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Another proof of the local curvature estimate for the Ricci flow

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Abstract By using the De Giorgi iteration method we will give a new simple proof of the recent result of Kotschwar et al. (J Funct Anal 271(9):2604–2630, 2016) and Sesum (Am J Math 127(6):1315–1324, 2005) on the local boundedness of the Riemannian curvature tensor of solutions of Ricci flow in terms of its inital value on a given ball and a local uniform bound on the Ricci curvature.

Keywords Ricci flow · Local boundedness · Riemmanian curvature · Ricci curvature

Mathematics Subject Classification (2010) Primary 58J35 · 35B45; Secondary 35K10

1 Introduction

There is a lot of interest on Ricci flow [2,3,10,13,14] because it is a very powerful tool in the study of the geometry of manifolds. Recently Perelman [15,16], by using the the Ricci flow technique solved the famous Poincare conjecture in geometry. Let $(M, g(t))$, $0 < t < T$, be a n-dimensional Riemannian manifold. We say that the metric $g(t) = (g_{ij}(t))$ evolves by the Ricci flow if it satisfies

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1.1}$$

on $M \times (0, T)$ where R_{ij} is the Ricci curvature of the metric $g(t) = (g_{ij}(t))$. Short time existence of solution of Ricci flow on compact Riemannian manifolds with any initial metric at $t = 0$ was proved by Hamilton in [9]. Short time existence of solution of Ricci flow on complete non-compact manifolds with bounded curvature initial metric at time $t = 0$ was proved by Shi in [18,19]. When M is a compact manifold, Hamilton [9] proved that either the Ricci flow solution exists globally or there exists a maximal existence time $0 < T < \infty$

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for the solution of Ricci flow and

$$\lim_{t\nearrow T}|Rm|_{g(t)}=\infty.$$

Hence in order to know whether the solution of Ricci flow can be extended beyond its interval of existence $(0,T)$, it is important to prove boundedness of the Riemannian curvature for the solution of Ricci flow near the time T. Uniform boundedness of the Riemannian curvature of the solution of Ricci flow on a compact manifold when the solution has uniform bounded Ricci curvature on $(0,T)$ was proved by Sesum in [17] using a blow-up contradiction argument and Perelman's noncollapsing result [15]. Local boundedness of the Riemannian curvature for κ-noncollapsing solutions of Ricci flow in term of its local $L^{\frac{n}{2}}$ norm when its local $L^{\frac{n}{2}}$ norm is sufficiently small was also proved by Ye in [20,21], using Moser iteration technique and the point picking technique of Perelman [15]. Similar result was also obtained by Dai et al. in [5].

Local boundedness of the Riemannian curvature of the solution of Ricci flow in terms of its inital value on a given ball and a local uniform bound on the Ricci curvature was proved by Kotschwar et al. using Moser iteration technique and results of Li [12] in [11]. A similar local Riemannian curvature result was proved recently by Chen [4] using the point picking technique of Perelman [15], Anderson's harmonic coordinates [1] and elliptic regularity results [8]. In this paper we will use the De Giorgi iteration method to give a new simple proof of this result.

We will assume that $(M,g(t))$ is a smooth solution of the Ricci flow (1.1) in $[0,T)$ for the rest of the paper. For any $x_0\in M$, $\rho>0$ and $0\le t<T$, we let $B_{g(t)}(x_0,\rho)=\{x\in M:\text{dist}_{g(t)}(x,x_0)<\rho\}$, $V_{x_0}(\rho,t)=\text{vol}_{g(t)}(B_{g(0)}(x_0,\rho))$, $V_{x_0}(\rho)=V_{x_0}(\rho,0)$, $|Ric|(x,t)=|Ric(x,t)|_{g(t)}$ and $|Rm|(x,t)=|Rm(x,t)|_{g(t)}$. We let dv_t be the volume element of the metric $g(t)$ and let $C>0$ denote a generic constant that may change from line to line. For any complete Riemannian manifold (M,g), we let $B(x_0,\rho)=\{x\in M:\text{dist}_g(x,x_0)<\rho\}$, $V_{x_0}(\rho)=\text{vol}_g(B(x_0,\rho))$ and dv be the volume element of the metric g.

