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# Higher dimensional Apollonian packings, revisited

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**Abstract** The Apollonian circle and sphere packings are well known objects that have attracted the attention of mathematicians throughout the ages. The historically natural generalization of the procedure for generating the packing breaks down in higher dimensions, as it leads to overlapping hyperspheres. There is, however, an alternative interpretation that allows one to extend the concept to higher dimensions and in a unique way. For relatively small dimensions (2 through at least 8), those packings can be thought of as ample cones for classes of K3 surfaces. We describe the packings in some detail for dimensions 4 (with plenty of pictures), 5, and 6.

**Keywords** Apollonius · Apollonian · Circle packing · Sphere packing · Hexlet · Soddy · K3 surface · Ample cone · Lattice

**Mathematics Subject Classification (2010)** 52C26 · 52C17 · 14J28 · 11H31 · 06B99

The Apollonian circle packing is generated as follows: begin with four mutually tangent circles. In the resulting curvilinear triangles, we inscribe a circle tangent to the three sides, thereby producing new curvilinear triangles. We continue the procedure indefinitely, as shown in Fig. 1. The points left over, the residual set, is a set of measure zero. It is a fractal of dimension $\approx 1.305688$ [9,16].

The procedure can be done in three dimensions as well: Begin with five mutually tangent spheres. For each of the five subsets of four mutually tangent spheres, it is possible to inscribe a new sphere that is tangent to the four. Again, continue indefinitely. In two dimensions, it is clear that none of the newly generated circles will overlap any of the earlier generations (other than tangentially), as the curvilinear triangles are separated. In three dimensions, the space between the initial four mutually tangent spheres is connected, so it is not a priori clear that the procedure will not lead to overlapping spheres. This does not happen, and the resulting

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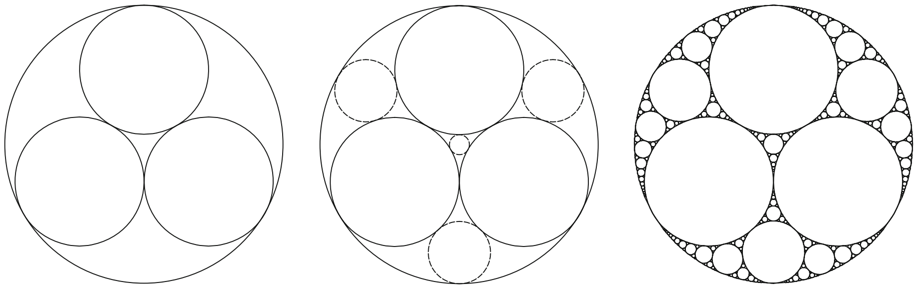

**Fig. 1** Generating the Apollonian packing

packing fills the space, except for a residual set of measure zero (with fractal dimension $\approx 2.42$ [8]).

In higher dimensions, this procedure in fact leads to overlapping hyperspheres, as was observed by Boyd [8]. This has led many to conclude there is no canonical way of generalizing the Apollonian packing to higher dimensions. Boyd gives alternative Apollonian-like packings whose initial configuration is a set of $N+2$ spheres of dimension $N-1$ in $\mathbb{R}^N$ but with gaps or separation (i.e. not mutually tangent) [8]. These packings are described via *separation* matrices.

In a recent work [6], we show that the Apollonian circle and spherical packings can be realized as the ample cone for classes of K3 surfaces. Given a K3 surface $X$ with Picard group $\text{Pic}(X) = \mathbf{e}_1\mathbb{Z} \oplus \cdots \oplus \mathbf{e}_\rho\mathbb{Z}$, the intersection matrix $J_X = [\mathbf{e}_i \cdot \mathbf{e}_j]$ uniquely determines the ample cone for $X$. Let $J_\rho$ be the $\rho \times \rho$ matrix with $-2$'s on the diagonal and 2's off the diagonal. Then there are K3 surfaces with intersection matrix $J_\rho$ for $\rho \leq 10$ [17], and the ample cone for $\rho = 4$ and 5 generate the Apollonian circle and sphere packings, respectively [6]. Thus, it seems natural to suggest that the canonical Apollonian packing in 4 dimensions (for example) should be the ample cone generated by $J_6$. Ample cones can have edges, meaning the hyperspheres intersect, though the angle of intersection can only be $\pi/2$ or $2\pi/3$. (See [4] for an example.) Thus, the ample cone for $J_6$ a priori might not be what we expect or desire.

In this paper, using the above as our inspiration, we give a formal definition of the Apollonian packing in any dimension $N \geq 2$. This definition is consistent with the Apollonian circle and sphere packings, and with the ample cone for classes of K3 surfaces with Picard number $N+2 = \rho \leq 10$ (and possibly higher, though no larger than 20). Though arithmetic geometry played a role in our inspiration, this paper will not rely on any arithmetic geometry, except in the remarks. In each dimension, there is a unique Apollonian packing. For those familiar with multiple Apollonian circle packings (e.g. Figs. 2 and 3), we will take the point of view that these are the same packing but viewed from a different perspective.

For dimensions 4 through 6, we show that the Apollonian packing shares many of the familiar properties of the circle and sphere packings, and so are in a sense what we desire. These are:

(a) The packings include a configuration of $N+2$ mutually tangent hyperspheres in $\mathbb{R}^N$.
(b) Every hypersphere in the packing is a member of $N+2$ mutually tangent hyperspheres in the packing.
(c) The hyperspheres do not intersect except tangentially.
(d) The hyperspheres fill $\mathbb{R}^N$.

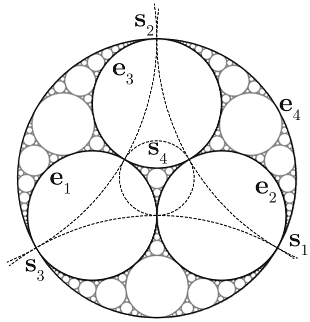


**Fig. 2** The circles of inversion (dotted lines) that generate the Apollonian packing. In this figure and figures throughout the paper, we will label a circle $H_{\mathbf{n}}$ with its normal vector $\mathbf{n}$

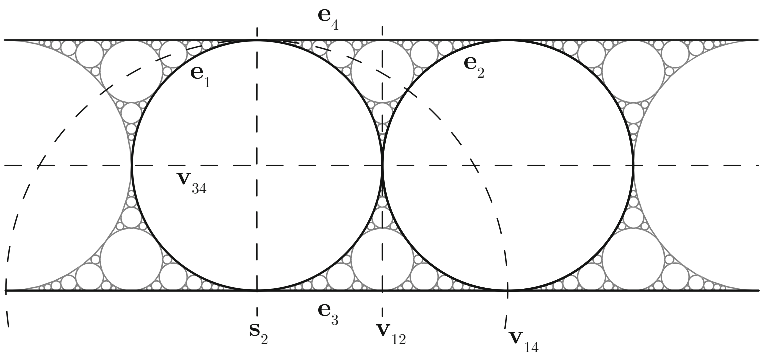


**Fig. 3** The strip packing and its symmetries

(e) Given a perspective where a configuration of $N + 2$ mutually tangent hyperspheres all have integer curvature, every hypersphere in the packing has integer curvature.

By "fill $\mathbb{R}^N$" in (d) we mean there is no space left where we can insert a hypersphere. By "curvature" we mean the inverse of the radius together with a sign, which is negative if the hypersphere contains the packing (e.g. the outside circle in Fig. 1) and positive otherwise. It is sometimes called *bend*.

We prove (a) and (e) for all $N$ (see Lemmas 5.1 and 4.1). Because our definition is consistent with the description of ample cones for K3 surfaces, (d) for $2 \leq N \leq 8$ follows from results due to Kovacs [14] and Morrison [17]. It is a priori possible that for some $N$, the packing has intersecting hyperspheres, though that intersection must be perpendicular. Assuming a variation of (d) we prove (c) (see Lemma 5.3), and in passing establish (b) (Corollary 5.4). The variation on property (d), which does not follow from the results of Kovacs and Morrison, is the main result in Sect. 6 and is established for $N = 4$, 5, and 6.

Besides a mathematical/written description of the packings, we generate multiple two-dimensional cross sections of the four-dimensional Apollonian packing (see Figs. 5, 6, 7, 8, 9, 10, 11). We also explain the classical obstruction and how it does not fit in this description.

## 1 Definitions and background

It has long been known that the Apollonian circle and sphere packings have an underlying hyperbolic structure (e.g. [15]), an observation that was foreshadowed by René Descartes' celebrated result a full two centuries before the discovery of hyperbolic geometry. We will model hyperbolic geometry with the *pseudosphere* embedded in Lorentz space. This is sometimes called the *vector model*. Boyd's *polyspherical* coordinates [7] (attributed to Clifford [10] and Darboux [11] in the late nineteenth century) are essentially the same, though not interpreted that way. Suggested references for the pseudosphere in Lorentz space include [3,18]. A nice summary appears in [12], who recommends the references [1,2,21]. (Dolgachev is also interested in the connection between Apollonian-like packings and arithmetic geometry; in particular the connection between the growth rate of orbits of curves on surfaces and the Hausdorff dimension of residual sets.)

### 1.1 The pseudosphere in Lorentz space

Lorentz space, $\mathbb{R}^{\rho-1,1}$, is the set of $\rho$-tuples over $\mathbb{R}$ equipped with the Lorentz product

$$\mathbf{u} \circ \mathbf{v} := u_1 v_1 + u_2 v_2 + \cdots + u_{\rho-1} v_{\rho-1} - u_\rho v_\rho.$$

The surface $\mathbf{x} \circ \mathbf{x} = -1$ is a hyperboloid of two sheets. Let us take the top sheet

$$\mathcal{H}: \qquad \mathbf{x} \circ \mathbf{x} = -1, \qquad x_\rho > 0,$$

which lies in the *light cone*

$$\mathcal{L}^+: \qquad \mathbf{x} \circ \mathbf{x} = 0, \qquad x_\rho > 0.$$

We call $\mathcal{H}$ the *pseudosphere*, as it can be thought of as a sphere of radius $i$. Many of the properties we cite herein have analogous results on the sphere of radius $r$, where $r$ is replaced with $i$ and the dot product is replaced by the Lorentz product. We define the distance $|AB|$ between two points on $\mathcal{H}$ by

$$\cosh(|AB|) = -A \circ B.$$

(Compare this with the similar result for a sphere of radius $r$: $r^2 \cos(|AB|/r) = A \cdot B$.) The pseudosphere $\mathcal{H}$ equipped with this metric is a model of $\mathbb{H}^{\rho-1}$.

Hyperplanes on $\mathcal{H}$ are the intersection of $\mathcal{H}$ with hyperplanes in $\mathbb{R}^{\rho-1,1}$ that go through the origin. That is, hyperplanes of the form $\mathbf{n} \circ \mathbf{x} = 0$ with $\mathbf{n} \in \mathbb{R}^{\rho-1,1}$. The hyperplane intersects $\mathcal{H}$ if and only if $\mathbf{n} \circ \mathbf{n} > 0$. Let us denote the hyperplane in $\mathbb{R}^{\rho-1,1}$ and its intersection with $\mathcal{H}$ by $H_\mathbf{n}$. The plane divides $\mathbb{R}^{\rho-1,1}$ and $\mathcal{H}$ into two halves, which we denote $H_\mathbf{n}^+$ and $H_\mathbf{n}^-$, where

$$H_\mathbf{n}^+ = \{\mathbf{x} : \mathbf{n} \circ \mathbf{x} \geq 0\}.$$

The angle $\theta$ between two hyperplanes $H_\mathbf{n}$ and $H_\mathbf{m}$ that intersect in $\mathcal{H}$ is given by

$$|\mathbf{n}||\mathbf{m}| \cos\theta = -\mathbf{n} \circ \mathbf{m}, \tag{1}$$

where $|\mathbf{n}| = \sqrt{\mathbf{n} \circ \mathbf{n}}$, and $\theta$ is the angle in the region $H_\mathbf{m}^+ \cap H_\mathbf{n}^+$. If the planes do not intersect, then

$$|\mathbf{n} \circ \mathbf{m}| = |\mathbf{n}||\mathbf{m}| \cosh\psi$$

where $\psi$ is the shortest distance between the two planes $H_\mathbf{m}$ and $H_\mathbf{n}$. The sign of $\mathbf{n} \circ \mathbf{m}$ is negative if $H_\mathbf{m}^+ \cap H_\mathbf{n}^+$ is the region between the two planes.

When $\mathbf{u} \circ \mathbf{u} < 0$, our notation $|\mathbf{u}| = \sqrt{\mathbf{u} \circ \mathbf{u}}$ is the positive imaginary square root. The notation $||\mathbf{u}||$ represents the absolute value of $|\mathbf{u}|$.

