqrt{k}}$. Suppose that the following statements hold:*

(1) $\ell(\gamma) = \ell(\gamma')$;
(2) $\kappa_c(\gamma) = \kappa_c(\gamma')$.

Then for any subarc γ_{xy} of γ with $\ell(\gamma_{xy}) \leq \frac{\ell(\gamma)}{2}$ and any subarc $\gamma'_{x'y'}$ of γ' with $\ell(\gamma'_{x'y'}) \leq \frac{\ell(\gamma')}{2}$, $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ if and only if $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$.

Proof The implication part follows easily by Theorem 1.3 and (2). Now suppose that $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$, and by (1) of Remark 1.4, we have $\ell(\gamma_{xy}) \leq \ell(\gamma'_{x'y'})$. However, the strict inequality does not occur since $\ell(\gamma) = \ell(\gamma')$. Therefore, $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$. □

In Theorem 3.6, as the main result of this paper, we give characterizations of a closed curve γ in a CAT(k) space which bounds a convex surface that is isometric to the disk bounded by a circle in the model space S_k. In order to prove Theorem 3.6, we first need Lemmas 3.3 and 3.5.

Lemma 3.3 *Let γ be a closed spherical curve at a distance r from q in a CAT(k) space and let γ' be a circle of radius r centered at q' in the model space S_k. Additionally, if $k > 0$ we assume $r < \frac{\pi}{2\sqrt{k}}$. If $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'}) \leq \frac{\ell(\gamma)}{2}$ and $d(x, y) = d'(x', y')$ where γ_{xy} is a subarc of γ and $\gamma'_{x'y'}$ is a subarc of γ', then $C\big(\{q'\} \cup \gamma'_{x'y'}\big)$ and $C\big(\{q\} \cup \gamma_{xy}\big)$ are isometric to each other.*

Proof We define a map j from $C\big(\{q'\}\cup\gamma'_{x'y'}\big)$ to $C\big(\{q\}\cup\gamma_{xy}\big)$ in such a way that every segment $[q', z']$ from q' to a point z' on $\gamma'_{x'y'}$ is transfered on to the geodesic segment $[q, z]$ from q to a point z on γ_{xy}, where z is the point such that $\ell(\gamma_{xz}) = \ell(\gamma'_{x'z'})$. As $d(x, y) = d'(x', y')$, by using Theorem 1.3, we have that j is an isometry. The lemma is then proved. □

If we set a map j which is similar to the one in Lemma 3.3, then the two following lemmas follow from Theorem 1.3 as well.

Lemma 3.4 *Let γ be a closed spherical curve at a distance r from q in a CAT(k) space and let γ' be a circle of radius r centered at q' in the model space S_k. Additionally, if $k > 0$ we assume $r < \frac{\pi}{2\sqrt{k}}$. Let γ_{xy} be a subarc of γ and $\gamma'_{x'y'}$ be a subarc of γ'. If one of the following statements holds:*

(1) $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ *and* $\angle_q(x, y) = \angle_{q'}(x', y')$;
(2) $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ *and* $d(x, y) = d'(x', y')$;
(3) $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ *and* $\angle_q(x, y) = \angle_{q'}(x', y')$,

then $C\big(\{q'\} \cup \gamma'_{x'y'}\big)$ and $C\big(\{q\} \cup \gamma_{xy}\big)$ are isometric to each other.

Let $x, y \in \gamma$ be two distinct points of a closed spherical curve γ in a CAT(k) space. Then we have two subarcs with endpoints x, y, and we let γ^{xy} denote the shorter one and call it the *minor subarc*.

Lemma 3.5 *Let γ be a closed spherical curve at a distance r from q in a CAT(k) space and let γ' be a circle of radius r centered at q' in the model space S_k. Additionally, if $k > 0$ we assume $r < \frac{\pi}{2\sqrt{k}}$. Suppose that the following statements hold:*

(1) $\ell(\gamma) = \ell(\gamma')$;
(2) $d(x, y) = d'(x', y')$ *whenever* $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ *for any subarc* γ_{xy} *of* γ *and any subarc* $\gamma'_{x'y'}$ *of* γ'.