Note that by Corollary 13.3 of [9] or Lemma 7.4 of [2],

$$\frac{\partial}{\partial t}|Rm|^2\le\Delta|Rm|^2-2|\nabla Rm|^2+C|Rm|^3 \tag{1.2}$$

in $(0,T)$ for some constant $C>0$ depending only on n. Since $|\nabla|Rm||\le|\nabla Rm|$, by (1.2),

$$\frac{\partial}{\partial t}|Rm|\le\Delta|Rm|+C|Rm|^2\quad\text{in }(0,T). \tag{1.3}$$

We will prove the following main result in this paper.

Theorem 1.1 (cf. Theorem 1 of [11]) *Let $g(t)$, $0\le t<T$, be a smooth solution of Ricci flow on a n-dimensional Riemannian manifold M. Suppose there exists $x_0\in M$ and constants $K>0$, $\rho>0$, such that*

$$|Ric|\le K\quad in\ B_{g(0)}\left(x_0,\frac{2\rho}{\sqrt{K}}\right)\times[0,T) \tag{1.4}$$

and

$$\Lambda_0:=\sup_{B_{g(0)}\left(x_0,\frac{2\rho}{\sqrt{K}}\right)}|Rm|(x,0)<\infty. \tag{1.5}$$

Then for any $n \geq 3$ *and* $p > \frac{n+2}{2}$ *there exist constants* $C_0 > 0$ *and* $C > 0$ *such that*

$$|Rm|(x,t) \leq C_0 \left\{ \frac{\rho^{\frac{2n}{n+2}} e^{C(\rho+tK)}}{K^{\frac{n}{n+2}} V_{x_0}\left(\rho/\sqrt{K}\right)^{\frac{2}{n+2}} \min(t, \rho^2/K)} \left[(K + E_p(t)^{\frac{1}{p}})t + 1\right] \right\}^{\frac{n+2}{2p}}$$

$$\left(1 + \sqrt{t V_{x_0}\left(2\rho/\sqrt{K}\right)} E_p(t)^{\frac{1}{2}}\right)^{\frac{n+4}{2p}} \tag{1.6}$$

holds for any $x \in B_{g(0)}\left(x_0, \rho/\sqrt{K}\right)$ *and* $0 < t < T$ *where*

$$E_p(t) = Ce^{CKt} t \left[\Lambda_0^{2p} V_{x_0}\left(2\rho/\sqrt{K}\right) + K^{2p}(1 + \rho^{-4p}) V_{x_0}\left(\rho/\sqrt{K}\right)\right]$$

and for $n = 2$ *and any* $p > \frac{5}{2}$ *there exist constants* $C_0 > 0$ *and* $C > 0$ *such that*

$$|Rm|(x,t) \leq C_0 \left\{ \frac{\rho^{\frac{4}{5}} e^{C(\rho+tK)}}{K^{\frac{2}{5}} V_{x_0}\left(\rho/\sqrt{2K}\right)^{\frac{2}{5}} \min(t, \rho^2/K)} \left[(K + (4\rho/\sqrt{K})^{\frac{1}{p}} E_p(t)^{\frac{1}{p}})t + 1\right] \right\}^{\frac{5}{2p}}$$

$$\times (1 + (4\rho/\sqrt{K})\sqrt{t V_{x_0}\left(2\rho/\sqrt{K}\right)} E_p(t)^{\frac{1}{2}})^{\frac{7}{2p}} \tag{1.7}$$

holds for any $x \in B_{g(0)}\left(x_0, \rho/\sqrt{K}\right)$ *and* $0 < t < T$.

Remark 1.2 Note that the bounds for the Riemannian curvature in (1.6) and (1.7) are slightly different from that of Theorem 1 of [11]. When $t \to \infty$, both the right hand side of (1.6), (1.7), and the bound in Theorem 1 of [11] are approximately equal to e^{CKt} for some constant $C > 0$. However, for $0 < t < \rho^2/K$ and t close to zero, the right hand side of (1.6) and (1.7) are approximately equal to $Ct^{-\frac{n+2}{2p}}$ and $Ct^{-\frac{5}{2p}}$ respectively for some constant $C > 0$, while the bound in Theorem 1 of [11] is approximately equal to $Ct^{-\beta}$ for some constant $\beta > 0$. Since the constant β in Theorem 1 of [11] is unknown, Theorem 1.1 is therefore a refinement of the result in Theorem 1 of [11].