We let

$$\mathcal{O}(\mathbb{R}) = \left\{T \in M_{\rho\times\rho} : T\mathbf{u}\circ T\mathbf{v} = \mathbf{u}\circ\mathbf{v} \text{ for all } \mathbf{u},\mathbf{v} \in \mathbb{R}^{\rho-1,1}\right\}$$
$$\mathcal{O}^{+}(\mathbb{R}) = \left\{T \in \mathcal{O}(\mathbb{R}) : T\mathcal{L}^{+} = \mathcal{L}^{+}\right\}.$$

Reflection through the plane $H_{\mathbf{n}}$ is given by

$$R_{\mathbf{n}}(\mathbf{x}) = \mathbf{x} - 2\mathrm{proj}_{\mathbf{n}}(\mathbf{x})\frac{\mathbf{n}}{|\mathbf{n}|} = \mathbf{x} - 2\frac{\mathbf{n}\circ\mathbf{x}}{\mathbf{n}\circ\mathbf{n}}\mathbf{n},$$

and is in $\mathcal{O}^{+}(\mathbb{R})$. Because all isometries (in any dimension and any geometry) are generated by reflections, the group $\mathcal{O}^{+}(\mathbb{R})$ is therefore the group of isometries of $\mathcal{H}$.

### 1.2 The Poincaré models

If we project $\mathcal{H}$ through the point $(0,\ldots,0,-1)$ and onto the hyperplane $x_\rho = 0$, then we get the Poincaré hyperball model of $\mathbb{H}^{\rho-1}$. (If we project $\mathcal{H}$ through the origin and onto $x_\rho = 1$, then we get the Klein model.) Let $\partial\mathbb{H}^{\rho-1}$ be the usual compactification of $\mathbb{H}^{\rho-1}$, which is the spherical boundary of the hyperball model and is isomorphic to $\mathbb{S}^{\rho-2}$. For a point $E \in \mathcal{L}^{+}$, the set of planes that includes $E$ and the origin generates a set of lines on $\mathcal{H}$, all with a common endpoint at infinity. In this way, we understand $\mathcal{L}^{+}/\mathbb{R}^{+}$ as representing $\partial\mathcal{H} \cong \partial\mathbb{H}^{\rho-1}$.

Let us use $E \in \mathcal{L}^{+}$ for our point at infinity for the Poincaré upper half hyperspace model, which we denote with $\mathcal{H}_E$. Let $\partial\mathcal{H}_E = \partial\mathcal{H} \setminus \{E\mathbb{R}^{+}\}$ be the bounding plane of $\mathcal{H}_E$. Then $\partial\mathcal{H}_E$ is isomorphic to $\mathbb{R}^{\rho-2}$. In [5], we give a direct map to this model, and prove that the metric

$$|PQ|_E^2 = \frac{-2P\circ Q}{(P\circ E)(Q\circ E)} \tag{2}$$

is a Euclidean metric on $\partial\mathcal{H}_E$. (In [5], there is no negative sign. This is because in that paper we use the intersection pairing, which is the negative of a Lorentz product.)

In the upper half space model $\mathcal{H}_E$, $H_{\mathbf{n}}$ is represented by a hemisphere or plane perpendicular to the boundary. Its intersection with $\partial\mathcal{H}_E$ is a $(\rho-3)$-sphere or plane, which we will represent with $H_{\mathbf{n},E}$ or $H_{\mathbf{n}}$, depending on whether the choice of $E$ is important.

**Lemma 1.1** *Let $H_{\mathbf{n},E}$ be a $(\rho-3)$-sphere in $\partial\mathcal{H}_E$. Then the radius of $H_{\mathbf{n},E}$ is given by*

$$\frac{|\mathbf{n}|}{|\mathbf{n}\circ E|},$$

*using the metric defined in Eq. (2).*

*Proof* The center of $H_{\mathbf{n},E}$ is the reflection of $E$ through the plane $H_{\mathbf{n}}$, so is $P = R_{\mathbf{n}}(E) = E - \frac{2\mathbf{n}\circ E}{\mathbf{n}\circ\mathbf{n}}\mathbf{n}$. Let $Q$ be any point on the intersection of $H_{\mathbf{n}}$ with $\partial\mathcal{H}_E$, so $Q\circ Q = 0$ and $Q\circ\mathbf{n} = 0$. The radius of $H_{\mathbf{n},E}$ therefore satisfies

$$\begin{aligned} r^2 = |PQ|_E^2 &= \frac{-2P\circ Q}{(P\circ E)(Q\circ E)} \\ &= \frac{-2E\circ Q}{-\frac{2\mathbf{n}\circ E}{\mathbf{n}\circ\mathbf{n}}(\mathbf{n}\circ E)(Q\circ E)} \\ &= \frac{\mathbf{n}\circ\mathbf{n}}{(\mathbf{n}\circ E)^2}, \end{aligned}$$

from which the result follows. □

The sign of $\mathbf{n} \circ E$ depends on the orientations of $\mathbf{n}$ and $E$. In particular, once $E$ is fixed, we can choose the orientation of $\mathbf{n}$ so that the curvature is $\mathbf{n} \circ E/|\mathbf{n}|$.

For $P \in \mathcal{H}$, the quantity $P \circ E$ can be thought of as a measure of how far away $P$ is from $\partial H_E$ in the Poincaré model:

**Lemma 1.2** *Let $P \in \mathcal{H}$ and $E \in \mathcal{L}^+$. In the Poincaré upper half hyperspace model of $\mathcal{H}$, the image of $P$ is a distance*

$$\frac{||P||}{|P \circ E|}$$

*away from $\partial H_E$, using the Euclidean metric in Eq. (2).*

*Proof* Let us find the plane $H_{\mathbf{n}}$ through $P$ with the property that the corresponding hypersphere $H_{\mathbf{n},E}$ has minimal radius. The center of such a hypersphere is an endpoint of the line in $\mathcal{H}$ through $P$ and $E$, so $\mathbf{n}$ is a linear combination of $P$ and $E$:

$$\mathbf{n} = aE + P.$$

Now

$$\begin{aligned} 0 = \mathbf{n} \circ P &= aE \circ P + P \circ P \\ \mathbf{n} \circ E &= P \circ E \\ \mathbf{n} \circ \mathbf{n} &= 2aE \circ P + P \circ P \\ &= -2P \circ P + P \circ P = -P \circ P. \end{aligned}$$

Thus, the radius of the hypersphere $H_{\mathbf{n},E}$ is

$$\frac{||P||}{|P \circ E|},$$

from which the result follows. □

### 1.3 The Apollonian circle packing

To generate the Apollonian packing, we begin with four circles. Let us think of those circles as representing planes in the Poincaré upper half space model of $\mathbb{H}^3$. They can therefore be denoted with $H_{\mathbf{e}_i}$ for $i = 1, \ldots, 4$ and vectors $\mathbf{e}_i$ in $\mathbb{R}^{3,1}$. Let us orient the vectors $\mathbf{e}_i$ so that the half space $H^+_{\mathbf{e}_i}$ includes the other circles $H_{\mathbf{e}_j}$, $j \neq i$. Note that with this choice of orientation, the curvature of $H_{\mathbf{e}_i}$ is positive if $E \in H^+_{\mathbf{e}_i}$, and negative otherwise. Let us also normalize their lengths so $\mathbf{e}_i \circ \mathbf{e}_i = 1$. Since the circles are mutually tangent, the angle between them (in pairs) is zero, so $\mathbf{e}_i \circ \mathbf{e}_j = \pm 1$ for $i \neq j$. Because of the orientations we chose, and by Eq. (1), we get $\mathbf{e}_i \circ \mathbf{e}_j = -1$. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^{3,1}$ expressed as linear combinations of the vectors $\mathbf{e}_i$, then $\mathbf{x} \circ \mathbf{y} = \mathbf{x}^t J \mathbf{y}$ where

$$J = [\mathbf{e}_i \circ \mathbf{e}_j] = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

Since $\det(J) \neq 0$, the set $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis of $\mathbb{R}^{3,1}$.

The next step in our generation of the packing is to inscribe circles in the curvilinear triangles formed by our initial four circles. We can think of this as inverting in the four circles shown in Fig. 2.

Let us denote these new circles as $H_{\mathbf{s}_i}$ for some vectors $\mathbf{s}_i \in \mathbb{R}^{3,1}$. The circle inscribed in the curvilinear triangle formed by $H_{\mathbf{e}_1}$, $H_{\mathbf{e}_2}$, and $H_{\mathbf{e}_3}$ is the image of $H_{\mathbf{e}_4}$ under inversion in the circle $H_{\mathbf{s}_4}$, etc.

Inversion in the circle $H_{\mathbf{s}_i}$ can be thought of as reflection in the plane $H_{\mathbf{s}_i}$ in $\mathbb{H}^3$. Since $H_{\mathbf{s}_i}$ is perpendicular to $H_{\mathbf{e}_j}$ for all $j \neq i$, we get the relations $\mathbf{s}_i \circ \mathbf{e}_j = 0$, from which we can solve for $\mathbf{s}_i$ (up to a multiple): $\mathbf{s}_1 = (-1, 1, 1, 1)$, etc. (Note that $\mathbf{s}_i \circ \mathbf{s}_i = 4$.) These inversions/reflections generate the Apollonian group

$$\Gamma_{Ap} = \langle R_{\mathbf{s}_1}, R_{\mathbf{s}_2}, R_{\mathbf{s}_3}, R_{\mathbf{s}_4} \rangle.$$

The image of the circles $H_{\mathbf{e}_i}$ under the action of $\Gamma_{Ap}$ is the Apollonian packing.

What is often overlooked is that there is an underlying lattice, the lattice

$$\Lambda = \mathbf{e}_1\mathbb{Z} \oplus \mathbf{e}_2\mathbb{Z} \oplus \mathbf{e}_3\mathbb{Z} \oplus \mathbf{e}_4\mathbb{Z}.$$

Let us consider the group

$$\mathcal{O}^+(\mathbb{Z}) = \{T \in \mathcal{O}^+(\mathbb{R}) : T\Lambda = \Lambda\}.$$

Since $\mathbf{s}_i \circ \mathbf{s}_i = 4$, it is not immediately obvious that $R_{\mathbf{s}_i} \in \mathcal{O}^+(\mathbb{Z})$, but it is easily verified. Thus $\Gamma_{Ap} \leq \mathcal{O}^+(\mathbb{Z})$. Note that of our choices for $\mathbf{s}_i$, we chose $\mathbf{s}_i \in \Lambda$ and *primitive*, meaning its coefficients have no common factor.

Since we are viewing the Apollonian packing as the boundary at infinity of an object in $\mathbb{H}^3$, let us change our perspective and choose our point at infinity (for the upper half space model) to be a point of tangency, say where $H_{\mathbf{e}_3}$ and $H_{\mathbf{e}_4}$ meet. This gives us the familiar strip packing in Fig. 3. There are a lot of advantages to studying this version. It is in particular easier to visualize its analog in higher dimensions.

Let $H_{\mathbf{v}_{ij}}$ be the plane that is tangent to $H_{\mathbf{e}_i}$ and $H_{\mathbf{e}_j}$, and is perpendicular to $H_{\mathbf{e}_k}$ for $k \neq i, j$. Several are noted in Fig. 3. The reflection $R_{\mathbf{v}_{ij}}$ just switches the $i$-th and $j$-th component of vectors written in the basis $\beta$. Thus $R_{\mathbf{v}_{ij}} \in \mathcal{O}^+(\mathbb{Z})$. In [6], we prove

$$\mathcal{O}^+(\mathbb{Z}) = \langle R_{\mathbf{e}_3}, R_{\mathbf{s}_2}, R_{\mathbf{v}_{12}}, R_{\mathbf{v}_{34}}, R_{\mathbf{v}_{14}} \rangle.$$

The group

$$\Gamma = \langle R_{\mathbf{s}_2}, R_{\mathbf{v}_{12}}, R_{\mathbf{v}_{34}}, R_{\mathbf{v}_{14}} \rangle$$

is the full group of symmetries of the packing.

### 1.4 Descartes' theorem

Lemma 1.1 gives us a simple proof of Descartes theorem.

**Theorem 1.3** (Descartes) *Given four mutually tangent circles with curvatures $k_1$, $k_2$, $k_3$, and $k_4$, those curvatures satisfy*

$$\mathbf{k}^t J^{-1} \mathbf{k} = 0,$$

*where* $\mathbf{k} = (k_1, k_2, k_3, k_4)$.