If x_1, x_2, x_3 *and* x_4 *are different consecutive vertices on* γ *and* x'_1, x'_2, x'_3 *and* x'_4 *are different consecutive vertices on* γ' *such that* $\ell(\gamma_{x_1x_2}) = \ell(\gamma'_{x'_1x'_2})$, $\ell(\gamma_{x_2x_3}) = \ell(\gamma'_{x'_2x'_3})$ *and* $\ell(\gamma_{x_3x_4}) = \ell(\gamma'_{x'_3x'_4})$, *then the geodesic segment* $[x_1, x_3]$ *meets the geodesic segment* $[x_2, x_4]$ *at a point.*

Proof By Lemma 3.3, we have that $C(\{q'\} \cup \gamma'^{x'_1x'_3})$ is isometric to $C(\{q\} \cup \gamma^{x_1x_3})$ and $C(\{q'\} \cup \gamma'^{x'_2x'_4})$ is isometric to $C(\{q\} \cup \gamma^{x_2x_4})$. Without loss of generality, we can suppose that $x_3 \in \gamma^{x_2x_4}$ and $x_4 \in \gamma^{x_3x_1}$. Let x' be the point of intersection between the segments $[x'_1, x'_3]$ and $[x'_2, x'_4]$ and let x be a point on geodesic segment $[x_1, x_3]$ such that $d(x, x_3) = d'(x', x'_3)$. Since $C(\{q'\} \cup \gamma'^{x'_2x'_4})$ is isometric to $C(\{q\} \cup \gamma^{x_2x_4})$ and $C(\{q'\} \cup \gamma'^{x'_1x'_3})$ is isometric to $C(\{q\} \cup \gamma^{x_1x_3})$, we have that x is the corresponding point of x'. Hence we obtain that $d(x_4, x) = d'(x'_4, x')$ and $d(x_2, x) = d'(x'_2, x')$. Consequently,

$$\begin{aligned} d(x_2, x_4) &\leq d(x_2, x) + d(x, x_4) \\ &= d'(x'_2, x') + d'(x', x'_4) \\ &= d'(x'_2, x'_4) \\ &= d(x_2, x_4), \end{aligned}$$

which means that x is a point on the geodesic segments $[x_2, x_4]$. Therefore, x is the point of intersection of two geodesic segments $[x_1, x_3]$ and $[x_2, x_4]$. □

Now we are ready to prove Theorem 3.6.

Theorem 3.6 *Let* γ *be a closed spherical curve at a distance* r *from* q *in a CAT*(k) *space and let* γ' *be a circle of radius* r *centered at* q' *in the model space* S_k. *Additionally, if* $k > 0$ *we assume* $r < \frac{\pi}{2\sqrt{k}}$. *Suppose that the following statements hold:*

(1) $\ell(\gamma) = \ell(\gamma')$;
(2) $d(x, y) = d'(x', y')$ *whenever* $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ *for any subarc* γ_{xy} *of* γ *and any subarc* $\gamma'_{x'y'}$ *of* γ'.

Then $C(\gamma')$ *is isometric to* $C(\gamma)$, *that is, the totally geodesic surface bounded by* γ *and the disk bounded by* γ' *are isometric to each other.*

Proof Let $x, y \in \gamma$ and $x', y' \in \gamma'$ be such that $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'}) = \frac{\ell(\gamma)}{2}$. By hypothesis, we thus have that $d'(x', y') = d(x, y)$. We define a map j_1 from $C\big(\{q'\} \cup \gamma'_{x'y'}\big)$ to $C\big(\{q\} \cup \gamma_{xy}\big)$ in such a way that every segment $[q', z']$ from q' to a point z' on $\gamma'_{x'y'}$ is transfered on to the geodesic segment $[q, z]$ from q to a point z on γ_{xy}, where z is the point such that $\ell(\gamma'_{x'z'}) = \ell(\gamma_{xz})$ and a map j_2 is defined from $C\big(\{q'\} \cup \gamma'_{y'x'}\big)$ to $C\big(\{q\} \cup \gamma_{yx}\big)$ similar to j_1. By Lemma 3.3, we get that j_1 and j_2 are isometries.