2 The main result

We first recall a result of [11]:

Proposition 2.1 (Proposition 1 of [11]) *Let* $g(t)$, $0 \leq t < T$, *be a smooth solution of Ricci flow on a* n*-dimensional Riemannian manifold* M. *Suppose there exists* $x_0 \in M$ *and constants* $K > 0$, $\rho > 0$, *such that* (1.4) *holds. Then for any* $n \geq 2$ *and* $q \geq 3$ *there exists a constant* $c = c(n,q) > 0$ *such that*

$$\int_{B_{g(0)}\left(x_0, \frac{\rho}{\sqrt{K}}\right)} |Rm|(x,t)^q \, dv_t$$
$$\leq ce^{cKt}\left\{\int_{B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right)} |Rm|(x,0)^q \, dv_0 + cK^q(1+\rho^{-2q})V_{x_0}\left(\rho/\sqrt{K}, t\right)\right\}$$

holds for any $0 \leq t < T$.

Proof A proof of this result is given in [11]. For the sake of completeness we will give a sketch of the proof of this result in this paper. By using (1.2), the inequalities (Chapter 6 of [2] or Lemma 1 of [11]),

$$\begin{cases} |\nabla Ric|^2 \leq \frac{1}{2}(\Delta - \partial_t)|Ric|^2 + CK^2|Rm| \\ \partial_t R^l_{ijk} = g^{lq}(\nabla_i \nabla_q R_{jk} + \nabla_j \nabla_i R_{kq} + \nabla_j \nabla_k R_{iq}) - g^{lq}(\nabla_i \nabla_j R_{kq} + \nabla_i \nabla_k R_{jq} + \nabla_j \nabla_q R_{ik}), \end{cases}$$

and a direct computation one can show that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{d}{dt}\left(\int_M |Rm|^p \phi^{2p} \, dv_t + \frac{1}{K}\int_M |Ric|^2|Rm|^{p-1}\phi^{2p} \, dv_t + c_1 K \int_M |Rm|^{p-1}\phi^{2p} \, dv_t\right)$$
$$\leq c_2 K \int_M |Rm|^p \phi^{2p} \, dv_t + c_2 K \int_M |Rm|^{p-1}\phi^{2p-2} \, dv_t$$

holds on $M \times (0, T)$ for any Lipschitz function ϕ with support in $B\left(x_0, \frac{2\rho}{\sqrt{K}}\right)$. Proposition 2.1 then follows by choosing an appropriate cut-off function ϕ for the set $B\left(x_0, \frac{\rho}{\sqrt{K}}\right)$ and integrating the above differential inequality over $(0, t)$, $0 < t < T$. □

Lemma 2.2 (cf. Theorem 14.3 of [12]) *Let* (M, g) *be a complete Riemannian manifold of dimension* $n \geq 3$ *with Ricci curvature satisfying*

$$R_{ij} \geq -(n-1)k_1 \quad \text{on } B(x_0, \rho)$$

for some constant $k_1 \geq 0$. *Then there exists constants* $c_1 > 0$ *and* $c_2 > 0$ *depending only on* n *such that for any function* $f \in H_c^{1,2}(B(x_0, \rho))$ *with compact support in* $B(x_0, \rho)$, f *satisfies*

$$\left(\int_{B(x_0,\rho)} |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} dv \leq c_1 \frac{\rho^2 e^{c_2 \rho \sqrt{k_1}}}{V_{x_0}(\rho)^{2/n}} \int_{B(x_0,\rho)} |\nabla f|^2 \, dv$$

Theorem 2.3 *Let* $g(t)$, $0 \leq t < T$, *be a smooth solution of Ricci flow on a* n*-dimensional Riemannian manifold* M. *Suppose there exists* $x_0 \in M$ *and constants* $K > 0$, $\rho > 0$, *such that* (1.4) *holds. Then for any* $n \geq 3$ *and* $p > \frac{n+2}{2}$ *there exist constants* $C_0 > 0$ *and* $C > 0$ *such that*

$$|Rm|(x,t) \leq C_0 \left\{\frac{\rho^{\frac{2n}{n+2}} e^{C(\rho + tK)}}{K^{\frac{n}{n+2}} V_{x_0}\left(\rho/\sqrt{K}\right)^{\frac{2}{n+2}} \min(t, \rho^2/K)} \left[\left(\left(\iint_{Q_0} |Rm|^{2p} \, dv_0 \, dt\right)^{\frac{1}{p}} + K\right) t + 1\right]\right\}^{\frac{n+2}{2p}}$$
$$\times \left(1 + \iint_{Q_0} |Rm|^p \, dv_0 \, dt\right)^{\frac{n+4}{2p}} \tag{2.1}$$