*Proof* Let $\mathbf{e}_i$ be as above. That is, let them represent the four circles. Recall that $\mathbf{e}_i \circ \mathbf{e}_i = 1$, so $\mathbf{e}_i \circ E = k_i$, where $E$ represents the point at infinity in the Poincaré upper half space model. Combining these four equalities, we get

$$JE = \mathbf{k}$$

$$E = J^{-1}\mathbf{k}.$$

Since $E \circ E = 0$, we get

$$(J^{-1}\mathbf{k})^t J J^{-1}\mathbf{k} = 0$$
$$\mathbf{k}^t J^{-1} J J^{-1}\mathbf{k} = 0,$$

from which the result follows. □

More generally, if $\mathbf{k}$ represents the curvatures of $\rho$ hyperspheres $H_{\mathbf{e}_i}$ in $\mathbb{R}^{\rho-2}$, and $J = [\mathbf{e}_i \circ \mathbf{e}_j]$ is not degenerate, then

$$\mathbf{k}^t J^{-1}\mathbf{k} = 0.$$

This was observed by Boyd [8].

We also get the following classic result:

**Lemma 1.4** *Suppose $\rho$ hyperspheres $H_{\mathbf{e}_i}$ in $\mathbb{R}^{\rho-2}$ have integer curvatures $k_i$, that $J = [\mathbf{e}_i \circ \mathbf{e}_j]$ is not degenerate, and that $\mathbf{e}_i \circ \mathbf{e}_i$ are all equal. Suppose $\gamma \in \mathcal{O}^+(\mathbb{Z})$. Then the curvature of $H_{\gamma\mathbf{e}_i}$ is an integer.*

*Proof* As above, we note that $JE = \mathbf{k}|\mathbf{e}_i|$. The curvature of $H_{\gamma\mathbf{e}_i}$ is

$$\frac{\gamma\mathbf{e}_i \circ E}{|\gamma\mathbf{e}_i|} = \frac{\mathbf{e}_i^t \gamma^t J E}{|\mathbf{e}_i|} = \mathbf{e}_i^t \gamma^t \mathbf{k}.$$

Since $\gamma$ and $\mathbf{k}$ have integer entries, this is an integer. □

*Remark 1* We get an integer packing if $E \in \Lambda$.

## 2 Intuition and the classical obstruction

In $\rho-2$ dimensions, it is possible to arrange $\rho$ mutually tangent $(\rho-3)$-spheres. As before, let us represent these spheres with $H_{\mathbf{e}_i}$ for $\rho$ vectors $\mathbf{e}_i \in \mathbb{R}^{\rho-1,1}$, normalized so that $\mathbf{e}_i \circ \mathbf{e}_i = 1$, and oriented so that $H^+_{\mathbf{e}_i}$ is the half space that contains the other hyperplanes/hyperspheres. Then the tangency conditions and the orientations mean the matrix $J_\rho = [\mathbf{e}_i \circ \mathbf{e}_j]$ has 1's along the diagonal and $-1$ off the diagonal. Note that $J$ has eigenvalues $\lambda = 2$ with multiplicity $\rho - 1$, and $\lambda = 2 - \rho$ with multiplicity 1, so $J$ has signature $(\rho - 1, 1)$ and hence yields a Lorentz product. (This is one way to show that it is possible to arrange $\rho$ mutually tangent hyperspheres in $\mathbb{R}^{\rho-2}$.)

### 2.1 The classical obstruction (part one)

The classical temptation is to invert $H_{\mathbf{e}_1}$ in the hypersphere $H_{\mathbf{s}_1}$ that is perpendicular to all the other hyperspheres $H_{\mathbf{e}_i}$ for $i \neq 1$. Solving for $\mathbf{s}_1$, we get

$$\mathbf{s}_1 = (3 - \rho)\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_\rho,$$

and hence

$$R_{\mathbf{s}_1}(\mathbf{x}) = \mathbf{x} - 2\frac{\mathbf{x} \circ \mathbf{s}_1}{\mathbf{s}_1 \circ \mathbf{s}_1}\mathbf{s}_1 = \mathbf{x} - \frac{2x_1}{\rho - 3}\mathbf{s}_1.$$

For $\rho = 4$ and 5, this is in $\mathcal{O}^+(\mathbb{Z})$, but not for $\rho \geq 6$, hence the perceived obstruction.

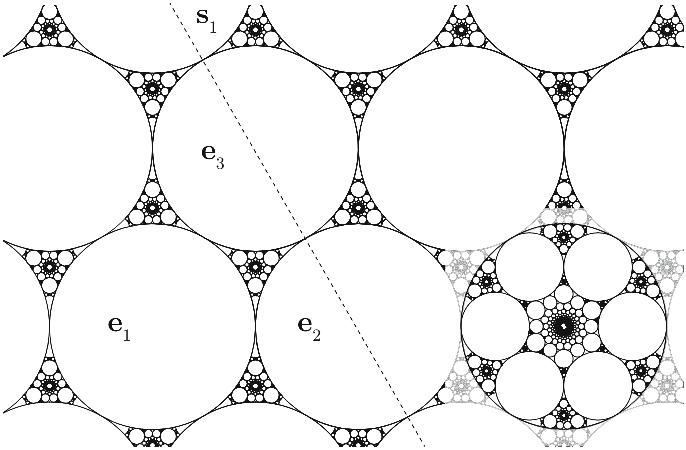


**Fig. 4** A cross section of the strip version of the sphere packing. The cross section goes through the centers of $H_{\mathbf{e}_1}$, $H_{\mathbf{e}_2}$, and $H_{\mathbf{e}_3}$, and is parallel to the planes $H_{\mathbf{e}_4}$ and $H_{\mathbf{e}_5}$. The dotted line represents $H_{\mathbf{s}_1}$, and $R_{\mathbf{s}_1}$ restricted to this cross section is reflection through the dotted line. The lower right inset is the inverse of this packing in the circle at that position. It is the cross section of the sphere packing that appears in Soddy's paper [19, Figure 1]

### 2.2 Intuition

We should think of the underlying lattice, $\Lambda_\rho = \mathbf{e}_1\mathbb{Z} \oplus \cdots \oplus \mathbf{e}_\rho\mathbb{Z}$, as a fundamental property of the packings. When we look at the strip version of the Apollonian packing (see Fig. 3), we see a Euclidean translational symmetry, which generates a one-dimensional sub-lattice of $\Lambda_4$. The strip version of the sphere packing has a similar symmetry. This is where we take three mutually tangent congruent spheres and sandwich them between two planes. The three spheres represent $H_{\mathbf{e}_1}$, $H_{\mathbf{e}_2}$, and $H_{\mathbf{e}_3}$, while the two planes represent $H_{\mathbf{e}_4}$ and $H_{\mathbf{e}_5}$. The two planes are tangent at the point at infinity, so we have a configuration of five mutually tangent spheres. The packing includes an infinite set of congruent spheres, laid out in a honeycomb pattern, and sandwiched between the planes $H_{\mathbf{e}_4}$ and $H_{\mathbf{e}_5}$. A cross section appears in Fig. 4. Again, we see a two-dimensional sub-lattice of $\Lambda_5$. Note that the reflection $R_{\mathbf{s}_1}$ is a symmetry of this two-dimensional sub-lattice.

By analogy, we should build the four-dimensional Apollonian packing as follows: Let us begin with four mutually tangent congruent spheres in $\mathbb{R}^3$, representing $H_{\mathbf{e}_1}$, ..., $H_{\mathbf{e}_4}$, which should be thought of as the analog of the three circles shown in Fig. 4. Let us use translations to extend this tetrahedral arrangement into a three-dimensional Euclidean lattice of congruent spheres in $\mathbb{R}^3$ arranged in a cannon-ball like packing. Let us think of this as a cross section of the (hypothetical) four-dimensional Apollonian packing. To get the four-dimensional packing, we thicken this configuration with a dimension and sandwich it between two hyperplanes, $H_{\mathbf{e}_5}$ and $H_{\mathbf{e}_6}$. To get the group of isometries, we first identify the isometries of the Euclidean lattice, lift these to isometries in $\mathbb{R}^{5,1}$, and then change our choice of point of tangency for the point at infinity, giving us more elements of the group of symmetries.

### 2.3 The classical obstruction (part two)

In the initial tetrahedral configuration of the spheres mentioned above, consider the bottom layer of three spheres $H_{\mathbf{e}_2}$, $H_{\mathbf{e}_3}$, and $H_{\mathbf{e}_4}$. These three spheres create a *cradle* on which we rest the fourth sphere, $H_{\mathbf{e}_1}$. Now let us extend the bottom layer into an infinite planar arrangement of spheres in a honeycomb pattern. In one of the infinitely many cradles created by this layer, we have nested $H_{\mathbf{e}_1}$. Note that the adjacent cradles now cannot be filled, as $H_{\mathbf{e}_1}$ is in the way. Filling in the second layer of spheres as prescribed by the lattice structure, we note that half of the cradles receive a sphere, while the other half remain empty. Now consider the layer below what we called the bottom layer, and in particular consider the other cradle formed by the spheres $H_{\mathbf{e}_2}$, $H_{\mathbf{e}_3}$, and $H_{\mathbf{e}_4}$. If we follow the pattern governed by our lattice, this cradle remains empty, and its adjacent cradles receive spheres. This requisite emptiness is the classical obstruction. The inversion $R_{\mathbf{s}_1}$ when restricted to this cross section, sends $H_{\mathbf{e}_1}$ into this cradle.

## 3 The Apollonian packing in four dimensions

As suggested in the previous section, we should begin with isometries of the cannon-ball sub-lattice in $\mathbb{R}^3$. There are of course the translations, but the fundamental building blocks are the $-1$ maps through the centers of the spheres, and through the points of tangency of pairs of tangential spheres. Such maps appear in [5]. Let $E, P \in \mathcal{L}^+$. Then the $-1$ map on $\partial\mathcal{H}_E$ through the point $P$ is given by

$$\phi = \phi_{P,E}(\mathbf{x}) = \frac{2((P \circ \mathbf{x})E + (E \circ \mathbf{x})P)}{P \circ E} - \mathbf{x}.$$

It is straight forward to verify that $\phi \in \mathcal{O}^+(\mathbb{R})$, $\phi^2 = id$, and that $P$ and $E$ are eigenvectors associated to the eigenvalue $\lambda = 1$. The space perpendicular to $E$ and $P$,

$$V^{\perp P,E} = \{\mathbf{x} \in \mathbb{R}^{\rho-1,1} : \mathbf{x} \circ E = \mathbf{x} \circ P = 0\},$$

is the eigenspace associated to $\lambda = -1$. We would like to verify that $\phi \in \mathcal{O}^+(\mathbb{Z})$ for appropriate choices of $P$ and $E$.

Note that $\mathbf{e}_i \circ (\mathbf{e}_i + \mathbf{e}_j) = 0$, so $\mathbf{e}_i + \mathbf{e}_j$ is on both $H_{\mathbf{e}_i}$ and $H_{\mathbf{e}_j}$, and $(\mathbf{e}_i + \mathbf{e}_j) \circ (\mathbf{e}_i + \mathbf{e}_j) = 0$. Thus, $\mathbf{e}_i + \mathbf{e}_j$ is the point of tangency between $H_{\mathbf{e}_i}$ and $H_{\mathbf{e}_j}$. Let

$$\mathcal{S} = \{\mathbf{e}_i + \mathbf{e}_j : i \neq j\}.$$

For $E \in \mathcal{S}$ and $\mathbf{e}_i \circ E \neq 0$, let $P_{i,E}$ be the center of the sphere $H_{\mathbf{e}_i}$ in $\partial\mathcal{H}_E$:

$$P_{i,E} = R_{\mathbf{e}_i}(E) = E + 4\mathbf{e}_i.$$

Let

$$\mathcal{T}_E = \{\mathbf{e}_i + \mathbf{e}_j : \mathbf{e}_i \circ E \neq 0, \mathbf{e}_j \circ E \neq 0, i \neq j\} \cup \{P_{i,E} : \mathbf{e}_i \circ E \neq 0\}.$$

**Lemma 3.1** *Suppose* $\mathbf{x} \in \Lambda$ *and* $E = \mathbf{e}_i + \mathbf{e}_j \in \mathcal{S}$ *for some fixed* $i$ *and* $j$. *Then*

$$\mathbf{x} \circ E \equiv 0 \pmod 2.$$

*Proof* It is enough to calculate $\mathbf{e}_k \circ E$ for all $k$. If $k \neq i, j$, then $\mathbf{e}_k \circ E = -2$. If $k = i$ or $j$, then $\mathbf{e}_k \circ E = 0$. □

**Corollary 3.2** *If $E \in \mathcal{S}$, and $P \in \mathcal{T}_E$, then $\phi_{P,E} \in \mathcal{O}^+(\mathbb{Z})$.*

*Proof* Case 1, $P = \mathbf{e}_i + \mathbf{e}_j \in \mathcal{T}_E$: Note that $E \circ P = -4$, so it is enough to verify that the numerator in the definition of $\phi_{P,E}$ is 0 modulo 4. This is straight forward, using the above lemma.