Now we are in a position to show that $C(\gamma')$ is isometric to $C(\gamma)$. We first note that $C(\gamma)$ exists and is unique by the definition of convex hull. Let j be a map from $C(\gamma') = C(\gamma'_{x'y'}) \cup C(\gamma'_{y'x'})$ to $C(\gamma)$ in such a way that the function j on $C(\gamma'_{x'y'})$ is j_1 and on $C(\gamma'_{y'x'})$ is j_2. To verify that j is an isometry from $C(\gamma')$ to $C(\gamma)$, we have to show that j is an isometry onto its image and $C(\gamma) = C\big(\gamma_{xy} \cup \gamma_{yx}\big) = C(\gamma_{xy}) \cup C(\gamma_{yx})$. It is obvious that j is surjective. Moreover, j is injective following from the conditions of isometric convex hulls and intersecting geodesics, as we proved in Lemma 3.5. Let $u_1', u_2' \in C(\gamma')$ and $u_1 = j(u_1')$ and $u_2 = j(u_2')$. We shall show that $d'(u_1', u_2') = d(u_1, u_2)$. There is nothing to prove if $u_1', u_2' \in C(\gamma'_{x'y'})$ or $u_1', u_2' \in C(\gamma'_{y'x'})$. Suppose that $u_1' \in C(\gamma'_{x'y'})$ and $u_2' \in C(\gamma'_{y'x'})$, let $[q', v_1']$, where $v_1' \in \gamma'_{x'y'}$, be the segment containing u_1' and let $[q', v_2']$, where $v_2' \in \gamma'_{y'x'}$, be the segment containing u_2'. On M, we let $[q, v_1]$ be the geodesic segment containing u_1 and let $[q, v_2]$ be the geodesic segment containing u_2 where $v_1 \in \gamma_{xy}$ and $v_2 \in \gamma_{yx}$. If $\ell(\gamma'_{v_1'v_2'}) \le \ell(\gamma')/2$, we then have $\gamma'_{v_1'v_2'} = \gamma'_{v_1'y'} \cup \gamma'_{y'v_2'}$. Since $C(\gamma'_{v_1'y'})$ is isometric to $C(\gamma_{v_1y})$ by j_1 and $C(\gamma'_{y'v_2'})$ is isometric to $C(\gamma_{yv_2})$ by j_2, we get that $C(\gamma'_{v_1'v_2'})$ is isometric to $C(\gamma_{v_1v_2})$ by j. Consequently, $d'(u_1', u_2') = d(u_1, u_2)$. Additionally, if $\ell(\gamma'_{v_2'v_1'}) \le \ell(\gamma')/2$, we do the same as in the case that $\ell(\gamma'_{v_1'v_2'}) \le \ell(\gamma')/2$. We also have that $d'(u_1', u_2') = d(u_1, u_2)$.

Now we shall verify that $C\big(\gamma_{xy} \cup \gamma_{yx}\big) = C(\gamma_{xy}) \cup C(\gamma_{yx})$. It suffices to show that $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is convex. Let x_1 and x_2 be two points in $C(\gamma_{xy}) \cup C(\gamma_{yx})$. It is clear that $[x_1, x_2] \in C(\gamma_{xy}) \cup C(\gamma_{yx})$ if both x_1 and x_2 are in the same convex hull. Without loss of generality, we can suppose that $x_1 \in C(\gamma_{xy})$ and $x_2 \in C(\gamma_{yx})$. Let $[q, w_1]$, where $w_1 \in \gamma_{xy}$, be the segment containing x_1 and $[q, w_2]$, where $w_2 \in \gamma_{yx}$, the segment containing x_2. Since j_1 is the isometry from $C(\gamma'_{x'y'})$ to $C(\gamma_{xy})$ and j_2 is the isometry from $C(\gamma'_{y'x'})$ to $C(\gamma_{yx})$, we let two points w_1' and w_2' in S_k be the points corresponding to w_1 and w_2, respectively, and let two points x_1' and x_2' in S_k be the points corresponding to x_1 and x_2, respectively. If $\ell(\gamma'_{w_1'w_2'}) \le \ell(\gamma')/2$, we then have $\gamma'_{w_1'w_2'} = \gamma'_{w_1'y'} \cup \gamma'_{y'w_2'}$. Since $C(\gamma'_{w_1'y'})$ is isometric to $C(\gamma_{w_1y})$ by j_1 and $C(\gamma'_{y'w_2'})$ is isometric to $C(\gamma_{yw_2})$ by j_2, we get that $C(\gamma'_{w_1'w_2'})$ is isometric to $C(\gamma_{w_1w_2})$ by j. Consequently, $d'(x_1', x_2') = d(x_1, x_2)$. Let $\hat{x}$ be the point of intersection of $[x', y']$ and $[x_1', x_2']$, and let $x = j(\hat{x}) = j_1(\hat{x}) = j_2(\hat{x})$. Then

$$d(x_1, x_2) = d'(x_1', x_2') = d'(x_1', \hat{x}) + d'(\hat{x}, x_2') = d(x_1, x) + d(x, x_2),$$

which means that $[x_1, x_2] = [x_1, x] \cup [x, x_2] \subset C(\gamma_{xy}) \cup C(\gamma_{yx})$. Therefore, $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is convex. If $\ell(\gamma'_{w_2'w_1'}) \le \ell(\gamma')/2$, we proceed in the same manner as in the case $\ell(\gamma'_{w_1'w_2'}) \le \ell(\gamma')/2$ to get that $C(\gamma_{xy}) \cup C(\gamma_{yx})$ is convex.