holds for any $x \in B_{g(0)}\left(x_0, \rho/\sqrt{K}\right)$ *and* $0 < t < T$ *where* $Q_0 = B_{g(0)}\left(x_0, 2\rho/\sqrt{K}\right) \times (t/4, t)$ *and for* $n = 2$ *and any* $p > \frac{5}{2}$ *there exist constants* $C_0 > 0$ *and* $C > 0$ *such that*

$$|Rm|(x,t) \le C_0 \left\{ \frac{\rho^{\frac{4}{5}} e^{C(\rho+tK)}}{K^{\frac{2}{5}} V_{x_0}\left(\rho/\sqrt{2K}\right)^{\frac{2}{5}} \min(t, \rho^2/K)} \left[\left(K + \left(\frac{4\rho}{\sqrt{K}} \iint_{Q_0} |Rm|^{2p}\, dv_0\, dt\right)^{\frac{1}{p}}\right) t + 1\right]\right\}^{\frac{5}{2p}} \times \left(1 + \frac{4\rho}{\sqrt{K}} \iint_{Q_0} |Rm|^p\, dv_0\, dt\right)^{\frac{7}{2p}} \tag{2.2}$$

holds for any $x \in B_{g(0)}\left(x_0, \rho/\sqrt{K}\right)$ *and* $0 < t < T$.

Proof **Case 1**: $n \ge 3$.

Let $v = |Rm|$, $0 < t < T$ and $p > \frac{n+2}{2}$. We will use a modification of the proof of Proposition 2.1 of [6] to prove this theorem. By (1.4),

$$\begin{aligned} & e^{-2Kt} g_{ij}(x,0) \le g(x,s) \le e^{2Kt} g_{ij}(x,0) && \forall x \in B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right), 0 \le s < T \\ \Rightarrow \quad & e^{-nKt} dv_0 \le dv_s \le e^{nKt} dv_0 && \forall x \in B_{g(0)}\left(x_0, \frac{2\rho}{\sqrt{K}}\right), 0 \le s \le t < T. \end{aligned} \tag{2.3}$$

Let $\rho_m = \left(\rho/\sqrt{K}\right)(1+2^{-m})$ and $t_m = (1-2^{-m-1})t/2$ for any $m \ge 0$. Then $\rho_0 = 2\rho/\sqrt{K}$ and $t_0 = t/4$. Moreover ρ_m decreases to $\rho/\sqrt{K}$ and t_m increases to $t/2$ as $m \to \infty$. Let $B_{\rho_m} = B_{g(0)}(x_0, \rho_m)$, $Q_m = B_{\rho_m} \times (t_m, t)$ and $Q_m^s = B_{\rho_m} \times (t_m, s)$ for any $t_m \le s \le t$. Then

$$B_{2\rho/\sqrt{K}} \times (t/4, t) = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_{m-1} \supseteq Q_m \supseteq \cdots \supseteq Q_\infty = B_{\rho/\sqrt{K}} \times (t/2, t).$$

We choose a sequence of Lipschitz continuous functions $\{\phi_m\}$ on $M \times (0,t)$ such that $0 \le \phi_m \le 1$ on $M \times (0,t)$, $\phi_m(x,s) = 1$ for $(x,s) \in Q_{m+1}$, $\phi_m(x,s) = 0$ for $(x,s) \in M \times (0,t) \setminus Q_m$, and satisfying

$$\begin{cases} |\nabla \phi_m| \le \dfrac{C\sqrt{K}2^m}{\rho} & \text{in } Q_m \\ 0 \le \phi_{m,t} \le \dfrac{C2^m}{t} & \text{in } Q_m. \end{cases} \tag{2.4}$$

Let $k > 0$ be a constant to be determined later and $k_m = k(1 - 2^{-m})$ for any $m \geq 0$. Multiplying (1.3) by $(v - k_{m+1})_+^{p-1}\phi_m^2$ and integrating over Q_m^s, $t_m \leq s \leq t$,

$$\begin{aligned}
&\frac{1}{p}\iint_{Q_m^s} \phi_m^2 \frac{\partial}{\partial t}(v - k_{m+1})_+^p \, dv_t\, dt + \iint_{Q_m^s} \nabla(v - k_{m+1})_+ \cdot [(p-1)(v - k_{m+1})_+^{p-2}\phi_m^2 \nabla(v - k_{m+1})_+ \\
&\quad + 2(v - k_{m+1})_+^{p-1}\phi_m \nabla\phi_m]\, dv_t\, dt \\
&\leq C\iint_{Q_m^s} v^2 (v - k_{m+1})_+^{p-1}\phi_m^2 \, dv_t\, dt \\
&\leq C\left(\iint_{Q_0} v^{2p}\, dv_t\, dt\right)^{\frac{1}{p}} \left(\iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m^2 \, dv_t\, dt\right)^{\frac{p-1}{p}}. \qquad (2.5)
\end{aligned}$$

