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ORIGINAL PAPER

Strong symplectic fillability of contact torus bundles

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Abstract In this paper, we study strong symplectic fillability and Stein fillability of some tight contact structures on negative parabolic and negative hyperbolic torus bundles over the circle. For the universally tight contact structure with twisting π in S^1-direction on a negative parabolic torus bundle, we completely determine its strong symplectic fillability and Stein fillability. For the universally tight contact structure with twisting π in S^1-direction on a negative hyperbolic torus bundle, we give a necessary condition for it being strongly symplectically fillable. For the virtually overtwisted tight contact structure on the negative parabolic torus bundle with monodromy $-T^n$ $(n < 0)$, we prove that it is Stein fillable. In addition, we give a partial answer to a conjecture of Golla and Lisca.

Keywords Contact structures · Strong symplectic fillability · Stein fillability · Torus bundle

Mathematics Subject Classification 57R17 (57M50)

1 Introduction

Tight contact structures on torus bundles are classified up to isotopy, see [10] and [15]. The study of symplectic fillability and Stein fillability of contact torus bundles has been conducted in the past decade. Symplectic fillability and Stein fillability of contact elliptic, positive hyperbolic, and positive parabolic torus bundles have been completely determined. See [2,3,5,9,18,24] and [1]. In this paper, we focus on strong symplectic fillability and Stein fillability of certain contact negative parabolic and negative hyperbolic torus bundles.

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If $A \in SL(2, \mathbb{Z})$, let M_A denote the T^2-bundle over S^1 with monodromy A. That is, M_A is obtained from $T^2 \times I = \mathbb{R}^2/\mathbb{Z}^2 \times [0, 1]$, with coordinates $(\mathbf{x}, t) = (\begin{pmatrix} x \\ y \end{pmatrix}, t)$ by identifying the two ends via the map $A : T^2 \times \{1\} \to T^2 \times \{0\}$, where $(\mathbf{x}, 1) \mapsto (A\mathbf{x}, 0)$. Two torus bundles M_A and M_B are orientation-preserving diffeomorphic if and only if A is conjugate in $SL(2, \mathbb{Z})$ to B or $JB^{-1}J^{-1}$, where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (cf. [21, Lemma 6.2]). If $|\mathrm{tr}(A)| < 2$ (resp. $|\mathrm{tr}(A)| = 2$ or $|\mathrm{tr}(A)| > 2$), then A and the torus bundle M_A are called elliptic (resp. parabolic or hyperbolic). If $\mathrm{tr}(A) > 0$ (resp. $\mathrm{tr}(A) < 0$), then A and the torus bundle M_A are called positive (resp. negative).

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function with strictly positive derivative. The 1-form

$$\sin\phi(t)\mathrm{d}x + \cos\phi(t)\mathrm{d}y, \ \ (x, y, t) \in \mathbb{R}^3,$$

defines a contact structure on $\mathbb{R}^3$. This contact structure descends to a contact structure on $T^2 \times \mathbb{R} = (\mathbb{R}^2/\mathbb{Z}^2) \times \mathbb{R}$ which we denote by $\tilde{\zeta}(\phi)$.

For each $A \in SL(2, \mathbb{Z})$, M_A is the quotient of $T^2 \times \mathbb{R} = (\mathbb{R}^2/\mathbb{Z}^2) \times \mathbb{R}$ with coordinates $(\mathbf{x}, t) = (\begin{pmatrix} x \\ y \end{pmatrix}, t)$ by the transformation $(\mathbf{x}, t) \mapsto (A\mathbf{x}, t - 1)$.

For each $\theta \in \mathbb{R}$, let Δ_θ denote the ray

$$\left\{ \begin{pmatrix} s\cos\theta \\ -s\sin\theta \end{pmatrix} : s \geq 0 \right\} \subset \mathbb{R}^2.$$

If $A(\Delta_{\phi(t)}) = \Delta_{\phi(t-1)}$ for all $t \in \mathbb{R}$, then the contact structure $\tilde{\zeta}(\phi)$ on $T^2 \times \mathbb{R}$ is invariant under the transformation $(\mathbf{x}, t) \mapsto (A\mathbf{x}, t - 1)$ and thus descends to a contact structure on M_A which we denote by $\zeta(\phi)$. By [2, Theorem 1], $\zeta(\phi)$ is weakly symplectically fillable.

Let m denote the integer satisfying

$$m\pi \leq \sup_{t\in\mathbb{R}}(\phi(t + 1) - \phi(t)) < (m + 1)\pi.$$

Up to fiber preserving isotopy, the contact structure $\zeta(\phi)$ on M_A depends only on m when $m \geq 1$. This is the universally tight contact structure on M_A with twisting $m\pi$ in S^1-direction (see [15, Theorem 0.1]). If A is negative parabolic or negative hyperbolic, the set of possible values for m is the set of positive odd numbers and the contact structure $\zeta(\phi)$ on M_A with the corresponding $m = 1$ is denoted by ξ_A. The contact structure $\zeta(\phi)$ on M_A with the corresponding $m \geq 3$ has positive Giroux torsion and is not strongly symplectically fillable due to [9, Corollary 3].

Let $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

For $n \in \mathbb{Z}$, M_{-T^n}, the torus bundle with monodromy $-T^n$, is also denoted by M_n. Then M_n, $n \in \mathbb{Z}$, constitute all negative parabolic torus bundles. The contact structure ξ_{-T^n} on M_n is also denoted by ξ_n. When $n \geq -3$, ξ_n is Stein fillable by [24]. In [11], Golla and Lisca constructed a strongly symplectically fillable contact structure on M_n for each $n \geq -4$. We first claim that for $-4 \leq n \leq -1$, the strongly symplectically fillable contact structure on M_n they constructed is (contactomorphic to) ξ_n (see Lemma 3.1). Thus ξ_{-4} is strongly symplectically fillable. In fact, ξ_{-4} is Stein fillable (see Proposition 3.2). For $n \leq -5$, we have the following result.