Case 2, $P = P_{i,E}$: Then $E \circ P = E \circ (E + 4\mathbf{e}_i) = -8$, so we check the numerator modulo 8:

$$\begin{aligned} 2((P \circ \mathbf{x})E + (E \circ \mathbf{x})P) &\equiv 2(((E + 4\mathbf{e}_i) \circ \mathbf{x})E + (E \circ \mathbf{x})(E + 4\mathbf{e}_i)) \\ &\equiv 2((E \circ \mathbf{x})E + (E \circ \mathbf{x})E) \equiv 0 \pmod 8. \end{aligned}$$ □

Because of symmetry, $\mathcal{O}^+(\mathbb{Z})$ clearly includes the map that switches the $i$-th and $j$-th component of $\mathbf{x}$ when written in the basis $\beta$. Geometrically, this is the reflection $R_{\mathbf{v}_{ij}}$ where $\mathbf{v}_{ij} = \mathbf{e}_i - \mathbf{e}_j$. Composition of these maps gives us the group of permutations of the components of $\mathbf{x}$.

We now have a large group of isometries that preserves the lattice. Of course, $\mathcal{O}^+(\mathbb{Z})$ also includes the reflections $R_{\mathbf{e}_i}$, but we want to avoid these, as we did in our definition for $\Gamma$ in the circle packing case. Let $\Gamma'$ be the subgroup of $\mathcal{O}^+(\mathbb{Z})$ generated by the maps $\phi_{E,P}$ for $E \in \mathcal{S}$ and $P \in \mathcal{T}_E$, and the reflections $R_{\mathbf{v}_{ij}}$, and let us look at the image of the hyperspheres $H_{\mathbf{e}_i,E}$ under the action of this group (in $\partial H_E \cong \mathbb{R}^{\rho-2}$).

For $\rho = 4$ and 5, it is clear that $\Gamma'$ is a subgroup of symmetries of the Apollonian circle packing, as we used those as inspiration to create $\Gamma'$. Thus, $\Gamma' \leq \Gamma$. On the other hand, the packings are generated by the inversions $R_{\mathbf{s}_i}$, and the reflections $R_{\mathbf{v}_{ij}}$. For $\rho = 4$, we have $R_{\mathbf{s}_2} = R_{\mathbf{v}_{34}} \circ \phi_{P_1,\mathbf{e}_3+\mathbf{e}_4}$, so $\Gamma' = \Gamma$. For $\rho = 5$, we have $R_{\mathbf{s}_1} = R_{\mathbf{v}_{23}} \circ R_{\mathbf{v}_{45}} \circ \phi_{\mathbf{e}_2+\mathbf{e}_3,\mathbf{e}_4+\mathbf{e}_5}$, so again $\Gamma' = \Gamma$.

From here until the end of this section, we fix $\rho = 6$ and $E = \mathbf{e}_5 + \mathbf{e}_6$.

We use $\Gamma'$ to generate what we will call the Apollonian strip packing in four dimensions:

$$\mathcal{A}_{4,E} = \{H_{\gamma\mathbf{e}_1,E} \subset \partial\mathcal{H}_E \cong \mathbb{R}^4 : \gamma \in \Gamma'\}. \tag{3}$$

We will be more precise in the following sections, but for now, we have enough to see (sort of) what the packing looks like. Like we did in Fig. 4 for the sphere packing, we look at cross sections. If we let $x_1 = x_2 = 0$ then we get the circle packing in Fig. 3. If we let $x_1 = 0$ then we get the sphere packing, and if we let $x_1 = 0$ and $x_5 = x_6$ then we get the cross section in Fig. 4. Our first interesting cross section is shown in Fig. 5. This is the cross section with $x_2 = x_3$ and $x_5 = x_6$. The condition $x_5 = x_6$ means we are looking at the cannon-ball packing in $\mathbb{R}^3$, the cross section parallel to and midway between the hyperplanes $H_{\mathbf{e}_5}$ and $H_{\mathbf{e}_6}$. In this 3-dimensional cross section, we are taking the plane that passes through the centers of $H_{\mathbf{e}_1}$ and $H_{\mathbf{e}_4}$, and through the point of tangency $\mathbf{e}_2 + \mathbf{e}_3$.

Let $T_{ij} \in \Gamma'$ be translation from $P_i$ to $P_j$ (for $i, j \in \{1, \ldots, 4\}$), so

$$T_{ij} = \phi_{P_{i,E},E} \circ \phi_{\mathbf{e}_i+\mathbf{e}_j,E}.$$

The translations $T_{12}$, $T_{13}$, and $T_{14}$ generate the three-dimensional sub-lattice of $\Lambda$ in $\partial\mathcal{H}_E$. We let $\mathbf{f}_i = T_{1j} \circ T_{1k}(\mathbf{e}_i) = T_{1k} \circ T_{1j}(\mathbf{e}_i)$, where $i, j, k$ is a permutation of 2, 3, 4; and we let $f_1 = T_{12} \circ T_{13} \circ T_{14}(\mathbf{e}_1)$. The canonical fundamental domain for the sub-lattice is the parallepiped with vertices the centers of the spheres $H_{\mathbf{e}_1}$, $H_{\mathbf{e}_2}$, $H_{\mathbf{e}_3}$, $H_{\mathbf{e}_4}$, $H_{\mathbf{f}_2}$, $H_{\mathbf{f}_3}$, $H_{\mathbf{f}_4}$, and $H_{\mathbf{f}_1}$. The cross section shown in Fig. 6 goes through the center of the parallelepiped, as well as $H_{\mathbf{e}_4}$ and $H_{\mathbf{f}_4}$, and is perpendicular to the hyperplanes $H_{\mathbf{e}_5}$ and $H_{\mathbf{e}_6}$. It is the cross section $x_2 = x_3$ and $x_1 + x_2 = 0$. As an Apollonian-like packing in two dimensions, it is Example

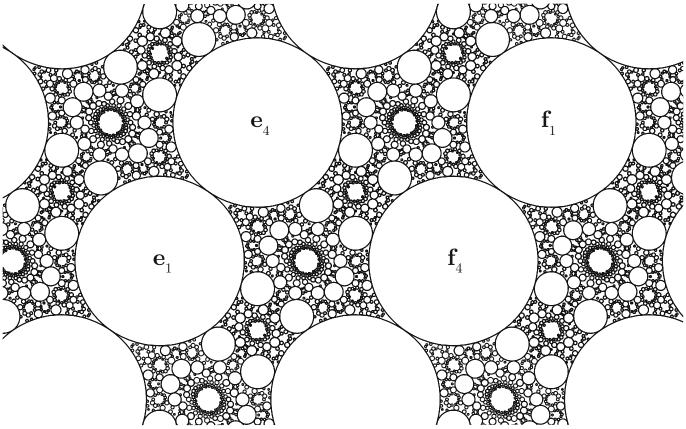


**Fig. 5** The cross section of the 4-dimensional strip version of the Apollonian packing (with $E = \mathbf{e}_5 + \mathbf{e}_6$), corresponding to $x_2 = x_3$ and $x_5 = x_6$. The limit point of circles midway between $H_{\mathbf{e}_1}$ and $H_{\mathbf{f}_4}$ is the point $\mathbf{e}_2 + \mathbf{e}_3$, which is where the hyperspheres $H_{\mathbf{e}_2}$ and $H_{\mathbf{e}_3}$ are tangent

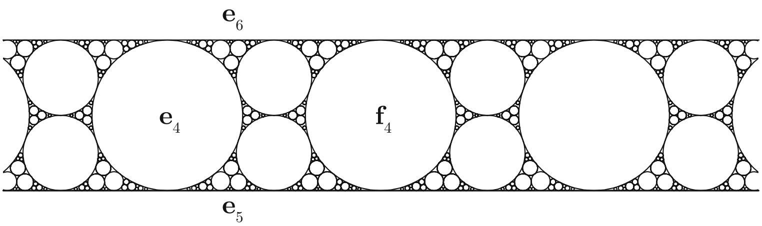


**Fig. 6** The cross section through the centers of $H_{\mathbf{e}_4}$ and $H_{\mathbf{f}_4}$, and in a plane perpendicular to $H_{\mathbf{e}_5}$ and $H_{\mathbf{e}_6}$

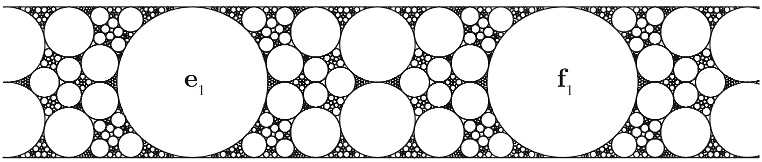


**Fig. 7** The cross section along the long diagonal of the parallelepiped and perpendicular to $H_{\mathbf{e}_5}$ and $H_{\mathbf{e}_6}$

2.6 in Boyd's paper [8]. It was also studied by Guettler and Mallows [13], who drew pictures, but seemed to be unaware of Boyd's result.

The strip cross section along the long diagonal of the parallepiped is shown in Fig. 7. This is the cross section $x_2 = x_3 = x_4$.

It is natural to consider the strip cross section through the centers of $H_{\mathbf{e}_1}$ and $H_{\mathbf{f}_4}$, and this is shown in Fig. 8. However, since $x_4 = 0$, this can also be thought of as a cross section of the sphere packing. In Fig. 4, our cross section is along the line perpendicular to $\mathbf{s}_1$ and through the center of $H_{\mathbf{e}_1}$. It is the cross section $x_2 = x_3$.

A couple more cross sections are shown in Figs. 9 and 10.

In the caption of Fig. 5, we note that the limit point midway between $H_{\mathbf{e}_1}$ and $H_{\mathbf{f}_4}$ is the point of tangency of the two hyperspheres $H_{\mathbf{e}_2}$ and $H_{\mathbf{e}_3}$; it is the point $\mathbf{e}_2 + \mathbf{e}_3$. The cross sections in Figs. 8, 9 and 10 all have points with the same type of feature, where one can

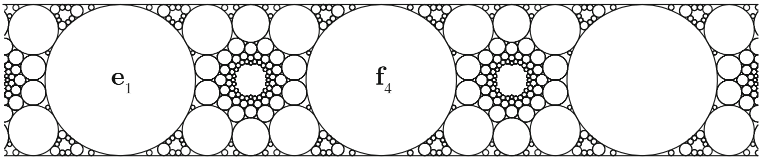


**Fig. 8** The strip cross section through the centers of $H_{\mathbf{e}_1}$ and $H_{\mathbf{f}_4}$. It is a cross section of the sphere packing

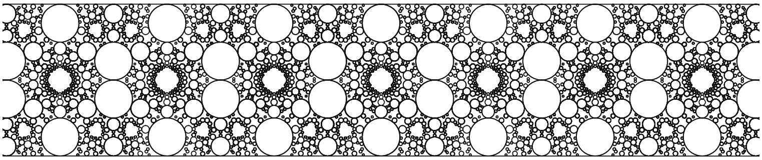

**Fig. 9** The strip cross section through the tangent points $\mathbf{e}_1 + \mathbf{e}_4$ and $\mathbf{e}_2 + \mathbf{e}_3$. The constraints are $x_1 = x_4$ and $x_2 = x_3$

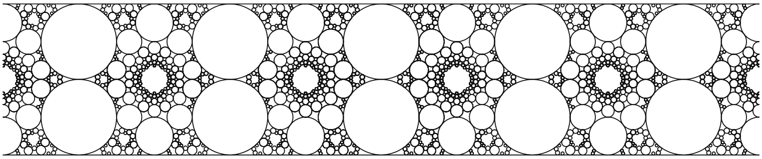

**Fig. 10** The strip cross section through the tangent point $\mathbf{e}_2 + \mathbf{e}_3$ and the center of the parallelepiped. The constraints are $x_2 = x_3$ and $x_1 + x_4 = 0$

imagine tangent spheres above and below the page. To better understand the packing near these points, we can invert (as we did with Fig. 4) so that these points are sent to infinity. If we do this to Fig. 8 then we get Fig. 4. If we do this to Fig. 9 then we get Fig. 5. If we do this to Fig. 10 then we get Fig. 11, which appears to be generated by a cross section of a square-based canon ball packing. Indeed, the triangular and square-based canon ball packings are the same. With a little imagination, this can be seen in Fig. 5: Four spheres creating a square base are $H_{\mathbf{e}_4}$; $H_{\mathbf{f}_4}$; the sphere $H_{\mathbf{e}_2}$, which is the sphere above the page at the point midway between $H_{\mathbf{e}_1}$ and $H_{\mathbf{f}_4}$; and the sphere $H_{\mathbf{f}_2}$, which is the sphere below the page and tangent to the page at the point midway between $H_{\mathbf{e}_4}$ and $H_{\mathbf{f}_1}$. One can also see it in the parallelpiped, which can be thought of as the union of an octahedron and two tetrahedrons (see Fig. 13 on page 22). The octahedron is the union of two square-based pyramids, and the vertices of that square are the centers of the four canon balls. The packing in Fig. 11 also appears in [20, Fig. 2].