Since $\frac{d}{dt}(dv_t) = -R dv_t$, by (1.4),

$$\begin{aligned}
&\iint_{Q_m^s} \phi_m^2 \frac{\partial}{\partial t}(v - k_{m+1})_+^p \, dv_t\, dt = \int_{t_m}^s \frac{d}{dt}\left(\int_{B_{\rho m}} (v - k_{m+1})_+^p \phi_m^2 \, dv_t\right) dt - 2\iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m \phi_{m,t}\, dv_t\, dt \\
&\quad + \iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m^2 R\, dv_t\, dt \\
&\geq \int_{B_{\rho m}} (v(x,s) - k_{m+1})_+^p \phi_m(x,s)^2 \, dv_s - C\frac{2^m}{t}\iint_{Q_m^s} (v - k_{m+1})_+^p \, dv_t\, dt \\
&\quad - CK\iint_{Q_m^s} (v - k_{m+1})_+^p \phi_m^2 \, dv_t\, dt. \qquad (2.6)
\end{aligned}$$

Since

$$\int_{B_{\rho m}} |\nabla(\phi_m (v - k_{m+1})_+^{\frac{p}{2}})|^2 \, dv_t \leq \frac{11}{10}\int_{B_{\rho m}} \phi_m^2 |\nabla(v - k_{m+1})_+^{\frac{p}{2}}|^2 \, dv_t + 11\int_{B_{\rho m}} (v - k_{m+1})_+^p |\nabla\phi_m|^2 \, dv_t$$

and

$$\begin{aligned}
&2\int_{B_{\rho m}} \phi_m (v - k_{m+1})_+^{p-1} \nabla\phi_m \cdot \nabla(v - k_{m+1})_+ \, dv_t \\
&= \frac{4}{p}\int_{B_{\rho m}} \phi_m (v - k_{m+1})_+^{\frac{p}{2}} \nabla\phi_m \cdot \nabla(v - k_{m+1})_+^{\frac{p}{2}} \, dv_t \\
&\geq -\frac{2}{p}\left(\frac{p-1}{p}\int_{B_{\rho m}} \phi_m^2 |\nabla(v - k_{m+1})_+^{\frac{p}{2}}|^2 \, dv_t + \frac{p}{p-1}\int_{B_{\rho m}} (v - k_{m+1})_+^p |\nabla\phi_m|^2 \, dv_t\right),
\end{aligned}$$

$$\begin{aligned}
&(p-1)\int_{B_{\rho m}} \phi_m^2 (v - k_{m+1})_+^{p-2} |\nabla(v - k_{m+1})_+|^2 \, dv_t \\
&\quad + 2\int_{B_{\rho m}} \phi_m (v - k_{m+1})_+^{p-1} \nabla\phi_m \cdot \nabla(v - k_{m+1})_+ \, dv_t \\
&\geq \frac{4(p-1)}{p^2}\int_{B_{\rho m}} \phi_m^2 |\nabla(v - k_{m+1})_+^{\frac{p}{2}}|^2 \, dv_t \\
&\quad - \frac{2}{p}\left(\frac{p-1}{p}\int_{B_{\rho m}} \phi_m^2 |\nabla(v - k_{m+1})_+^{\frac{p}{2}}|^2 \, dv_t + \frac{p}{p-1}\int_{B_{\rho m}} (v - k_{m+1})_+^p |\nabla\phi_m|^2 \, dv_t\right) \\
&\geq \frac{2(p-1)}{p^2}\left(\frac{10}{11}\int_{B_{\rho m}} |\nabla((v - k_{m+1})_+^{\frac{p}{2}}\phi_m)|^2 \, dv_t - 10\int_{B_{\rho m}} (v - k_{m+1})_+^p |\nabla\phi_m|^2 \, dv_t\right) \\
&\quad - \frac{2}{p-1}\int_{B_{\rho m}} (v - k_{m+1})_+^p |\nabla\phi_m|^2 \, dv_t
\end{aligned}$$

$$= \frac{20(p-1)}{11p^2}\int_{B_{\rho m}} |\nabla((v-k_{m+1})_+^{\frac{p}{2}}\phi_m)|^2\, dv_t$$
$$- \frac{CK4^m}{\rho^2}\left(\frac{20(p-1)}{p^2}+\frac{2}{p-1}\right)\int_{B_{\rho m}} (v-k_{m+1})_+^p\, dv_t. \tag{2.7}$$