Theorem 1.1 *If $n \leq -5$, then ξ_n is not strongly symplectically fillable.*

Remark 1.2 The negative parabolic torus bundle M_n can be considered as a non-orientable S^1-bundle over the Klein bottle. The contact structure ξ_n on M_n is transverse to the S^1-fibers away from a single torus. By Theorem 1.1, (M_n, ξ_n), $n \leq -5$, are examples of contact 3-manifolds without Giroux torsion that are weakly but not strongly symplectically fillable. Niederkrüger and Wendl constructed such examples by considering S^1-invariant contact structures on $S^1 \times \Sigma$ with Σ a closed oriented surface of genus at least 2 (see [22, Corollary 5]).

Remark 1.3 Let P_n denote the positive parabolic torus bundle with monodromy T^n and η_n denote the universally tight contact structure on P_n with twisting 2π in S^1-direction. P_n can be considered as an oriented S^1-bundle over the torus with Euler number n, and η_n is transverse to the S^1-fibers away from two parallel tori. If $n \geq 0$, then η_n is Stein fillable since it can be obtained from η_0 by Legendrian surgery (see [2, Proposition 13] and its proof). If $n < 0$, then η_n is not strongly symplectically fillable since it has positive Giroux torsion.

According to [15, Theorem 0.1], for each $n < 0$, there is a unique, up to isotopy, virtually overtwisted tight contact structure on M_n. We denote this contact structure on M_n by ξ_n'. We obtain the following.

Proposition 1.4 *If $n < 0$, then ξ_n' is Stein fillable.*

Given $(d_1, \ldots, d_k) \in \mathbb{Z}^k$, $k \geq 1$, we define

$$A(d_1, \ldots, d_k) := T^{-d_k} S \cdots T^{-d_1} S = \begin{pmatrix} d_k & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d_1 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

By [21, proposition 6.3], for $A \in SL(2, \mathbb{Z})$, the torus bundle M_A is negative hyperbolic if and only if A is conjugate in $SL(2, \mathbb{Z})$ to $-A(d_1, \ldots, d_k)$ for some $d_1, \ldots, d_k$ with $d_i \geq 2$ for all i and $d_i \geq 3$ for some i.

Let

$$d = (n_1 + 3, \underbrace{2, \ldots, 2}_{m_1}, n_2 + 3, \underbrace{2, \ldots, 2}_{m_2}, \ldots, n_s + 3, \underbrace{2, \ldots, 2}_{m_s}), \quad m_i, n_i \geq 0, s \geq 1,$$

and

$$\rho(d) = (m_s + 3, \underbrace{2, \ldots, 2}_{n_s}, m_{s-1} + 3, \underbrace{2, \ldots, 2}_{n_{s-1}}, \ldots, m_1 + 3, \underbrace{2, \ldots, 2}_{n_1}),$$

then by [21, Theorem 7.3] we have $-M_{-A(d)} = M_{-A(\rho(d))}$. If d is embeddable (see [11] for the definition), then by [11, Theorems 1.2 and 2.5] and [12], $\xi_{-A(d)}$ is strongly symplectically fillable. For general d, we give a necessary condition for $\xi_{-A(d)}$ to be strongly symplectically fillable.

Theorem 1.5 *If $\xi_{-A(d)}$ is strongly symplectically fillable, then*

$$n_1 + n_2 + \cdots + n_s \leq m_1 + m_2 + \cdots + m_s + 4.$$

If $s = 1$, then this necessary condition is also sufficient.

Proposition 1.6 *If $d = (n_1 + 3, \underbrace{2, \ldots, 2}_{m_1})$ with $m_1, n_1 \geq 0$, then $\xi_{-A(d)}$ is strongly symplectically fillable if and only if $n_1 \leq m_1 + 4$.*

We also give a partial answer to [11, Conjecture 1]. Suppose that $D = C_1 \cup \cdots \cup C_l$ is a symplectic divisor in a symplectic 4-manifold X. If each C_i is a 2-sphere, then the divisor D is called spherical. If $C_i \cdot C_j = 0$ for $j - i \not\equiv -1, 0, 1 \pmod{l}$ and $C_i \cdot C_j = 1$ for $j - i \equiv -1, 1 \pmod{l}$, then the divisor D is called circular. The plumbing graph of a circular, spherical symplectic divisor is illustrated in [21, Theorem 6.1, IV].

Proposition 1.7 *Let (X, ω) be a closed symplectic 4-manifold obtained as a symplectic blowup of $\mathbb{CP}^2$ with the standard Kähler form. Suppose that*

$$D = C_1 \cup \cdots \cup C_l \subset X$$

is a circular, spherical symplectic divisor such that $C_i \cdot C_i \in \{0, 1\}$ for some $i \in \{1, \ldots, l\}$ and the intersection matrix of D is nonsingular. Then, any contact structure induced on the boundary of a concave neighbourhood of D is universally tight.

In Sect. 2, we give some preliminaries. In Sect. 3, we identify the strongly symplectically fillable contact structure on M_n constructed in [11] (Lemma 3.1) and prove Theorems 1.1, 1.5 and Proposition 1.6. In Sect. 4, we prove Proposition 1.4. In Sect. 5, we prove Proposition 1.7.

2 Preliminaries

2.1 Legendrian surgery on (M_A, ξ_A)

A fiber torus of M_A is a torus $T^2 \times \{p\} \subset M_A$ where $p \in [0, 1]$. If $A \in SL(2, \mathbb{Z})$ is negative parabolic or negative hyperbolic, then in (M_A, ξ_A), each fiber torus $T^2 \times \{p\}$ is pre-Lagrangian (i.e., linearly foliated) by the construction of ξ_A (see Sect. 1). Using the same method as in the proof of [2, Proposition 11], we can deduce the following proposition. Note that M_A corresponds to $T_{A^{-1}}$ in [2].