## 4 A formal definition of Apollonian packings

Let $\beta = \{\mathbf{e}_1, \ldots, \mathbf{e}_\rho\}$ be a basis for a $\rho$-dimensional vector space. Define the bilinear form $\circ$ by

$$\mathbf{e}_i \circ \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i \neq j. \end{cases}$$

As noted above, the set $\beta$ together with $\circ$ represent a configuration of $\rho$ mutually tangent hyperspheres in $\mathbb{R}^{\rho-2}$; the signature of $J = [\mathbf{e}_i \circ \mathbf{e}_j]$ is $(\rho - 1, 1)$; $\circ$ is a Lorentz product; and

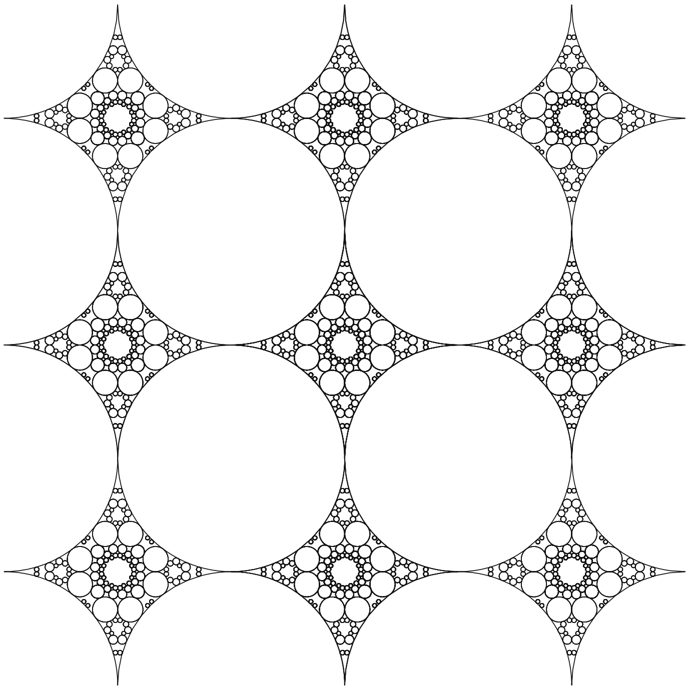

**Fig. 11** The cross section in Fig. 10, inverted in the point $\mathbf{e}_2 + \mathbf{e}_3$

$\beta$ is a basis of $\mathbb{R}^{\rho-1,1}$. Let $\Lambda = \Lambda_\rho = \mathbf{e}_1\mathbb{Z} \oplus \cdots \oplus \mathbf{e}_\rho\mathbb{Z}$. Fix $D \in \Lambda$ such that $D \circ D < 0$ and $D \circ \mathbf{n} \neq 0$ for any $\mathbf{n} \in \Lambda$ such that $\mathbf{n} \circ \mathbf{n} = 1$. (Such a $D$ exists, as we will see.) Let $\mathcal{L}^+$ be the cone that contains $D$:

$$\mathcal{L}^+ = \{\mathbf{x} \in \mathbb{R}^{\rho-1,1} : \mathbf{x} \circ \mathbf{x} = 0, \mathbf{x} \circ \mathbf{D} < 0\}.$$

Let

$$\mathcal{E}_1 = \{\mathbf{n} \in \Lambda : \mathbf{n} \circ \mathbf{n} = 1, \mathbf{n} \circ D < 0\}$$

and

$$\mathcal{K} = \bigcap_{\mathbf{n} \in \mathcal{E}_1} H_{\mathbf{n}}^-.$$

That is, for every $\mathbf{n} \in \Lambda$ with $\mathbf{n} \circ \mathbf{n} = 1$, we consider the half space that contains $D$ and is bounded by $H_{\mathbf{n}}$, and take the intersection of all these half spaces. Thus $\mathcal{K}$ is a polyhedral cone with an infinite number of faces. For $\rho = 4$ and 5, the faces do not intersect (the circles/spheres do not intersect except tangentially), but there is also no open space at infinity (the circles/spheres are space filling).

Let

$$\mathcal{E}_1^* = \{\mathbf{n} \in \mathcal{E}_1 : H_{\mathbf{n}} \text{ is a face of } \mathcal{K}\}.$$

Then the Apollonian packing $\mathcal{A}_\rho$ is the set of hyperplanes

$$\mathcal{A}_\rho = \{H_{\mathbf{n}} \subset \mathcal{H} \cong \mathbb{H}^{\rho-1} : \mathbf{n} \in \mathcal{E}_1^*\}.$$

Given a point $E \in \mathcal{L}^+$ and setting it as our point at infinity, we define the *perspective* with respect to $E$ to be the set

$$\mathcal{A}_{\rho,E} = \{H_{\mathbf{n},E} \in \partial\mathcal{H}_E \cong \mathbb{R}^{\rho-2} : \mathbf{n} \in \mathcal{E}_1^*\}.$$

So for example, the Apollonian circle packing shown in Fig. 3 is $\mathcal{A}_{4,\mathbf{e}_3+\mathbf{e}_4} = \mathcal{A}_{4,(0,0,1,1)}$, while the one shown is Fig. 2 is $A_{4,(1,1,1,3+2\sqrt{3})}$. The strip version of the sphere packing is $\mathcal{A}_{5,\mathbf{e}_4+\mathbf{e}_5}$, while the model built by Soddy (see [19, Figure 2]) is $\mathcal{A}_{5,2\mathbf{e}_1+\mathbf{e}_5}$.

For fixed $\rho$, $\mathcal{A}_\rho$ exists and is unique. There are infinitely many perspectives $\mathcal{A}_{\rho,E}$. It is a priori not clear that the spheres in $\mathcal{A}_{\rho,E}$ are space filling and do not intersect except tangentially, though it is clear that these properties are independent of the choice of $E$.

As we did for the Apollonian circle packing ($\rho = 4$), let us define

$$\mathcal{O}^+(\mathbb{Z}) = \mathcal{O}_\rho^+(\mathbb{Z}) = \{T \in \mathcal{O}^+(\mathbb{R}) : T\Lambda = \Lambda\},$$

and define the group of symmetries of $\mathcal{K}$ to be

$$\Gamma = \Gamma_\rho = \{T \in \mathcal{O}^+(\mathbb{Z}) : T\mathcal{K} = \mathcal{K}\}.$$

To describe $\mathcal{A}_\rho$ for small values of $\rho$, we describe $\Gamma$ (or a sufficiently large subgroup of $\Gamma$).

We first establish property (e) (as outlined in the Introduction):

**Lemma 4.1** *Suppose we have a configuration of $\rho = N + 2$ mutually tangent hyperspheres in $\mathcal{A}_\rho$ and suppose these $\rho$ hyperspheres all have integer curvature. Then every hypersphere in $\mathcal{A}_\rho$ has integer curvature.*

*Proof* Because the $\rho$ hyperspheres are in $\mathcal{A}_\rho$, they have normal vectors $\{\mathbf{f}_1, \ldots, \mathbf{f}_\rho\}$ that are in $\Lambda$. Let us define $\Lambda' = \mathbf{f}_1\mathbb{Z} + \cdots + \mathbf{f}_\rho\mathbb{Z}$, so $\Lambda' \subset \Lambda$. Note that both $\pm f_i \in \Lambda$, so let us choose the orientation so that $H^+_{\mathbf{f}_i}$ contains $H_{\mathbf{f}_j}$ for all $j \neq i$. This is how we chose the orientations of $\mathbf{e}_i$, so the matrix

$$J' = [\mathbf{f}_i \circ \mathbf{f}_j]$$

is the same as $J$. In particular, $\det(J') = \det(J)$, so $\Lambda' = \Lambda$ and $\beta' = \{\mathbf{f}_1, \ldots, \mathbf{f}_\rho\}$ is a basis of $\Lambda$. As in the proof of Theorem tDescartes, $J'E = \mathbf{k}$ where $\mathbf{k}$ is the vector of curvatures of the hyperspheres. If $H_\mathbf{n} \in \mathcal{A}_\rho$, then $\mathbf{n} \in \mathcal{E}_1^*$, so $\mathbf{n} \in \Lambda$ and $|\mathbf{n}| = 1$. Thus the curvature of $H_\mathbf{n}$ is $\mathbf{n} \circ E = \mathbf{n}^t J'E = \mathbf{n}^t\mathbf{k}$, so is an integer. □

*Remark 2* Given a different initial configuration of $\rho$ circles $H_{\mathbf{e}_1}$, …, $H_{\mathbf{e}_\rho}$, we can define a $J = [\mathbf{e}_i \circ \mathbf{e}_j]$, where we again choose $\mathbf{e}_i \circ \mathbf{e}_i = 1$, but $\mathbf{e}_i \circ \mathbf{e}_j$ is defined by the separation between the spheres $H_{\mathbf{e}_i}$ and $H_{\mathbf{e}_j}$. Boyd's polyspherical coordinates yield the negative of $J$, which he calls a separation matrix [8]. For example, in Fig. 6, we can select the ordered basis $\beta = \{\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_4, \mathbf{f}_4\}$. We calculate $f_4 = (-1, 1, 1, 0, 1, 1)$ and $f_4 \circ e_4 = -3$, giving us the separation between $\mathbf{e}_4$ and $\mathbf{f}_4$. (Since the cross section goes through the centers of the hyperspheres $H_{\mathbf{e}_4}$ and $H_{\mathbf{f}_4}$, the curvatures of the circles are the same as the curvatures of the hyperspheres, so the separation in four dimensions is the same as in the two dimensional cross section.) We therefore get the matrix

$$J = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -3 \\ -1 & -1 & -3 & 1 \end{bmatrix},$$

which is the negative of Boyd's separation matrix in his Example 2.6.

Boyd also observed that the off diagonals can be half integers and still lead to Apollonian like packings, so it seems reasonable to look at $-2J$ instead. That is, require $\mathbf{e}_i$ to have square norm $\mathbf{e}_i \circ \mathbf{e}_i = -2$. (Note that $R_{\mathbf{e}_i}$ is still in $\mathcal{O}^+(\mathbb{Z})$, but this is not guaranteed if $\mathbf{e}_i \circ \mathbf{e}_i < -2$.) With this new Lorentz product, if $\mathbf{e}_i \circ \mathbf{e}_j \in \mathbb{Z}$ for all $i$ and $j$, then the lattice $\Lambda$ is *even*, meaning for any $\mathbf{x} \in \Lambda$, $\mathbf{x} \circ \mathbf{x}$ is even. The set $\mathcal{E}_1$ is replaced with $\mathcal{E}_{-2}$ and yields the same $\mathcal{K}$.

If $X$ is a K3 surface, then the Picard group for $X$ is a lattice $\Lambda$ together with the intersection pairing. The lattice is even and the matrix $J$ is the intersection matrix. If we choose $D$ to be ample (so $\mathbf{n} \cdot D \neq 0$ for any $\mathbf{n} \in \mathcal{E}_{-2}$), then the set $\mathcal{E}^*_{-2}$ is the set of divisor classes of irreducible $-2$ curves on $X$, and $\mathcal{K}$ is the ample cone [14].

Given an even lattice $\Lambda$ of dimension $\rho \leq 10$, there exists a K3 surface $X$ with $\mathrm{Pic}(X) = \Lambda$ [17]. Thus, the Apollonian packing in dimensions two through 8 ($4 \leq \rho \leq 10$) can be thought of as representing the ample cones for classes of K3 surfaces.