By (2.5), (2.6) and (2.7),

$$\int_{B_{\rho m}} v(x,s)^p\phi_m(x,s)^2\, dv_s + \iint_{Q_m^s} |\nabla((v-k_{m+1})_+^{\frac{p}{2}}\phi_m)|^2\, dv_t\, dt$$
$$\le C\left(\iint_{Q_0} v^{2p}\, dv_t\, dt\right)^{\frac{1}{p}}\left(\iint_{Q_m^s}(v-k_{m+1})_+^p\, dv_t\, dt\right)^{\frac{p-1}{p}}$$
$$+ C\left(K+\frac{2^m}{t}+\frac{K4^m}{\rho^2}\right)\iint_{Q_m^s}(v-k_{m+1})_+^p\, dv_t\, dt \tag{2.8}$$

By (2.3) and (2.8),

$$\sup_{t_m\le s\le t}\int_{B_{\rho m}} v(x,s)^p\phi_m(x,s)^2\, dv_0 + \iint_{Q_m} |\nabla((v-k_{m+1})_+^{\frac{p}{2}}\phi_m)|_{g(0)}^2\, dv_0\, dt$$
$$\le Ce^{CKt}\left\{A_1 Y_m^{\frac{p-1}{p}} + \left(K+\frac{2^m}{t}+\frac{K4^m}{\rho^2}\right)Y_m\right\} \tag{2.9}$$

where

$$A_1 = \left(\iint_{Q_0} v^{2p}\, dv_0\, dt\right)^{\frac{1}{p}} \tag{2.10}$$

and

$$Y_m = \iint_{Q_m}(v-k_m)_+^p\, dv_0\, dt.$$

By Lemma 2.2,

$$\int_{B_{\rho m}} |\nabla((v-k_m)_+^{\frac{p}{2}}\phi_m)|_{g(0)}^2\, dv_0 \ge \frac{CV_{x_0}(\rho_m)^{\frac{2}{n}}}{\rho_m^2 e^{C\rho_m\sqrt{K}}}\left(\int_{B_{\rho m}}[(v-k_m)_+^{\frac{p}{2}}\phi_m]^{\frac{2n}{n-2}}\, dv_0\right)^{\frac{n-2}{n}}$$
$$\ge \frac{CKV_{x_0}(\rho/\sqrt{K})^{\frac{2}{n}}}{\rho^2 e^{C\rho}}\left(\int_{B_{\rho m}}[(v-k_m)_+^{\frac{p}{2}}\phi_m]^{\frac{2n}{n-2}}\, dv_0\right)^{\frac{n-2}{n}}. \tag{2.11}$$

By the Holder inequality,

$$Y_{m+1} = \iint_{Q_{m+1}}(v-k_{m+1})_+^p\, dv_0\, dt$$
$$\le \iint_{Q_m}(v-k_{m+1})_+^p\phi_m^2\, dv_0\, dt$$
$$\le \left(\iint_{Q_m}[(v-k_m)_+^p\phi_m^2]^{\frac{n+2}{n}}\, dv_0\, dt\right)^{\frac{n}{n+2}}|E_m|^{\frac{2}{n+2}} \tag{2.12}$$

where $E_m = \{(x,s) \in Q_m : v(x,s) > k_{m+1}\}$. By the Holder inequality and (2.11) (cf. proof of proposition 3.1 of chapter 1 of [7]),