Proposition 2.1 *Assume that $A \in SL(2, \mathbb{Z})$ is negative parabolic or negative hyperbolic. Let L be a simple closed curve on T^2 such that $L \times \{p\} \subset T^2 \times \{p\}$ $(p \in [0, 1])$ is Legendrian in (M_A, ξ_A). If AT_L is negative parabolic or negative hyperbolic, where $T_L \in SL(2, \mathbb{Z})$ corresponds to a right-handed Dehn twist along L in T^2, then the Legendrian surgery along $L \times \{p\}$ in (M_A, ξ_A) yields the contact manifold (M_{AT_L}, ξ_{AT_L}).*

In $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, let $\mu = \{\begin{pmatrix} t \\ 0 \end{pmatrix} : 0 \le t \le 1\}$ and $\lambda = \{\begin{pmatrix} 0 \\ t \end{pmatrix} : 0 \le t \le 1\}$. Then μ (resp. λ) is a linear simple closed curve in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ of slope 0 (resp. ∞). Here we use the parameter t to orient μ and λ. The right-handed Dehn twists $T_\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = T$ and $T_\lambda = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

2.2 b_2^+ and b_2^-

The following lemma is obvious.

Lemma 2.2 *Let X_1, X_2 be two compact oriented 4-manifolds. Let N_i be a component of ∂X_i, $i = 1, 2$. Suppose $f : N_1 \to N_2$ is an orientation-reversing diffeomorphism. The manifold obtained by gluing X_1 and X_2 via f is denoted by X. Then we have $b_2^+(X) \ge b_2^+(X_1) + b_2^+(X_2)$ and $b_2^-(X) \ge b_2^-(X_1) + b_2^-(X_2)$.*

2.3 Minimal strong symplectic fillings and Stein cobordism

The following proposition is due to John Etnyre.

Proposition 2.3 [4] *Let N' be a minimal strong symplectic filling of a contact 3-manifold (Y_1, ξ_1), W' be a Stein cobordism from (Y_1, ξ_1) to a contact 3-manifold (Y_2, ξ_2). Then $N' \cup W'$ is a minimal strong symplectic filling of (Y_2, ξ_2).*

Proof If there was a symplectic sphere Σ of self-intersection -1 in $N' \cup W'$, then according to [23, Proposition 7.1], Σ is an (almost) complex sphere. It cannot intersect W', otherwise the strictly pluri-subharmonic function would then have a maximum when restricted to the sphere and that can't happen. Thus the sphere Σ would have to be entirely contained in N', but that's not possible either since N' was minimal. □

3 Universally tight contact torus bundles

3.1 Identification of the contact structures constructed on M_n $(-4 \le n \le -1)$ in [11].

Lemma 3.1 *For every integer $-4 \le n \le -1$, the strongly symplectically fillable contact structure on M_n constructed in [11] is (contactomorphic to) ξ_n.*

Proof By [11, Lemma 2.3], for each $n \ge -4$ (here n corresponds to $-n$ in [11, Lemma 2.3]), there is a spherical complex divisor $D \subset \mathbb{CP}^2\#(5+n)\overline{\mathbb{CP}}^2$ which is the proper transform of a complex line and a smooth conic in general position in $\mathbb{CP}^2$, obtained by blowing up at $4+n$ generic points of the conic and one generic point of the complex line, such that the boundary of a closed regular neighborhood of D is $-M_n$. Since the intersection matrix of D is nonsingular and not negative definite, by [17, Theorem 1.3] (see also [11, Theorem 2.5]), there is a closed regular neighborhood of D and a symplectic form ω on $\mathbb{CP}^2\#(5+n)\overline{\mathbb{CP}}^2$ such that $\partial W = -M_n$ is a concave boundary of (W, ω). The contact structure on M_n constructed in [11] is induced by the ω-concave structure on $\partial W = -M_n$. We prove the lemma for the case $n = -4$. The proof for other cases are similar.

According to [7, Theorem 1.1(part B)] and [8], the contact structure on M_{-4} is supported by an open book decomposition whose page is shown in Fig. 1 and whose monodromy is the composition of Dehn twists along the $\pm$-labelled simple closed curves, where the Dehn twists along the $+$(resp. $-$)-labelled curves are right (resp. left) handed. Repeatedly using [24, Lemma 4.4.2], the above open book decomposition is stably equivalent to the open book decomposition shown in Fig. 2 with monodromy $\psi_{-4} = t_{\delta_1} t_{\delta_2} t_{\alpha_1}^{-6} t_{\alpha_2}^{-2}$, where t_γ denotes a right-handed Dehn twist along the simple closed curve γ. By part 3(d) of the proof of [24, Theorem 4.3.1], the latter open book decomposition supports the contact structure ξ_{-4} on M_{-4}. So the lemma holds. □

Proposition 3.2 *For every integer $-4 \le n \le -1$, the contact structure ξ_n on M_n is Stein fillable.*

Proof We only prove the proposition for the case $n = -4$. It suffices to show that the monodromy ψ_{-4} admits a factorization into a product of right-handed Dehn twists. Using