## 5 The details

In Sect. 3, we described a packing $\mathcal{A}_6$ by looking at the orbit of a hypersphere under the action of a group of isometries $\Gamma'$. In this section, we aim to justify what we did. We begin with a $D$ and use that to define $\mathcal{K}$. We show that $R_{\mathbf{v}_{ij}}$ and $\phi_{P,E}$ are in $\Gamma$ so $\Gamma' \leq \Gamma$. We use $\Gamma'$ to describe an a priori different cone $\mathcal{K}'$ and use $\Gamma' \leq \Gamma$ to conclude $\mathcal{K} \subset \mathcal{K}'$. We then use a descent argument to show $\mathcal{K}' = \mathcal{K}$. The descent argument is dimension specific, so in this section we state its main consequence (Statement 1), which we prove in Sect. 6 for $\rho = 6$, 7, and 8.

Let us choose

$$D = \sum_{i=1}^{\rho} \mathbf{e}_i$$

and use this to define $\mathcal{K}$. We need to know that $D \circ \mathbf{n} \neq 0$ for any $\mathbf{n} \in \mathcal{E}_1$.

For any $\mathbf{n} \in \mathcal{E}_1$, there exists an $E \in \mathcal{S}$ so that $\mathbf{n} \circ E \neq 0$, for otherwise, $\mathbf{n} \circ \mathbf{e}_i = 0$ for all $i$, which has the unique solution $\mathbf{n} = \mathbf{0}$. Let us use this $E$ for our point at infinity. By Lemmas 1.1 and 3.1, the radius of the sphere $H_{\mathbf{n},E} \in \partial\mathcal{H}_E$ is no more than $1/2$. But $D$ is too high in the Poincaré model $\mathcal{H}_E$, since by Lemma 1.2, its distance above $\partial\mathcal{H}_E$ is

$$\frac{||D||}{|D \circ E|} = \frac{\sqrt{\rho(\rho-2)}}{2(\rho-2)} = \frac{1}{2}\sqrt{\frac{\rho}{\rho-2}} > 1/2.$$

Thus $D \circ \mathbf{n} \neq 0$ for any $\mathbf{n} \in \mathcal{E}_1$. In the above calculation, and some that follow, it is useful to note that $D \circ \mathbf{e}_i = 2 - \rho$, so $D \circ D = \rho(2-\rho)$ and $D \circ E = 2(2-\rho)$.

Suppose $\gamma \in \mathcal{O}^+(\mathbb{Z})$. Then by definition we have that if $\gamma\mathcal{K} = \mathcal{K}$ then $\gamma \in \Gamma$. The condition that $\gamma\mathcal{K} = \mathcal{K}$ is equivalent with $\gamma\mathcal{E}_1 = \mathcal{E}_1$, which is satisfied if and only if

$$\gamma\mathcal{E}_1^* = \mathcal{E}_1^*.$$

But

$$\begin{aligned}\gamma\mathcal{E}_1 &= \{\gamma\mathbf{n} : \mathbf{n} \in \Lambda, \mathbf{n} \circ \mathbf{n} = 1, \mathbf{n} \circ D < 0\} \\ &= \{\mathbf{m} \in \Lambda : \mathbf{m} \circ \mathbf{m} = 1, \gamma^{-1}\mathbf{m} \circ D < 0\} \\ &= \{\mathbf{m} \in \Lambda : \mathbf{m} \circ \mathbf{m} = 1, \mathbf{m} \circ \gamma D < 0\}.\end{aligned}$$

Note that $R_{\mathbf{v}_{ij}} D = D$ (it switches the $i$-th and $j$-th component), so $R_{\mathbf{v}_{ij}} \in \Gamma$.

**Lemma 5.1** *The planes $H_{\mathbf{e}_i}$ are faces of $\mathcal{K}$.*

*Proof* Let $\mathbf{n} \in \mathcal{E}_1$ and let $\psi$ be the distance from $D/||D|| \in \mathcal{H}$ to $H_{\mathbf{n}}$. Then $2\psi$ is the distance between $D/||D||$ and its image $R_{\mathbf{n}}$ under reflection through $H_{\mathbf{n}}$, so

$$\begin{aligned}\cosh(2\psi) &= \frac{D \circ R_{\mathbf{n}}(D)}{D \circ D} \\ &= \frac{1}{D \circ D} D \circ \left(D - \frac{2\mathbf{n} \circ D}{\mathbf{n} \circ \mathbf{n}} \mathbf{n}\right) \\ &= 1 - \frac{2(\mathbf{n} \circ D)^2}{D \circ D} \\ &= 1 + \frac{2(\rho - 2)^2}{\rho(\rho - 2)} \left(\sum_{i=1}^{\rho} n_i\right)^2 .\end{aligned}$$

Since $\mathbf{n} \circ D \neq 0$, this is minimal when the sum is one, which occurs when $\mathbf{n} = \mathbf{e}_i$. Thus, the planes $H_{\mathbf{e}_i}$ are all faces of $\mathcal{K}$, as there are no planes $H_{\mathbf{n}}$ with $\mathbf{n} \in \mathcal{E}_1$ that are closer to $D/||D||$. □

As a consequence, the initial configuration of $\rho$ mutually tangent hyperspheres represent faces of $\mathcal{K}$, so the packing $\mathcal{A}_\rho$ contains those hyperspheres. This establishes property (a) outlined in the Introduction.

**Lemma 5.2** *Suppose $E \in \mathcal{S}$ and $P \in \mathcal{T}_E$. Then $\phi_{P,E} \in \Gamma$.*

*Proof* Let $\mathbf{n} \in \mathcal{E}_1^*$, and let us first suppose that $\mathbf{n} \circ E \neq 0$. Then in the Poincaré model $\mathcal{H}_E$, the point $D$ is above the highest point on $H_{\mathbf{n},E}$, as we saw earlier. Now $\phi_{P,E}(D) \circ E = D \circ \phi_{P,E}^{-1}(E) = D \circ E$, so the image of $D$ is at the same height and hence is still above $H_{\mathbf{n},E}$. Thus, $\phi_{P,E}(\mathbf{n}) \in \mathcal{E}_1$.

The case when $\mathbf{n} \circ E = 0$ is a bit more difficult. Without loss of generality, we may assume $E = e_{\rho-1} + e_\rho$. We first note that

$$0 = \mathbf{n} \circ E = -2 \sum_{i=1}^{\rho-2} n_i, \tag{4}$$

so

$$\mathbf{n} \circ \mathbf{e}_\rho = -\sum_{i=1}^{\rho} n_i + 2n_\rho = n_\rho - n_{\rho-1}.$$

Since both $H_{\mathbf{n}}$ and $H_{\mathbf{e}_\rho}$ contain $E$, the two intersect, so $\mathbf{n} \circ \mathbf{e}_\rho = 0$ or $\pm 1$ (see Eq. 1). We note that

$$\begin{aligned}1 = \mathbf{n} \circ \mathbf{n} &= \sum_{i=1}^{\rho} n_i^2 - 2\sum_{i \neq j} n_i n_j \\ &\equiv \sum_{i=1}^{\rho} n_i^2 \pmod 2 \\ &\equiv \left(\sum_{i=1}^{\rho} n_i\right)^2 \pmod 2\end{aligned}$$

$$\equiv n_{\rho-1}^2 + n_\rho^2 \pmod 2 \qquad \text{(using Equation 4).}$$

Thus $n_{\rho-1} \not\equiv n_\rho \pmod 2$, so $\mathbf{n} \circ \mathbf{e}_\rho = \pm 1$. That is, $H_{\mathbf{n}}$ and $H_{\mathbf{e}_\rho}$ are tangent at $E$. There are therefore at most two faces of $\mathcal{K}$ through $E$. Since $H_{\mathbf{e}_{\rho-1}}$ and $H_{\mathbf{e}_\rho}$ are two faces of $\mathcal{K}$ through $E$, we get that $\mathbf{n} = \mathbf{e}_{\rho-1}$ or $\mathbf{e}_\rho$.

Finally,

$$\phi_{P,E}(\mathbf{e}_\rho) = \frac{2(\mathbf{e}_\rho \circ P)E}{P \circ E} - \mathbf{e}_\rho = E - \mathbf{e}_\rho = \mathbf{e}_{\rho-1},$$

and $\phi_{P,E}(\mathbf{e}_{\rho-1}) = \mathbf{e}_\rho$. Thus $\phi_{P,E}(\mathbf{n}) \in \mathcal{E}_1^*$ for all $\mathbf{n} \in \mathcal{E}_1^*$, so $\phi_{P,E} \in \Gamma$. □

Let

$$\Gamma' = \langle \{R_{\mathbf{v}_{ij}}, \phi_{P,E} : i \neq j, E \in \mathcal{S}, P \in \mathcal{T}_E\} \rangle$$

and define

$$\mathcal{K}' = \bigcap_{\mathbf{n} \in \Gamma'(\mathbf{e}_\rho)} H_{\mathbf{n}}^-.$$

Then clearly $\mathcal{K} \subset \mathcal{K}'$, as $\Gamma'(\mathbf{e}_\rho) \subset \mathcal{E}_1$. We wish to show that $\mathcal{K}' = \mathcal{K}$. It is enough to show that the packing that corresponds to $\mathcal{K}'$ is space filling. If it is not, then there exists a gap in the packing where we can fit a sphere. This sphere represents a halfspace in $\mathcal{H}$ that is contained in $\mathcal{K}'$. Let us formalize this property with the following:

**Statement 1** Let $\mathbf{n} \in \Lambda_\rho$ and suppose $D \in H_{\mathbf{n}}^-$. Then $H_{\mathbf{n}}^+ \not\subset \mathcal{K}'$.

Establishing the veracity of this statement for $4 \leq \rho \leq 8$ is the main result of the next section.

**Lemma 5.3** *Suppose Statement 1 is true for a given $\rho$. Then $\mathcal{K}' = \mathcal{K}$. Furthermore, the hyperplanes in $\mathcal{A}_\rho$ do not intersect, so the hyperspheres in $\mathcal{A}_{\rho,E}$ intersect tangentially or not at all.*

*Proof* Suppose there exists $\mathbf{m} \in \mathcal{E}_1^*$ that is not in $\Gamma'(\mathbf{e}_\rho)$. Then $H_{\mathbf{m}}$ is a face of $\mathcal{K}$ but is not a face of $\mathcal{K}'$. If $H_{\mathbf{m}}$ does not intersect any faces of $\mathcal{K}'$ except tangentially, then $H_{\mathbf{m}}^+ \subset \mathcal{K}'$, contradicting Statement 1. Thus, $H_{\mathbf{m}}$ intersects $H_{\gamma \mathbf{e}_\rho}$ transversely for some $\gamma \in \Gamma'$. Consequently, $\mathbf{m} \circ \gamma \mathbf{e}_\rho = 0$, as this product is an integer and is in the interval $(-1, 1)$ (see Eq. 1). Hence, $\gamma^{-1}\mathbf{m} \circ \mathbf{e}_\rho = 0$. Note that $\gamma^{-1}\mathbf{m} \in \mathcal{E}_1^*$, since $\gamma \in \Gamma'$. Let $\mathbf{m}' = \gamma^{-1}\mathbf{m}$, so

$$0 = \mathbf{m}' \circ \mathbf{e}_\rho = m'_\rho - \sum_{i=1}^{\rho-1} m'_i.$$

But then

$$\begin{aligned} 1 = \mathbf{m}' \circ \mathbf{m}' &= \sum_{i=1}^{\rho} (m'_i)^2 - 2\sum_{i \neq j} m'_i m'_j \\ &\equiv \left(\sum_{i=1}^{\rho} m'_i\right)^2 \pmod 2 \\ &\equiv (\mathbf{m}' \circ \mathbf{e}_\rho)^2 \equiv 0 \pmod 2, \end{aligned}$$

a contradiction. Thus no such $\mathbf{m}$ exists, so $\mathcal{K}' = \mathcal{K}$. The same argument shows the hyperplanes in $\mathcal{A}_\rho$ do not intersect, so the hyperspheres in $\mathcal{A}_{\rho,E}$ intersect tangentially or not at all. □

This shows that the definition of $\mathcal{A}_{4,E}$ given in Eq. 3 is consistent with the formal definition given in Sect. 4. It also establishes properties (c) and (d) as outlined in the Introduction. Note that the proof of Lemma 5.3 depends on showing $\mathcal{K}' = \mathcal{K}$, so does not follow from the results of Kovacs and Morrison. Finally, we have property (b):

**Corollary 5.4** *Suppose Statement 1 is true for a given $\rho$. Then every hypersphere in $\mathcal{A}_\rho$ is a member of $\rho$ mutually tangent hyperspheres in $\mathcal{A}_\rho$.*

*Proof* Let $H_{\mathbf{n}}$ be a hypersphere in $\mathcal{A}_\rho$. Then $\mathbf{n} = \gamma(\mathbf{e}_\rho)$ for some $\gamma \in \Gamma'$. But then $H_{\mathbf{n}}$ is a member of the $\rho$ mutually tangent spheres $H_{\gamma\mathbf{e}_1}, \ldots, H_{\gamma\mathbf{e}_\rho}$. □

## 6 The descent argument

In this section, we establish Statement 1 for $\rho = 6$, 7, and 8. Our approach is a method of descent on curvature, which is roughly equivalent to establishing a fundamental domain for $\Gamma'$.