$$\begin{aligned}
&\iint_{Q_m} [(v-k_m)_+^p \phi_m^2]^{\frac{n+2}{n}}\, dv_0\, dt \\
&= \iint_{Q_m} [(v-k_m)_+^{\frac{p}{2}} \phi_m]^2 \cdot [(v-k_m)_+^{\frac{p}{2}} \phi_m]^{\frac{4}{n}}\, dv_0\, dt \\
&\le \int_{t_m}^t \left(\int_{B_{\rho_m}} [(v-k_m)_+^{\frac{p}{2}} \phi_m]^{\frac{2n}{n-2}}\, dv_0 \right)^{\frac{n-2}{n}} \cdot \left(\int_{B_{\rho_m}} (v-k_m)_+^p \phi_m^2\, dv_0 \right)^{\frac{2}{n}} dt \\
&\le \frac{C\rho^2 e^{C\rho}}{K V_{x_0}(\rho/\sqrt{K})^{\frac{2}{n}}} \left(\iint_{Q_m} |\nabla((v-k_m)_+^{\frac{p}{2}} \phi_m)|_{g(0)}^2\, dv_0\, dt \right) \\
&\quad \cdot \left(\sup_{t_n \le s \le t} \int_{B_{\rho_m}} (v-k_m)_+^p \phi_m^2\, dv_0 \right)^{\frac{2}{n}}. \qquad (2.13)
\end{aligned}$$

By (2.9), (2.12) and (2.13),

$$Y_{m+1} \le \frac{C\rho^{\frac{2n}{n+2}} e^{C(\rho+tK)}}{K^{\frac{n}{n+2}} V_{x_0}(\rho/\sqrt{K})^{\frac{2}{n+2}}} \left\{ A_1 Y_m^{\frac{p-1}{p}} + \left(K + \frac{2^m}{t} + \frac{K4^m}{\rho^2} \right) Y_m \right\} |E_m|^{\frac{2}{n+2}}. \qquad (2.14)$$

Now (cf. proof on P.645 of [6]),

$$\begin{aligned}
Y_m &= \iint_{Q_m} (v-k_m)_+^p\, dv_0\, dt \ge (k_{m+1}-k_m)^p |E_m| = \frac{k^p}{2^{(m+1)p}} |E_m| \\
&\Rightarrow |E_m| \le \frac{2^{(m+1)p}}{k^p} Y_m. \qquad (2.15)
\end{aligned}$$

Hemce by (2.14) and (2.15),

$$\begin{aligned}
Y_{m+1} &\le \frac{C\rho^{\frac{2n}{n+2}} e^{C(\rho+tK)}}{K^{\frac{n}{n+2}} V_{x_0}(\rho/\sqrt{K})^{\frac{2}{n+2}}} \left(\frac{2^{mp}}{k^p} \right)^{\frac{2}{n+2}} \left\{ A_1 Y_m^{1+\frac{2}{n+2}-\frac{1}{p}} + \left(K + \frac{2^m}{t} + \frac{K4^m}{\rho^2} \right) Y_m^{1+\frac{2}{n+2}} \right\} \\
&\le \frac{C_1(A_1 t + Kt + 1)\rho^{\frac{2n}{n+2}} e^{C(\rho+tK)}}{k^{\frac{2p}{n+2}} K^{\frac{n}{n+2}} V_{x_0}\left(\rho/\sqrt{K}\right)^{\frac{2}{n+2}} \min(t, \rho^2/K)} \cdot b^m \max\left(Y_m^{1+\alpha}, Y_m^{1+\frac{2}{n+2}} \right) \quad \forall m \ge 0
\end{aligned} \qquad (2.16)$$

for some constant $C_1 > 0$ where $b = 4 \cdot 2^{\frac{2p}{n+2}}$ and $\alpha = \frac{2}{n+2} - \frac{1}{p}$. Then $0 < \alpha < \frac{2}{n+2}$. We now let $\beta > 1/\alpha$ and

$$k = \left\{ \frac{C_1(A_1 t + Kt + 1)\rho^{\frac{2n}{n+2}} e^{C(\rho+tK)} b^\beta}{K^{\frac{n}{n+2}} V_{x_0}\left(\rho/\sqrt{K}\right)^{\frac{2}{n+2}} \min(t, \rho^2/K)} \right\}^{\frac{n+2}{2p}} \left(1 + \int_{t/4}^t \int_{B_{2\rho/\sqrt{K}}} v^p\, dv_0\, dt \right)^{\frac{n+4}{2p}}. \qquad (2.17)$$

We claim that

$$Y_{m+1} \le b^{-\frac{(\alpha\beta-1)[(1+\alpha)^{m+1}-(1+\alpha)]}{\alpha^2} - \frac{m}{\alpha}} < 1 \quad \forall m \ge 2. \qquad (2.18)$$