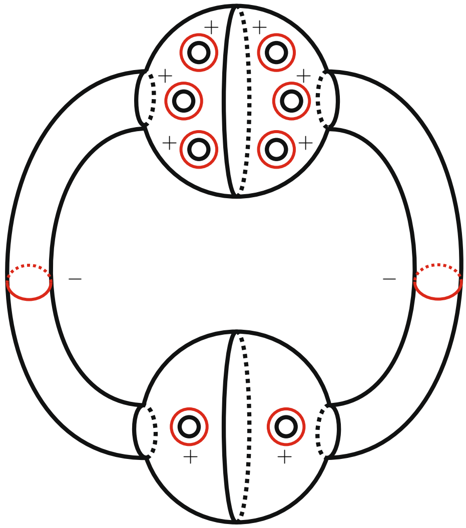


Fig. 1 A compact genus one surface with eight boundary components

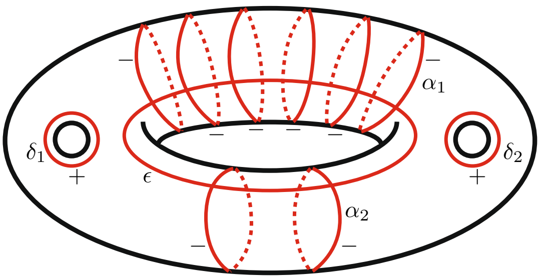


Fig. 2 A compact genus one surface with two boundary components

chain relations and braid relations in the mapping class group, we can factor the monodromy ψ_{-4} into a product of right-handed Dehn twists as follows:

$$\begin{aligned}\psi_{-4} &= t_{\delta_1} t_{\delta_2} t_{\alpha_1}^{-6} t_{\alpha_2}^{-2} \\ &= t_{\alpha_1}^{-4} t_{\alpha_1} t_\epsilon t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_2} t_{\alpha_1}^{-1} t_{\alpha_2}^{-2} t_{\alpha_1}^{-1} \\ &= t_{\alpha_1}^{-3} t_\epsilon t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_1}^{-1} t_{\alpha_2}^{-1} t_{\alpha_1}^{-1} \\ &= (t_{\alpha_1}^{-2} t_\epsilon t_{\alpha_1}^2) t_{\alpha_2} t_\epsilon (t_{\alpha_1} t_{\alpha_2} t_{\alpha_1} t_\epsilon t_{\alpha_1}^{-1} t_{\alpha_2}^{-1} t_{\alpha_1}^{-1}).\end{aligned}$$

□

3.2 Proof of Theorem 1.1

The torus bundle M_n has monodromy $\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$. Let $T^2 \times \{p\}$ ($p \in [0, 1]$) be a pre-Lagrangian fiber torus in the contact manifold (M_n, ξ_n) such that $\mu \times \{p\} \subset T^2 \times \{p\}$ is Legendrian (for the definition of μ, see Sect. 2). Denote $\mu \times \{p\}$ by K. Note that the contact framing of K coincides with its framing induced by the pre-Lagrangian fiber torus containing it.

Suppose now that $n \leq -5$. Let $(M_n \times [0, 1], \omega_n)$ be a symplectization of (M_n, ξ_n). Attaching $(-n-4)$ Weinstein 2-handles to $(M_n \times [0, 1], \omega_n)$ along $(-n-4)$ parallel copies of $K \times \{1\}$ in the fiber torus $T^2 \times \{p\} \times \{1\}$ in $M_n \times \{1\}$, by [6, Proposition 2.1], we obtain a Stein cobordism W such that the concave end is (M_n, ξ_n). By Proposition 2.1, the convex end of W is (M_{-4}, ξ_{-4}).

Lemma 3.3 $b_2^-(W) = -n - 4$ *and* $b_2^+(W) = 0$.

Proof If n is odd, then $H_1(M_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_4$. If n is even, then $H_1(M_n; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. So $b_1(M_n) = 1$, and hence $b_2(M_n) = 1$. Denote a generator of $H_2(M_n \times [0, 1]; \mathbb{R}) \cong \mathbb{R}$ by h_0. The inclusion $i : M_n \times [0, 1] \to W$ induces an injection $i_* : H_2(M_n; \mathbb{R}) \to H_2(W; \mathbb{R})$. By abuse of notation, $i_*(h_0)$ is still denoted by h_0. Obviously, for any element h in $H_2(W; \mathbb{R})$, $h_0 \cdot h = 0$.

Let $[K] \in H_1(M_n; \mathbb{Z})$ denote the homlogy class of K. In $H_1(M_n, \mathbb{Z})$, we have $2[K] = 0$. There is an oriented annulus in M_A whose boundary consists of two copies of the oriented K. In fact it comes from $\mu \times [0, 1] \subset T^2 \times [0, 1]$ by quotient. The framing of K induced by the annulus coincides with that induced by the fiber torus $T^2 \times \{p\}$ containing K.

Let $K_1, \cdots, K_{-n-4}$ be the $(-n-4)$ parallel copies of $K \times \{1\}$ along which we attach the $(-n-4)$ Weinstein 2-handles. For each $i = 1, \ldots, -n-4$, there is an oriented annulus A_i in $M_n \times \{1\}$ with $\partial A_i = 2K_i$. Let S_i be the oriented surface which is the union of A_i and two copies of the core disk of the Weinstein 2-handle attached along K_i. Let $[S_i] \in H_2(W; \mathbb{R})$ denote the homology class of S_i. Then $h_0, [S_1], \ldots, [S_{-n-4}]$ freely generate $H_2(W; \mathbb{R})$. Since the 2-handles are attached to $K_1, \ldots, K_{-n-4}$ with framing -1 with respect to the framing induced by the annuli $A_1, \ldots, A_{-n-4}$, for $i, j = 1, \cdots, -n-4$, we have

$$[S_i] \cdot [S_j] = \begin{cases} -4, & i = j, \\ 0, & i \neq j. \end{cases}$$

Thus $b_2^-(W) = -n - 4$ and $b_2^+(W) = 0$. □

Suppose that W_n is a strong symplectic filling of the contact manifold (M_n, ξ_n). Without loss of generality, we assume that W_n is minimal. Then, by Proposition 2.3, the union of W_n and W is a minimal strong symplectic filling of (M_{-4}, ξ_{-4}). By [11, Theorem 3.5] and its proof, all minimal strong symplectic fillings of the contact manifold (M_{-4}, ξ_{-4}) have vanishing b_2^-. Indeed, the union of a minimal strong symplectic filling and the symplectic cap in [11, Figure 3] is either $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ or $S^2 \times S^2$. Since that symplectic cap, $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ and $S^2 \times S^2$ all have $b_2^- = 1$, by Lemma 2.2, the minimal strong symplectic filling of (M_{-4}, ξ_{-4}) has $b_2^- = 0$. Hence $b_2^-(W_n \cup W) = 0$. So by Lemma 2.2, $b_2^-(W) = 0$. This contradicts Lemma 3.3 since $n \leq -5$. Thus (M_n, ξ_n) is not strongly symplectically fillable for $n \leq -5$.