Suppose $H_{\mathbf{n},E}$ is a hypersphere in $\partial\mathcal{H}_E$ with $E = \mathbf{e}_{\rho-1} + \mathbf{e}_\rho$. Its curvature is unchanged under the action of $\Gamma'_E$, the stabilizer of $E$ in $\Gamma'$, as those maps are Euclidean isometries on $\partial\mathcal{H}_E$. In $\partial\mathcal{H}_E$, there are many points of tangency $E'$ between spheres (e.g. the set $S$), which are essentially no different than $E$. Our intuitive idea is to use $\Gamma'_E$ to move $H_{\mathbf{n},E}$ close to one of these points $E'$, and check to see if the curvature of $H_{\mathbf{n},E'}$ with respect to $E'$ is strictly smaller, thereby giving us a method of descent.

We will begin in three dimensions (the sphere packing, $\rho = 5$), where our geometric intuition is strongest (and the packing is not too trivial), with a view to describing our geometric arguments algebraically, so that we may lift them to higher dimensions.

### 6.1 The case $\rho = 5$ (the sphere packing)

Let $E = \mathbf{e}_4 + \mathbf{e}_5$ be our point at infinity for the Poincaré model. Consider the cross section of $\mathcal{H}_E$ given by $x_4 = x_5$. This is the cross section shown in Fig. 4. Recall, we define the translations $T_{ij}$ in $\mathcal{H}_E$ by

$$T_{ij} = \phi_{P_{i,E},E} \circ \phi_{\mathbf{e}_i+\mathbf{e}_j,E}.$$

This is the translation that sends $P_{i,E}$ to $P_{j,E}$. The canonical fundamental domain for the group $G_1 = \langle T_{12}, T_{13}\rangle$ on this cross section is the parallelogram shown in Fig. 12, which we will use as a reference. Consider now the group $G_2 = \langle R_{\mathbf{v}_{12}}, R_{\mathbf{v}_{23}}, \phi_{\mathbf{e}_2+\mathbf{e}_3,E}\rangle$, which includes $G_1$. The two reflections $R_{\mathbf{v}_{12}}$ and $R_{\mathbf{v}_{23}}$ give us two natural faces for a fundamental domain for $G_2$, namely the faces $H_{\mathbf{v}_{12}}$ and $H_{\mathbf{v}_{23}}$. Let $Q_1$ be the center of $\Delta P_{1,E}P_{2,E}P_{3,E}$, the point of intersection between these two faces and on this cross section. Then $Q_1$ has coordinates $(x, x, x, y, y)$, satisfies $Q_1 \circ Q_1 = 0$, and is oriented so that $Q_1 \circ D < 0$, from which we conclude $Q_1 = (4, 4, 4, -1, -1)$. For the third face of a fundamental domain for $G_2$, let us use the plane

$$H_{\mathbf{n}} = \{\mathbf{x} \in \mathbb{R}^{4,1} : \mathbf{x} \circ Q_1 = \mathbf{x} \circ \phi_{\mathbf{e}_2+\mathbf{e}_3,E}(Q_1)\}.$$

This is the plane midway between $Q_1$ and $\phi_{\mathbf{e}_2+\mathbf{e}_3,E}(Q_1)$, which has normal vector the difference of $Q_1$ and its image. Solving for $\mathbf{n}$, we find $\mathbf{n} = \mathbf{s}_1$ (up to a multiple). Using a method of descent (using distance from $Q_1$), we find every point in the cross section is the image of a point in the region bounded by these three planes and under the action of

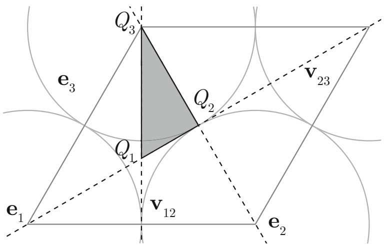


**Fig. 12** With $\rho = 5$: A fundamental domain for $G_2$ (shaded triangle $\Delta Q_1 Q_2 Q_3$), inside a fundamental domain for $G_1$ (the parallelogram)

an element of $G_2$. Those familiar with Dirichlet domains (which I learned from [12]) may recognize this construction.

Note that we could have made the argument easier by using $R_{\mathbf{s}_1} = R_{\mathbf{v}_{23}} \circ \phi_{\mathbf{e}_2+\mathbf{e}_3,E}$ instead of $\phi_{\mathbf{e}_2+\mathbf{e}_3,E}$, but we want to stay away from $R_{\mathbf{s}_1}$, as its canonical analog for larger $\rho$ is not in $\Gamma'_\rho$.

The vertices of this region are the points $Q_i$ which satisfy the linear equations $x_4 = x_5$, $Q_i \circ Q_i = 0$, $Q_i \circ D < 0$, and two of the three equations

$$\begin{aligned} x_1 &= x_2 \qquad (\text{on } H_{\mathbf{v}_{12}}) \\ x_2 &= x_3 \qquad (\text{on } H_{\mathbf{v}_{23}}) \\ x_1 &= 0 \qquad (\text{on } H_{\mathbf{s}_1}). \end{aligned} \tag{5}$$

We get $Q_1$ (using the first two equations), $Q_2 = \mathbf{e}_2 + \mathbf{e}_3 = (0, 1, 1, 0, 0)$ (using the first and third), and $Q_3 = P_{3,E} = 4\mathbf{e}_3 + E = (0, 0, 4, 1, 1)$ (using the second and third equations).

Suppose there exists $\mathbf{m} \in \Lambda$ such that $\mathbf{m} \circ D < 0$ and $H^+_{\mathbf{m}} \subset \mathcal{K}'$ (so contradicting Statement 1). Then the center of $H_{\mathbf{m},E}$ lies between the planes $H_{\mathbf{e}_4}$ and $H_{\mathbf{e}_5}$. Using $G_3 = \langle G_2, R_{\mathbf{v}_{45}} \rangle$, we can move this center to a point in the right prism bounded by the plane $H_{\mathbf{e}_5}$, $H_{\mathbf{v}_{45}}$, and the planes we found above: $H_{\mathbf{v}_{12}}$, $H_{\mathbf{v}_{23}}$, and $H_{\mathbf{s}_1}$. The prism has the vertices $Q_1$, $Q_2$, and $Q_3$, as well as the points $Q'_i$ which satisfies $Q'_i \circ \mathbf{e}_5 = 0$, and pairs of the equations in (5) (and of course, $Q'_i \circ Q'_i = 0$ and $Q'_i \circ D < 0$). We get the vertices

$$\begin{aligned} Q'_1 &= (1, 1, 1, -1, 2) \\ Q'_2 &= (0, 0, 1, 0, 1) = \mathbf{e}_3 + \mathbf{e}_5 \\ Q'_3 &= (0, 2, 2, -1, 3). \end{aligned}$$

Our intuition is that, once we have moved $H_{\mathbf{m},E}$ so that its center is inside this prism, then the curvature of $H_{\mathbf{m},E'}$ should be smaller for some $E'$. The intuitive choice is $E' = \mathbf{e}_3 + \mathbf{e}_5$. We note that $R_{\mathbf{v}_{34}}(E') = E$ (it switches the third and fourth components). Thus, all we need to check is that the prism lies entirely within $H_{\mathbf{v}_{34},E}$, for if it does, then the center of this moved $H_{\mathbf{m},E}$ will also be inside it. Its reflection through that plane, which is inversion in the sphere, will be a sphere with strictly smaller curvature, as desired.

Recall that $\mathbf{v}_{34} = \mathbf{e}_3 - \mathbf{e}_4$, and note that $(\mathbf{e}_3 - \mathbf{e}_4) \circ E = -2$, so the inside of $H_{\mathbf{v}_{34},E}$ is $H^+_{\mathbf{v}_{34},E}$. Thus, we need only check that $(\mathbf{e}_3 - \mathbf{e}_4) \circ \mathbf{x} \geq 0$ for all $\mathbf{x}$ in the prism. Since the prism is convex, it is enough to check it for its vertices $Q_i$ and $Q'_i$, which just means checking the third component is larger than the fourth component. This is easily verified.

This gives us a fundamental domain for $\langle \Gamma', R_{\mathbf{e}_5} \rangle$ in $\mathcal{H}$, namely the region above the half hypersphere $H_{\mathbf{v}_{34}}$ (in the Poincaré model $\mathcal{H}_E$), and in the infinite prism in $\mathcal{H}_E$ that lies above the three dimensional prism in $\partial \mathcal{H}_E$ described above. This fundamental domain has finite hypervolume, so this group has finite index in $\mathcal{O}^+(\mathbb{Z})$. A fundamental domain for $\Gamma'$ is the region with the face $H_{\mathbf{e}_5}$ removed.

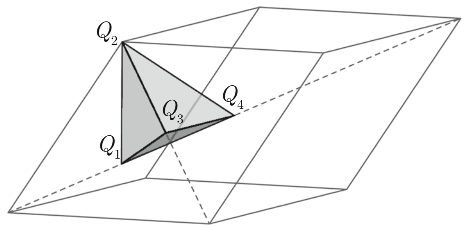


**Fig. 13** With $\rho = 6$: A parallelepiped that is the fundamental domain for $G_1$, together with the tetrahedron $Q_1Q_2Q_3Q_4$, which is a fundamental domain for $G_2$

In terms of our descent argument for $H_{\mathbf{m},E}$, since the property is preserved under the action of elements in $\Gamma'$, we have found an $\mathbf{m}'$ with the same property but such that $H_{\mathbf{m}',E}$ has strictly smaller curvature than $H_{\mathbf{m},E}$. Since $\mathbf{m} \in \Lambda$, $\mathbf{m} \circ E$ and $\mathbf{m}' \circ E$ are integers. Therefore descent cannot continue indefinitely, so at some point we get an $\mathbf{m}'$ with curvature 0 (so $H_{\mathbf{m}',E}$ has no center), meaning $H_{\mathbf{m}'}$ goes through $E$. No such $\mathbf{m}'$ exists, as we cannot fit such a half space between the planes $H_{\mathbf{e}_4}$ and $H_{\mathbf{e}_5}$. Thus, Statement 1 holds for $\rho = 5$.

## 6.2 The case $\rho = 6$

Let $E = \mathbf{e}_5 + \mathbf{e}_6$ be our point at infinity in the Poincaré model $\mathcal{H}_E$. Consider the cross section given by $x_5 = x_6$, which has a lattice of congruent spheres in a canon-ball stacking. Let $G_1 = \langle T_{12}, T_{13}, T_{14}\rangle$ and let its canonical fundamental domain be the parallelepiped shown in Fig. 13. Let $G_2 = \langle R_{\mathbf{v}_{12}}, R_{\mathbf{v}_{23}}, R_{\mathbf{v}_{34}}, \phi_{\mathbf{e}_3+\mathbf{e}_4,E}\rangle$. As before, let $Q_1$ be the center of the tetrahedron, the point of intersection of $H_{\mathbf{v}_{12}}$, $H_{\mathbf{v}_{23}}$, $H_{\mathbf{v}_{34}}$ and $H_{\mathbf{v}_{56}}$. Solving (together with $Q_1 \circ Q_1 = 0$ and $Q_1 \circ D < 0$), we get $Q_1 = (2, 2, 2, 2, -1, -1)$. We use this to get the plane

$$H_{\mathbf{n}} = \{\mathbf{x} \in \mathbb{R}^{5,1} : \mathbf{x} \circ Q_1 = \mathbf{x} \circ \phi_{\mathbf{e}_3+\mathbf{e}_4,E}(Q_1)\},$$

giving us $\mathbf{n} = (1, 1, -1, -1, -1, -1)$ and the equation $x_1 + x_2 = 0$. We therefore have, as an analog of Eq. (5), the following:

$$\begin{aligned} x_1 &= x_2 && (\text{from } R_{\mathbf{v}_{12}}) \\ x_2 &= x_3 && (\text{from } R_{\mathbf{v}_{23}}) \\ x_3 &= x_4 && (\text{from } R_{\mathbf{v}_{34}}) \\ x_1 + x_2 &= 0 && (\text{from } \phi_{\mathbf{e}_3+\mathbf{e}_4,E} \text{ and using } Q_1). \end{aligned} \tag{6}$$

The vertices are $Q_i$ where $Q_i$ is the solution to all but the $(4-i)$-th constraint (together with $Q_i \circ Q_i = 0$ and $Q_i \circ D < 0$). Solving, we get

$$\begin{aligned} Q_1 &= (2, 2, 2, 2, -1, -1) \\ Q_2 &= (0, 0, 0, 4, 1, 1) = P_{4,E} \\ Q_3 &= (0, 0, 1, 1, 0, 0) = \mathbf{e}_3 + \mathbf{e}_4 \\ Q_4 &= (-2, 2, 2, 2, 1, 1). \end{aligned}$$

Note that $Q_4$ is the center of the parallelepiped. This gives us a tetrahedral fundamental domain for $G_2$, as pictured in Fig. 13.