Then we have that $C(\gamma')$ is isometric to $C(\gamma)$. This completes the proof of the theorem. □

Theorem 3.7 gives three characterizations of a closed curve γ in a CAT(k) space, which bounds a surface that is isometric to the disk bounded by a circle γ' in the model space S_k with the same radius.

Theorem 3.7 *Let γ be a closed spherical curve at a distance r from q in a CAT(k) space and let γ' be a circle of radius r centered at q' in the model space S_k. Additionally, if $k > 0$ we assume $r < \frac{\pi}{2\sqrt{k}}$. Suppose that one of the following statements holds:*

(1) *$\ell(\gamma) = \ell(\gamma')$; $\angle_q(x, y) = \angle_{q'}(x', y')$ whenever $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ for any subarc γ_{xy} of γ and any subarc $\gamma'_{x'y'}$ of γ';*
(2) *$\kappa_c(\gamma) = \kappa_c(\gamma')$; $d(x, y) = d'(x', y')$ whenever $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ for any subarc γ_{xy} of γ and any subarc $\gamma'_{x'y'}$ of γ';*
(3) *$\kappa_c(\gamma) = \kappa_c(\gamma')$; $\angle_q(x, y) = \angle_{q'}(x', y')$ whenever $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'})$ for any subarc γ_{xy} of γ and any subarc $\gamma'_{x'y'}$ of γ'.*

Then the totally geodesic surface bounded by γ and the disk bounded by γ' are isometric to each other.

Proof We suppose that the statement (1) holds. Let γ_{xy} be a subarc of γ and let $\gamma'_{x'y'}$ be a subarc of γ' such that $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'}) = \frac{\ell(\gamma)}{2}$. By (1) of Lemma 3.4, we have that $C(\{q'\} \cup \gamma'_{x'y'})$ is isometric to $C(\{q\} \cup \gamma_{xy})$. We also prove in the same manner as in Theorem 3.6 that $C(\gamma')$ is isometric to $C(\gamma)$.

We suppose that the statement (2) (or (3)) holds. Let γ_{xy} be a subarc of γ and $\gamma'_{x'y'}$ be a subarc of γ' such that $\kappa(\gamma_{xy}) = \kappa(\gamma'_{x'y'}) = \frac{\kappa(\gamma)}{2}$. By (2) (or (3)) of Lemma 3.4, we have that $C(\{q'\} \cup \gamma'_{x'y'})$ is isometric to $C(\{q\} \cup \gamma_{xy})$. We can mimic the same idea as in Theorem 3.6 to prove that $C(\gamma')$ is isometric to $C(\gamma)$. □

By Theorem 3.7, we have a next following corollary.

Corollary 3.8 *Let γ be a closed spherical curve at a distance $r < \frac{\pi}{2\sqrt{k}}$ from a point q with perimeter $s < \frac{2\pi}{\sqrt{k}}$ in a CAT(k) space, for $k > 0$, and let γ' be a circle of radius r centered at a point q' in the model space S_k. Suppose that the total curvature of γ is $2\pi\sqrt{1 - \frac{ks^2}{4\pi^2}}$ and $d(x, y) = d'(x', y')$ (or $\angle_q(x, y) = \angle_{q'}(x', y')$) whenever $\ell(\gamma_{xy}) = \ell(\gamma'_{x'y'})$ for any subarc γ_{xy} of γ and any subarc $\gamma'_{x'y'}$ of γ', then the geodesic surface bounded by γ and the disk bounded by γ' are isometric to each other.*

It is worth remarking that the assumption of the space M being a CAT(k) space is crucial in Theorem 3.6 and Theorem 3.7. For example, we consider a set $A := R^2 - \{(x, y) \in R^2 : 0.2 < x^2 + y^2 < 0.5\}$ as a subspace of the 2-dimensional Euclidean space, S_0. We have that A is not a CAT(0) space. Let γ be the unit circle centered at the origin in A and γ' be the unit circle centered at the origin in S_0. We have that γ and γ' satisfy Theorem 3.6 and (2) of Theorem 3.7 but the region bounded by γ and the disk bounded by γ' are not isometric to each other.

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