3.3 Proof of Theorem 1.5

Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL(2, \mathbb{Z})$. Assume that $\mathrm{tr}(A) = x + w \leq -3$, i.e., A is negative hyperbolic. Let $T^2 \times \{p\}$ $(p \in [0, 1])$ be a pre-Lagrangian fiber torus in the contact manifold (M_A, ξ_A) such that $\lambda \times \{p\} \subset T^2 \times \{p\}$ is Legendrian (for the definition of λ, see Sect. 2). Denote $\lambda \times \{p\}$ by K. Let $(M_A \times [0, 1], \omega_A)$ be a symplectization of (M_A, ξ_A). Attaching a Weinstein 2-handle to $(M_A \times [0, 1], \omega_A)$ along $K \times \{1\}$ in the fiber torus $T^2 \times \{p\} \times \{1\}$ in $M_A \times \{1\}$, by [6, Proposition 2.1], we obtain a Stein cobordism W' such that the concave end is (M_A, ξ_A). Let $A' = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} x-y & y \\ z-w & w \end{pmatrix}$. Suppose that $\mathrm{tr}(A') = x + w - y \leq -3$. Then by Proposition 2.1, the convex end of W' is $(M_{A'}, \xi_{A'})$. Denote

$\mu \times \{0\}$ ($\subset T^2 \times \{0\} \subset M_A$) by μ_0 and $\lambda \times \{0\}$ ($\subset T^2 \times \{0\} \subset M_A$) by λ_0. Denote $\mu \times \{1\}$ ($\subset T^2 \times \{1\} \subset M_A$) by μ_1 and $\lambda \times \{1\}$ ($\subset T^2 \times \{1\} \subset M_A$) by λ_1.

Lemma 3.4 *Assume that* $\mathrm{tr}(A') \leq -3$, *then* $b_2^+(W') = 0$ *and* $b_2^-(W') = 1$.

Proof Since $b_1(M_A) = 1$, $b_2(M_A) = 1$. Denote a generator of $H_2(M_A \times [0, 1]; \mathbb{R}) \cong \mathbb{R}$ by h_0. The inclusion $i : M_A \times [0, 1] \to W'$ induces an injection $i_* : H_2(M_A; \mathbb{R}) \to H_2(W'; \mathbb{R})$. By abuse of notation, $i_*(h_0)$ is still denoted by h_0. Obviously, for any element h in $H_2(W'; \mathbb{R})$, $h_0 \cdot h = 0$.

Let $[\mu_0], [\lambda_0], [K] \in H_1(M_A; \mathbb{Z})$ denote the homology classes of μ_0, λ_0, K. In $H_1(M_A; \mathbb{Z})$, we have $[\mu_0] = x[\mu_0]+z[\lambda_0]$, $[\lambda_0] = y[\mu_0]+w[\lambda_0]$. Thus $(2-x-w)[\lambda_0] = 0$. Since $[K] = [\lambda_0]$, $(2-x-w)[K] = 0$. Let C be a 2-chain in M_A with $\partial C = (2-x-w)K$. Let S be the oriented surface which is the union of $C \times \{1\}$ ($\subset M_A \times \{1\}$) and $2 - x - w$ copies of the core disk of the attached Weinstein 2-handle. Let $[S] \in H_2(W'; \mathbb{R})$ denote the homology class of S. Then $h_0, [S]$ freely generate $H_2(W'; \mathbb{R})$. By Lemma 3.5 below, $[S] \cdot [S] = -(2 - x - w)(2 - x - w + y) = -(2 - \mathrm{tr}(A))(2 - \mathrm{tr}(A')) < 0$. Therefore, $b_2^+(W') = 0$ and $b_2^-(W') = 1$. □

Lemma 3.5 $[S] \cdot [S] = -(2 - x - w)(2 - x - w + y)$.

Proof Without loss of generality, we assume that $0 < p < 1$. Since $(2 - x - w)[K] = 0$ in $H_1(M_A; \mathbb{Z})$, K is a rationally null-homologous knot in M_A. Denote a closed regular neighborhood of K in M_A by $\nu(K)$. Let $\lambda' \subset \partial\nu(K)$ be a longitude for K determined by the framing induced by the fiber torus $T^2 \times \{p\}$ containing K. Let $\mu' \subset \partial\nu(K)$ be a meridian of $\nu(K)$ oriented such that the intersection number $\mu' \cdot \lambda' = 1$ on $\partial\nu(K)$. In $H_1(M_A \backslash \mathrm{Int}(\nu(K)); \mathbb{Z})$, we have $[\mu_1] = x[\mu_0]+z[\lambda_0]$, $[\lambda_1] = y[\mu_0]+w[\lambda_0]$, $[\lambda_1] = [\lambda_0] = [\lambda']$ and $[\mu'] = [\mu_1]-[\mu_0]$. So in $H_1(M_A \backslash \mathrm{Int}(\nu(K)); \mathbb{Z})$, $(2-x-w)[\lambda']+y[\mu'] = 0$. Since the 2-handle is attached to $K \times \{1\}$ with framing -1 with respect to the framing induced by the fiber torus $T^2 \times \{p\} \times \{1\}$ containing it, the Lemma follows from [19, Lemma 5.1]. □