Extending to $\partial\mathcal{H}_E$, we get a prism with vertices $Q_i$ and their corresponding points on $H_{\mathbf{e}_6}$:

$$Q_1' = (2, 2, 2, 2, -3, 5)$$

$$Q'_2 = (0, 0, 0, 1, 0, 1) = \mathbf{e}_4 + \mathbf{e}_6$$
$$Q'_3 = (0, 0, 2, 2, -1, 3)$$
$$Q'_4 = (-1, 1, 1, 1, 0, 2).$$

As before, we assume that $E' = \mathbf{e}_4 + \mathbf{e}_6$ will be our best choice, so we consider the reflection $R_{\mathbf{v}_{45}}$. Again, $\mathbf{v}_{45} \circ E = (\mathbf{e}_4 - \mathbf{e}_5) \circ E = -2$, so we wish to check that the vertices of the prism are in $H^+_{\mathbf{v}_{45}}$. That is, we verify that $\mathbf{v}_{45} \circ Q_i \geq 0$ and $\mathbf{v}_{45} \circ Q'_i \geq 0$ for all $i$, which again just means checking that the fourth component is larger than the fifth. We come to the same conclusions: Statement 1 is true for $\rho = 6$; and we have fundamental domains for a subgroup with finite index in $\mathcal{O}^+(\mathbb{Z})$, and for $\Gamma'$. The fundamental domain for $\Gamma'$ is geometrically finite, which may be of interest to some.

### 6.3 The case $\rho = 7$

We are ready to tackle a case without a picture. We let $E = \mathbf{e}_6 + \mathbf{e}_7$. The parallelepiped will be important, but not at this step. We let $Q_1$ be the intersection of the planes $H_{\mathbf{v}_{12}}$, $H_{\mathbf{v}_{23}}$, $H_{\mathbf{v}_{34}}$, $H_{\mathbf{v}_{45}}$, and $H_{\mathbf{v}_{67}}$. As before, we require $Q_1 \circ Q_1 = 0$ and $Q_1 \circ D < 0$. As before, we solve for $\mathbf{n}$ and find our set of equations is

$$x_1 = x_2, \quad x_2 = x_3, \quad x_3 = x_4, \quad x_4 = x_5$$
$$x_1 + x_2 + x_3 = 0, \tag{7}$$

giving us

$$Q_1 = (4, 4, 4, 4, 4, -3, -3) \qquad Q'_1 = (1, 1, 1, 1, 1, -2, 3)$$
$$Q_2 = (0, 0, 0, 0, 4, 1, 1) = P_{5,E} \qquad Q'_2 = (0, 0, 0, 0, 1, 0, 1) = \mathbf{e}_5 + \mathbf{e}_7$$
$$Q_3 = (0, 0, 0, 1, 1, 0, 0) = \mathbf{e}_4 + \mathbf{e}_5 \qquad Q'_3 = (0, 0, 0, 2, 2, -1, 3)$$
$$Q_4 = (-4, -4, 8, 8, 8, 3, 3) \qquad Q'_4 = (-4, -4, 8, 8, 8, -1, 15)$$
$$Q_5 = (-4, 2, 2, 2, 2, 3, 3) \qquad Q'_5 = (-2, 1, 1, 1, 1, 1, 3).$$

Because we have no picture, we should give some thought as to whether these five vertices generate a four-dimensional polytope. This is easily verified by noting that the five equations in (7) together with $x_6 = x_7$ yield the unique solution $E$. That means that each equation is a hyperplane in $\partial\mathcal{H}_E \cong \mathbb{R}^5$ and do not have a common point of intersection. That we could solve for the points $Q_i$ means no two hyperplanes are parallel, so they bound a polytope.

As before, our intuition is that the point is now close enough to $E' = \mathbf{e}_5 + \mathbf{e}_7$, which is the image of $E$ under $R_{\mathbf{v}_{56}}$. We check that all the vertices are in $H^+_{\mathbf{v}_{56}}$, which means the fifth component is greater than or equal to the sixth. This is true for all except $Q_5$.

This is where the parallelepiped comes in again. When $\rho = 5$, the vertices of the parallelepiped (parallelogram) are $P_{1,E}$, $T_{12}(P_{1,E}) = P_{2,E}$, $T_{13}(P_{1,E}) = P_{3,E}$, and $T_{12}T_{13}(P_{1,E}) = T_{13}T_{12}(P_{1,E})$. Notice the $1-2-1$ pattern (think binomial coefficients). When $\rho = 6$, the endpoints of the long diagonal are $P_{1,E}$ and the center of $H_{\mathbf{f}_1}$; and the two rings of vertices $P_{i,E}$ for $i = 2, 3, 4$, and the centers of $H_{\mathbf{f}_i}$ for $i = 2, 3, 4$. Note again the $1-3-3-1$ pattern.

For $\rho = 7$, we have the endpoints of the long diagonal $P_{1,E}$ and $T_{12}T_{13}T_{14}T_{15}(P_{1,E})$; the first ring of four vertices $T_{1i}(P_{1,E}) = P_{i,E}$ for $i = 2, 3, 4, 5$; its complement at the other end; and the ring in the center, which are the six points $T_{1i}T_{1j}(P_{1,E}) = T_{1j}T_{1i}(P_{1,E})$ for $\{i, j\} \subset \{2, 3, 4, 5\}$. We found that $E' = \mathbf{e}_5 + \mathbf{e}_7$ was not enough, as it is not close enough to $Q_5$, so we pick another point using a point of tangency between the hyperplane

$H_{\mathbf{e}_7}$ and a sphere centered at a point on the middle ring. Intuition guides us to pick $E' = T_{14}T_{15}(\mathbf{e}_1) + \mathbf{e}_7 = T_{14}\mathbf{e}_5 + \mathbf{e}_7$. This leads us to consider reflection in the plane with normal vector $T_{14}\mathbf{e}_5 - \mathbf{e}_6 = T_{14}(\mathbf{e}_5 - \mathbf{e}_6)$, which is $T_{14}R_{\mathbf{v}_{56}}T_{14}^{-1}$, so is in $\Gamma'$.

We now have two hyperballs $H^+_{\mathbf{v}_{56}}$ and $H^+_{T_{14}\mathbf{v}_{56}}$ that we hope together will cover the right prism. The edges of this right prism that include the vertex $Q_5$ are $Q_5Q_5'$ and the edges $Q_5Q_i$ for $i = 1, \dots, 4$. We can find the points where $H_{\mathbf{v}_{56}}$ cut these edges and if we can verify that they are in $H^+_{T_{14}\mathbf{v}_{56}}$, then we will be done. We note that $H^+_{T_{14}\mathbf{v}_{56}}$ includes $Q_5'$, $Q_3$, and $Q_4$, so we need only check the edges $Q_1Q_5$ and $Q_2Q_5$.

The line $Q_1Q_5$ is the intersection of the span of $\{Q_1, Q_5, E\}$ with $\partial\mathcal{H}$. We write $P = xQ_1 + yQ_5 + zE$, solve $P \circ \mathbf{v}_{56} = 0$, $P \circ P = 0$, orient $P$ so that $P \circ D < 0$, and verify that $P \circ T_{14}\mathbf{v}_{56} \geq 0$. We do the same for the line $Q_2Q_5$.

Since $H^+_{\mathbf{v}_{56}}$ and $H^+_{T_{14}\mathbf{v}_{56}}$ cover the prism, we conclude as before to get our method of descent and our fundamental domains.

### 6.4 The case $\rho = 8$

Cutting to the chase: $E = \mathbf{e}_7 + \mathbf{e}_8$; the equations are

$$x_1 = x_2 \qquad x_2 = x_3 \qquad x_3 = x_4 \qquad x_4 = x_5 \qquad x_5 = x_6$$
$$x_1 + x_2 + x_3 + x_4 = 0,$$

giving us

$$\begin{aligned}
Q_1 &= (1, 1, 1, 1, 1, 1, -1, -1) & Q_1' &= (2, 2, 2, 2, 2, 2, -5, 7)\\
Q_2 &= (0, 0, 0, 0, 0, 4, 1, 1) = P_{6,E} & Q_2' &= (0, 0, 0, 0, 0, 1, 0, 1) = \mathbf{e}_6 + \mathbf{e}_8\\
Q_3 &= (0, 0, 0, 0, 1, 1, 0, 0) = \mathbf{e}_5 + \mathbf{e}_6 & Q_3' &= (0, 0, 0, 0, 2, 2, -1, 3)\\
Q_4 &= (-1, -1, -1, 3, 3, 3, 1, 1) & Q_4' &= (-2, -2, -2, 6, 6, 6, -1, 11)\\
Q_5 &= (-1, -1, 1, 1, 1, 1, 1, 1) & Q_5' &= (-2, -2, 2, 2, 2, 2, 1, 5)\\
Q_6 &= (-3, 1, 1, 1, 1, 1, 3, 3) & Q_6' &= (-6, 2, 2, 2, 2, 2, 5, 9).
\end{aligned}$$

This time there are two points not captured in $H^+_{\mathbf{v}_{67}}$, namely $Q_6$ and $Q_6'$. As before, we look at the reflection $R_{T_{15}\mathbf{v}_{67}}$. Only $Q_1$, $Q_2$, and $Q_1'$ are not in $H^+_{T_{15}\mathbf{v}_{67}}$, so we need only check the lines $Q_1Q_6$, $Q_2Q_6$, and $Q_1'Q_6'$. Again, we find the balls overlap on these line segments, so together they cover the prism.

### 6.5 The case $\rho = 9$ (no conclusion)

The relative simplicity of the case $\rho = 8$ is a bit deceptive, as this line of reasoning breaks down in the case $\rho = 9$. We get the equations

$$x_1 = x_2 \qquad x_2 = x_3 \qquad x_3 = x_4 \qquad x_4 = x_5 \qquad x_5 = x_6 \qquad x_6 = x_7$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

giving us the vertices of the prism:

$$\begin{aligned}
Q_1 &= (4, 4, 4, 4, 4, 4, 4, -5, -5) & Q_1' &= (1, 1, 1, 1, 1, 1, 1, -3, 4)\\
Q_2 &= (0, 0, 0, 0, 0, 0, 4, 1, 1) = P_{7,E} & Q_2' &= (0, 0, 0, 0, 0, 1, 0, 1) = \mathbf{e}_7 + \mathbf{e}_9\\
Q_3 &= (0, 0, 0, 0, 0, 1, 1, 0, 0) = \mathbf{e}_6 + \mathbf{e}_7 & Q_3' &= (0, 0, 0, 0, 0, 2, 2, -1, 3)\\
Q_4 &= (-4, -4, -4, -4, 16, 16, 16, 5, 5) & Q_4' &= (-4, -4, -4, -4, 16, 16, 16, -3, 29)
\end{aligned}$$

$Q_5 = (-4, -4, -4, 6, 6, 6, 6, 5, 5)$ $Q'_5 = (-2, -2, -2, 3, 3, 3, 3, 1, 7)$
$Q_6 = (-12, -12, 8, 8, 8, 8, 8, 15, 15)$ $Q'_6 = (-12, -12, 8, 8, 8, 8, 8, 11, 27)$
$Q_7 = (-4, 1, 1, 1, 1, 1, 1, 5, 5)$ $Q'_7 = (-8, 2, 2, 2, 2, 2, 2, 9, 13).$

The ball $H^+_{\mathbf{v}_{78}}$ covers all except $Q_6$, $Q_7$, $Q'_6$, and $Q'_7$. Since the parallelepiped now has seven rings, we do not expect to be able to cover the prism with just two balls. However, even when adding another ball from the middle ring, we still do not have enough to cover the prism. This case seems sufficiently different that we will leave its analysis for another time.

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