In order to prove this claim we observe first that by (2.16) and (2.17),

$$Y_1 \le b^{-\beta} < 1. \tag{2.19}$$

By (2.16), (2.17) and (2.19),

$$Y_2 \le \frac{C_1(A_1 t + Kt + 1)\rho^{\frac{2n}{n+2}} e^{C(\rho + tK)}}{k^{\frac{2p}{n+2}} K^{\frac{n}{n+2}} \min(t, \rho^2/K)} \cdot bY_1^{1+\alpha} \le b^{1-\beta-\beta(1+\alpha)} < 1. \tag{2.20}$$

Repeating the above argument we get,

$$Y_{m+1} \le b^{\sum_{i=0}^{m} i(1+\alpha)^{m-i} - \beta \sum_{i=0}^{m} (1+\alpha)^i} = b^{-\frac{(\alpha\beta-1)[(1+\alpha)^{m+1}-(1+\alpha)]}{\alpha^2} - \frac{m}{\alpha}} \quad \forall m \ge 2$$

and (2.18) follows. Letting $m \to \infty$ in (2.18),

$$\lim_{m\to\infty} Y_{m+1} = 0.$$

Hence $v \le k$ in Q_∞ with k given by (2.17) and A_1 given by (2.10). Thus (2.1) follows.
Case 2: $n = 2$.
Let $\widetilde{M} = M \times \mathbb{R}$, $\widetilde{g}(x, t) = g(x, t) + dx^2$, and let $\widetilde{Rm}$, $\widetilde{Ric}$, $\widetilde{R}$, be the Riemannian curvature, Ricci curvature and scalar curvature of $(\widetilde{M}, \widetilde{g}(t))$. Then

$$|\widetilde{Rm}|(x, y, t) = |Rm|(x, t), \quad |\widetilde{Ric}|(x, y, t) = |Ric|(x, t), \quad \widetilde{R}(x, y, t) = R(x, t) \tag{2.21}$$

for all $x \in M$, $y \in \mathbb{R}$, $0 \le t < T$ and

$$\frac{\partial \widetilde{g}_{ij}}{\partial t} = -2\widetilde{R}_{ij} \quad \text{in } (0, T).$$

Let $\widetilde{V}_{x_0}(r) = \mathrm{vol}_{\widetilde{g}(0)}(B_{\widetilde{g}(0)}((x_0, 0), r))$ for any $r > 0$ and $d\widetilde{v}_0 = dv_0\, dy$ be the volume element of $\widetilde{g}(0)$. By case 1,

$$|\widetilde{Rm}|(x, y, t) \le C_0 \left\{ \frac{\rho^{\frac{6}{5}} e^{C(\rho + tK)}}{K^{\frac{3}{5}} \widetilde{V}_{(x_0,0)}\left(\rho/\sqrt{K}\right)^{\frac{2}{5}} \min(t, \rho^2/K)} \left[\left(\left(\iint_{\widetilde{Q}_0} |Rm|^{2p}\, d\widetilde{v}_0\, dt \right)^{\frac{1}{p}} + K \right) t + 1 \right] \right\}^{\frac{5}{2p}} \times \left(1 + \iint_{\widetilde{Q}_0} |Rm|^p\, d\widetilde{v}_0\, dt \right)^{\frac{7}{2p}} \tag{2.22}$$

holds for any $(x, y) \in B_{\widetilde{g}(0)}\left((x_0, 0), \rho/\sqrt{K}\right)$ and $0 < t < T$ where $\widetilde{Q}_0 = B_{\widetilde{g}(0)}\left((x_0, 0), 2\rho/\sqrt{K}\right) \times (t/4, t)$. Since $B_{\widetilde{g}(0)}\left((x_0, 0), \rho/\sqrt{K}\right) \supset B_{g(0)}\left(x_0, \rho/\sqrt{2K}\right) \times \left(-\rho/\sqrt{2K}, \rho/\sqrt{2K}\right)$,

$$\widetilde{V}_{(x_0,0)}\left(\rho/\sqrt{K}\right) \ge \left(\sqrt{2}\rho/\sqrt{K}\right) V_{x_0}\left(\rho/\sqrt{2K}\right). \tag{2.23}$$

Since $B_{\widetilde{g}(0)}\left((x_0, 0), 2\rho/\sqrt{K}\right) \subset B_{g(0)}\left(x_0, 2\rho/\sqrt{K}\right) \times \left(-2\rho/\sqrt{K}, 2\rho/\sqrt{K}\right)$, by (2.21), (2.22) and (2.23), we get (2.2) and the theorem follows. □

Remark 2.4 By Proposition 2.1, Theorem 2.3, Holder's inequality and (2.3), Theorem 1.1 follows.

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