Let

$$d = (n_1 + 3, \underbrace{2, \ldots, 2}_{m_1}, n_2 + 3, \underbrace{2, \ldots, 2}_{m_2}, \ldots, n_s + 3, \underbrace{2, \ldots, 2}_{m_s}), \quad m_i, n_i \geq 0, s \geq 1.$$

Suppose that $s \geq 2$. Let

$$d' = (\underbrace{2, \ldots, 2}_{m_1+1}, n_2 + 3, \underbrace{2, \ldots, 2}_{m_2}, \ldots, n_s + 3, \underbrace{2, \ldots, 2}_{m_s}).$$

Suppose that $T^2 \times \{p\}$ ($p \in [0, 1]$) is a pre-Lagrangian fiber torus in the contact manifold $(M_{-A(d)}, \xi_{-A(d)})$ such that $\lambda \times \{p\} \subset T^2 \times \{p\}$ is Legendrian. Let $(M_{-A(d)} \times [0, 1], \omega_{-A(d)})$ be a symplectization of $(M_{-A(d)}, \xi_{-A(d)})$. As before, attaching $n_1 + 1$ Weinstein 2-handles (if $s = 1$, attaching n_1 Weinstein 2-handles) to $(M_{-A(d)} \times [0, 1], \omega_{-A(d)})$ along parallel copies of $\lambda \times \{p\} \times \{1\}$ in the fiber torus $T^2 \times \{p\} \times \{1\}$ in $M_{-A(d)} \times \{1\}$, we obtain a Stein cobordism such that the concave end is $(M_{-A(d)}, \xi_{-A(d)})$ and the convex end is $(M_{-A(d')}, \xi_{-A(d')})$ by Proposition 2.1. Let

$$d'' = (n_2 + 3, \underbrace{2, \ldots, 2}_{m_2}, \ldots, n_s + 3, \underbrace{2, \ldots, 2}_{m_s+m_1+1}),$$

then $(M_{-A(d')}, \xi_{-A(d')}) = (M_{-A(d'')}, \xi_{-A(d'')})$. So we can attach Weinstein 2-handles as before. After successively attaching $n_1 + n_2 + \cdots + n_s + s - 1$ Weinstein 2-handles, we obtain a Stein cobordism W with the concave end $(M_{-A(d)}, \xi_{-A(d)})$ and the convex end $(M_{-A(d_0)}, \xi_{-A(d_0)})$, where

$$d_0 = (3, \underbrace{2, \ldots, 2}_{m_1+m_2+\cdots+m_s+s-1}).$$

By Lemma 3.4 and Lemma 2.2, we have $b_2^-(W) \geq n_1 + n_2 + \cdots + n_s + s - 1$.

Suppose that $W_{-A(d)}$ is a minimal strong symplectic filling of the contact manifold $(M_{-A(d)}, \xi_{-A(d)})$. Then by Proposition 2.3, $W_{-A(d)} \cup W$ is a minimal strong symplectic filling of $(M_{-A(d_0)}, \xi_{-A(d_0)})$. Let $c = m_1 + m_2 + \cdots + m_s + s + 2$. Then $M_{-A(d_0)} = -M_{-A(c)}$. By [11, Theorem 3.1], the contact manifold $(M_{-A(d_0)}, \xi_{-A(d_0)})$ admits a unique minimal strong symplectic filling up to orientation preserving diffeomorphism, which is the complement of the interior of a closed regular neighborhood $\tilde{W}$ of a spherical complex divisor D in $\mathbb{C}P^2\#(c+2)\overline{\mathbb{C}P^2}$. By the construction of D (see [11, Lemma 2.4]), $b_2^-(\tilde{W}) = 1$. Hence by Lemma 2.2, $b_2^-(W) \leq b_2^-(W_{-A(d)} \cup W) \leq b_2^-(\mathbb{C}P^2\#(c+2)\overline{\mathbb{C}P^2}) - b_2^-(\tilde{W}) = c + 2 - 1 = c + 1$. So $n_1 + n_2 + \cdots + n_s + s - 1 \leq m_1 + m_2 + \cdots + m_s + s + 2 + 1$, i.e., $n_1 + n_2 + \cdots + n_s \leq m_1 + m_2 + \cdots + m_s + 4$, concluding the proof of Theorem 1.5.

3.4 Proof of Proposition 1.6

It suffices to show that if $n_1 \leq m_1 + 4$, then $\xi_{-A(d)}$ is strongly symplectically fillable.

If $2 \leq n_1 \leq m_1 + 4$, then $(n_1 - 1, 1, \underbrace{2, \ldots, 2}_{n_1-2}, 1) \prec \rho(d) = (m_1 + 3, \underbrace{2, \ldots, 2}_{n_1})$. Since $(n_1 - 1, 1, \underbrace{2, \ldots, 2}_{n_1-2}, 1)$ is a blowup of $(0, 0)$, d is embeddable. We refer the reader to [11] for the notation "$\prec$" and the definition of "blowup". If $n_1 = 1$, then $(0, 0) \prec \rho(d) = (m_1+3, 2)$. So d is also embeddable. By [11, Theorem 1.2 and Theorem 2.5(v)], for $n_1 \geq 1$, $\xi_{-A(d)}$ is strongly symplectically fillable. If $n_1 = 0$, then $\rho(d) = (m_1 + 3)$. By [11, Theorem 1.2 and Theorem 2.5(iv)] ([11, Theorem 1.2] is still true if $d = (3, \underbrace{2, \ldots, 2}_{m_1})$), $\xi_{-A(d)}$ is strongly symplectically fillable.

4 Virtually overtwisted contact torus bundles

In this section, we prove Proposition 1.4. Throughout this section, n is a negative integer. Since $-T^n = (-S)^2T^n$ is conjugate to $(-S)T^n(-S)$, by [16, Theorem A.4], M_n can be obtained by surgery of S^3 along the framed link shown in the left of Fig. 3. We isotope the framed link to the right of Fig. 3. So by changing the 2-handles attached along the 0-framed unknots to 1-handles, we can turn the topological surgery diagram in Fig. 3 to a Legendrian link diagram in standard form (cf. [13] or [14]) shown in Fig. 4. The Legendrian knot K_0 in Fig. 4 has $\mathrm{tb}(K_0) = 1$. Performing $-n$ positive or negative stabilizations on K_0, we obtain a Legendrian knot K_0' which has $\mathrm{tb}(K_0') = n + 1$. Attaching a Weinstein 2-handle along K_0' yields a Stein domain N which is a Stein filling of a contact structure ξ on M_n. It is easy to know that $H_1(N; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

Let $\widetilde{M}_n$ denote the double cover of M_n corresponding to the epimorphism

$$\phi : \pi_1(M_n) \xrightarrow{pr} \pi_1(S^1) \cong \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2,$$

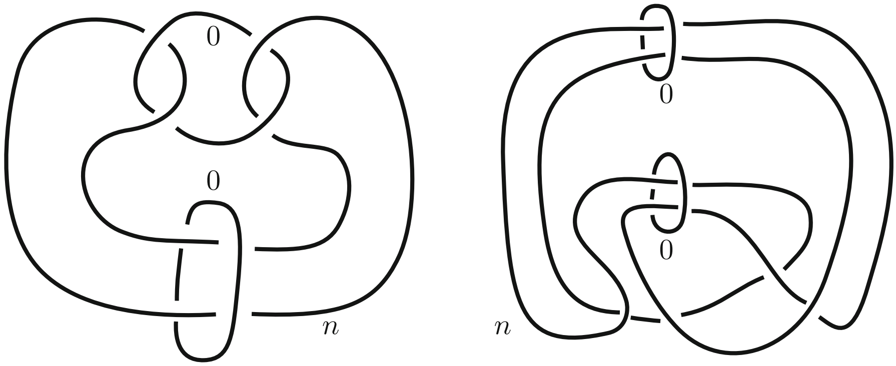
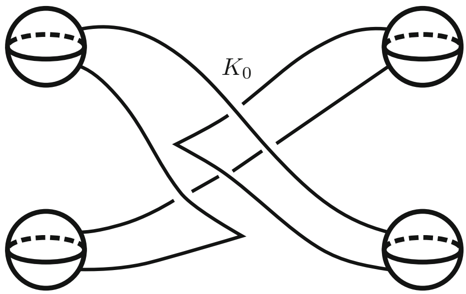


Fig. 3 An isotopy of topological surgery diagrams

Fig. 4 A Legendrian link diagram in standard form, where K_0 is a Legendrian knot with $\text{tb}(K_0) = 1$, $\text{rot}(K_0) = 0$



where pr is induced by the projection of the T^2-bundle and $\beta : \mathbb{Z} \to \mathbb{Z}_2$ denotes the homomorphism which sends 1 to the generator of $\mathbb{Z}_2$. Then $\widetilde{M}_n$ is a T^2-bundle over S^1 with monodromy T^{2n}.

Lemma 4.1 *If ξ is a universally tight contact structure on M_n, then the lift of ξ to $\widetilde{M}_n$ is not strongly symplectically fillable.*

Proof According to Honda's classification [15, Theorem 0.1], we divide the proof into two cases. If ξ is a universally tight contact structure on M_n with twisting in S^1-direction $\beta_{S^1} \geq \pi$, then the lift of ξ to $\widetilde{M}_n$ is universally tight with $\beta_{S^1} \geq 2\pi$. Explicitly, the lift of ξ can be written as given by the following 1-form on $(T^2 \times \mathbb{R})/\sim$: $\alpha_m = \sin(\phi(t))dx + \cos(\phi(t))dy$, with $m \in \mathbb{Z}^+$, $\phi'(t) > 0$, $2m\pi \leq \sup_{t\in R}(\phi(t+1) - \phi(t)) < (2m+1)\pi$, and $\ker \alpha_m$ is invariant under the action $(\mathbf{x}, t) \to (T^{2n}\mathbf{x}, t-1)$. See the second paragraph in page 99 of [15] or Section 1. If ξ is a universally tight contact structure on M_n with minimal twisting in the S^1-direction given by μ' if μ' is odd, or by $(\mu', \pm)$ if μ' is even, where μ' is a positive integer, then the lift of ξ to $\widetilde{M}_n$ is a universally tight contact structure on $\widetilde{M}_n$ with minimal twisting in the S^1-direction given by (μ', l') for some integer l', which is contactomorphic to a universally tight contact structure on $\widetilde{M}_n$ with $\beta_{S^1} \geq 2\pi$.

Since $n < 0$, it is straightforward to check that a universally tight contact structure on $\widetilde{M}_n$ with $\beta_{S^1} \geq 2\pi$ has positive Giroux torsion. So the lemma follows from [9, Corollary 3]. □

Let $c : H_1(N; \mathbb{Z}) \to \mathbb{Z}$ denote a homomorphism which sends a generator of the $\mathbb{Z}$-summand of $H_1(N; \mathbb{Z})$ to a generator of $\mathbb{Z}$. Let $\tilde{N}$ denote the double cover of N corresponding to the epimorphism

$$\psi : \pi_1(N) \xrightarrow{h} H_1(N;\mathbb{Z}) \xrightarrow{c} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2,$$

where h denotes the Hurewicz homomorphism. Since N is Stein, the homomorphism $j : \pi_1(M_n) \to \pi_1(N)$ induced by inclusion is surjective. Thus the boundary of $\tilde{N}$, $\partial\tilde{N}$, is a double cover of M_n corresponding to the epimorphism $\psi \circ j : \pi_1(M_n) \to \mathbb{Z}_2$.

Lemma 4.2

$$\psi \circ j = \phi.$$

Proof Note that $h \circ j = j_0 \circ h_0$, where $h_0 : \pi_1(M_n) \to H_1(M_n;\mathbb{Z})$ is the Hurewicz homomorphism and $j_0 : H_1(M_n;\mathbb{Z}) \to H_1(N;\mathbb{Z})$ is induced by inclusion. Thus $\psi \circ j = \beta \circ c \circ h \circ j = \beta \circ c \circ j_0 \circ h_0$. If n is odd, then $H_1(M_n;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_4$. If n is even, then $H_1(M_n;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The homomorphism $c \circ j_0 : H_1(M_n;\mathbb{Z}) \to \mathbb{Z}$ sends a torsion element of $H_1(M_n;\mathbb{Z})$ to 0. Since $c \circ j_0$ is surjective, it sends a generator of the $\mathbb{Z}$-summand of $H_1(M_n;\mathbb{Z})$ to a generator of $\mathbb{Z}$. Note that $\phi = \beta \circ pr = \beta \circ p_0 \circ h_0$, where $p_0 : H_1(M_n;\mathbb{Z}) \to H_1(S^1;\mathbb{Z}) \cong \mathbb{Z}$ is induced by the projection of the T^2-bundle. p_0 sends a torsion element of $H_1(M_n;\mathbb{Z})$ to 0. Since p_0 is surjective, it sends a generator of the $\mathbb{Z}$-summand of $H_1(M_n;\mathbb{Z})$ to a generator of $\mathbb{Z}$. Thus $\beta \circ p_0 = \beta \circ c \circ j_0$. Hence $\psi \circ j = \beta \circ c \circ j_0 \circ h_0 = \beta \circ p_0 \circ h_0 = \phi$. □

By Lemma 4.2, $\partial\tilde{N} = \widetilde{M_n}$. Lift the contact structure ξ to $\partial\tilde{N}$ and denote the resulting contact structure by $\tilde{\xi}$. Since the Stein structure on N lifts to $\tilde{N}$, $\tilde{\xi}$ is a Stein fillable contact structure on $\widetilde{M_n}$. By Lemma 4.1, ξ is not universally tight. It follows that ξ is just the virtually overtwisted contact structure ξ_n' and ξ_n' is Stein fillable.

5 Circular spherical symplectic divisors

Let (X, ω) be a closed symplectic 4-manifold obtained as a symplectic blowup of $\mathbb{CP}^2$ with the standard Kähler form. For a circular, spherical symplectic divisor $D = C_1 \cup \cdots \cup C_l \subset X$ ($l \geq 2$), define $e_i = C_i \cdot C_i$, $i = 1, \ldots, l$. The boundary of a closed regular neighborhood of D is M_A, a torus bundle over S^1, with $A = A(-e_1, \ldots, -e_l)$ (see the proof of [21, Theorem 6.1]).

Now we start the proof of Proposition 1.7. Assume that $e_i \in \{0, 1\}$ for some $i \in \{1, \ldots, l\}$. Then the intersection matrix of D is not negative definite. Since the intersection matrix of D is nonsingular, we can apply [11, Theorem 2.1] to see that there is a closed regular neighborhood V of D and a deformation ω_1 of ω such that ∂V is a concave boundary of (V, ω_1). Denote the induced contact structure on $\partial V = M_A$ by ξ. The contact manifold $(-M_A, \xi)$ admits a strong symplectic filling P_0 given by the complement of $\mathrm{Int}(V)$.

The intersection matrix of D is nonsingular implies that $b_1(M_A) = 1$ (see the proof of [11, Theorem 2.5]). Hence $\mathrm{tr}(A) \neq 2$.

Lemma 5.1 *$(-M_A, \xi)$ cannot admit a strong symplectic filling P with $b_1(P) = 1$.*

Proof Suppose that $(-M_A, \xi)$ admits a strong symplectic filling P with $b_1(P) = 1$. By [20, Theorems 1.1 and 1.4], $V \cup P$ is rational or ruled. So $b_1(V \cup P)$ is even. By the Mayer-Vietoris sequence of (V, P), the fact that $i_* : H_1(M_A;\mathbb{Z}) \to H_1(V;\mathbb{Z}) \cong \mathbb{Z}$ induced by inclusion is surjective and $b_1(V) = b_1(P) = 1$, we conclude that $b_1(V \cup P) = 1$, a contradiction. □

We prove Proposition 1.7 by a case by case argument.

If A is elliptic and ξ is not universally tight, then ξ is one of the three virtually overtwisted contact structures listed in elliptic case of [15, Theorem 0.1] (one for A^{-1} conjugate to S, two for A^{-1} conjugate to $(T^{-1}S)^2$). By [5, Theorem 1.1], these three contact structures are not weakly symplectically semi-fillable, contradicting the fact that $(-M_A, \xi)$ admits the strong symplectic filling P_0.

If A is hyperbolic with $\text{tr}(A) > 2$, since $(-M_A, \xi)$ is strongly symplectically fillable, it is a tight contact structure which is minimally twisting in the S^1-direction. Therefore there is a Stein filling P of $(-M_A, \xi)$ with $b_1(P) = 1$(see the proof of [1, Proposition 11]). This contradicts Lemma 5.1.

If A is hyperbolic with $\text{tr}(A) < -2$ and $(-M_A, \xi)$ is not universally tight, then it is a tight contact structure which is minimally twisting in the S^1-dirction. By [11, Lemma 4.3], there is a Stein filling P of $(-M_A, \xi)$ with $b_1(P) = 1$, contradicting Lemma 5.1.

If A is parabolic with $\text{tr}(A) = -2$ and $(-M_A, \xi)$ is not universally tight, then it is the virtually overtwisted contact structure in the preceding section by Honda's classification [15]. By the preceding section, it has a Stein filling P with $b_1(P) = 1$, contradicting Lemma 5.1.

This ends the proof of Proposition 1.7.

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