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ORIGINAL PAPER

Mixed curvature measures of translative integral geometry

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Abstract The curvature measures of a set X with singularities are measures concentrated on the normal bundle of X, which describe the local geometry of the set X. For given finitely many convex bodies or, more generally, sets with positive reach, the translative integral formula for curvature measures relates the integral mean of the curvature measures of the intersections of the given sets, one fixed and the others translated, to the mixed curvature measures of the given sets. In the case of two sets of positive reach, a representation of these mixed measures in terms of generalized curvatures, defined on the normal bundles of the sets, is known. For more than two sets, a description of mixed curvature measures in terms of rectifiable currents has been derived previously. Here we provide a representation of mixed curvature measures of sets with positive reach based on generalized curvatures. The special case of convex polyhedra is treated in detail.

Keywords Convex body · Set of positive reach · Convex polyhedron · Curvature measure · Translative integral geometry · Mixed functionals and measures · Geometric measure theory

Mathematics Subject Classification (2010) 53C65 · 52A20

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1 Introduction

The *reach* of a set $X \subset \mathbb{R}^d$, denoted reach X, is the supremum of all $r \geq 0$ such that for each point $z \in \mathbb{R}^d$ with $\operatorname{dist}(X, z) < r$ there is a unique nearest point $\Pi_X(z)$ in X. Sets with positive reach were studied by Federer [1] who showed that they satisfy a local Steiner formula, that is, for any $0 < r <$ reach X and any Borel set $B \subset \mathbb{R}^d$,

$$\mathcal{H}^d(X_r \cap \Pi_X^{-1}(B)) = \sum_{k=0}^{d} \kappa_{d-k} r^{d-k} \mathbf{C}_k(X, B), \tag{1}$$

where $X_r := \{z \in \mathbb{R}^d : \operatorname{dist}(z, X) \leq r\}$, $\kappa_j := \pi^{\frac{j}{2}}/\Gamma(1 + \frac{j}{2})$ and $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure. The coefficients $\mathbf{C}_k(X, \cdot)$ are signed Radon measures, called *curvature measures* of order k of X if $0 \leq k \leq d-1$, and $\mathbf{C}_d(X, \cdot) = \mathcal{H}^d(X \cap \cdot)$. The curvature measures possess the usual properties of curvature measures of sets with C^2 smooth boundaries and of convex sets, in particular, they satisfy the Gauss-Bonnet formula and the Principal Kinematic Formula (see [1]). Sets with positive reach constitute a common generalization of smooth submanifolds and convex sets. Although they have been studied for quite some time now, a complete structural understanding of sets with positive reach is still missing; see [11] for recent work on sets with positive reach and further references.

The main difference to the smooth case is that the Gauss map is not defined uniquely on the boundary of a set X with positive reach. Therefore, the *unit normal bundle*

$$\operatorname{nor} X := \{(x, u) \in \mathbb{R}^d \times S^{d-1} : x \in X,\ u \in \operatorname{Nor}(X, x)\}$$

is used instead [here $\operatorname{Nor}(X, x)$ is the normal cone of X at $x \in X$, defined as the dual convex cone to the tangent cone $\operatorname{Tan}(X, x)$], and the role of the Gauss map from the smooth case is played by the projection $(x, u) \mapsto u$ to the second component. Thus, in generalization of the curvature measures $\mathbf{C}_k(X, \cdot)$ on $\mathbb{R}^d$, it is convenient to consider curvature measures as measures on $\mathbb{R}^d \times S^{d-1}$ which are supported by the unit normal bundle of X. Such measures are determined by the refined local Steiner formula which states that, for any $0 < r <$ reach X and any bounded Borel set $A \subset \mathbb{R}^d \times S^{d-1}$,

$$\mathcal{H}^d((X_r \setminus X) \cap \xi_X^{-1}(A)) = \sum_{k=0}^{d-1} \kappa_{d-k} r^{d-k} C_k(X; A), \tag{2}$$

where

$$\xi_X : z \mapsto \left(\Pi_X(z), \frac{z - \Pi_X(z)}{\|z - \Pi_X(z)\|}\right), \quad \text{for } z \in X_r \setminus X.$$

The coefficients $C_k(X; \cdot)$ are signed Radon measures on $\mathbb{R}^d \times S^{d-1}$, their first component projections agree with the curvature measures from (1) and they are called *generalized curvature measures* [17], support measures [14] or curvature-direction measures. In the following, we shall also use the short name curvature measures for the measures in (2).

One starting point of the present work are kinematic formulas of integral geometry for sets $X, Y \subset \mathbb{R}^d$ of positive reach. Let G_d denote the Euclidean motion group of $\mathbb{R}^d$ and let μ_d denote the suitably normalized Haar measure on G_d. For bounded Borel sets $\alpha, \beta \subset \mathbb{R}^d$ and $k \in \{0, \ldots, d\}$, the principal kinematic formula for curvature measures states that

$$\int_{\mathrm{G}_d} \mathbf{C}_k(X \cap gY, \alpha \cap g\beta)\, \mu_d(dg) = \sum_{\substack{0 \leq i, j \leq d \\ i+j=d+k}} c(d, i, j) \mathbf{C}_i(X, \alpha) \mathbf{C}_j(Y, \beta),$$

where $c(d, i, j)$ are explicitly known constants (see [8,14]). In many applications in stochastic geometry it is, however, necessary to consider integration with respect to translations only. In particular, this is crucial for the investigation of stationary random sets which are not isotropic (see [14]). The basic formula of *translative* integral geometry thus deals with the integrals

$$\int_{\mathbb{R}^d} \mathbf{C}_k(X \cap (Y + z), \alpha \cap (\beta + z))\, dz,$$

which are expressed as a sum of mixed curvature measures depending on both sets X and Y. More generally, using the generalized curvature measures and an arbitrary nonnegative, Borel measurable function $h : \mathbb{R}^{2d} \times S^{d-1} \to [0, \infty]$ with compact support (allowing to include directional information), we are interested in the translative integrals

$$\int_{\mathbb{R}^d} \int h(x, x - z, u)\, C_k(X \cap (Y + z); d(x, u))\, dz,$$

for $k \in \{0, \dots, d-1\}$, which again can be expressed in terms of integrals of mixed curvatures measures of X and Y. The iterated version of such a relation works with a finite number q of sets, $q - 1$ of them being shifted independently. For $q \geq 2$ and given subsets $X_1, \dots, X_q$ of $\mathbb{R}^d$ with positive reach, the iterated translative integral formula involves the *mixed curvature measures*

$$C_{r_1,\dots,r_q}(X_1, \dots, X_q; \cdot)\,,$$

for $r_1, \dots, r_q \in \{0, \dots, d\}$ with $(q - 1)d \leq r_1 + \cdots + r_q \leq qd - 1$, which are signed Borel measures on $\mathbb{R}^{qd} \times S^{d-1}$, and reads

$$\begin{aligned}\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \int h(x, x - z_2, \dots, x - z_q, u)\, C_k(\underline{X}(\underline{z}); d(x, u))\, dz_q \dots dz_2 \\ = \sum_{\substack{0 \leq r_1,\dots,r_q \leq d \\ r_1+\cdots+r_q=(q-1)d+k}} \int h(x_1, \dots, x_q, u)\, C_{r_1,\dots,r_q}(X_1, \dots, X_q; d(x_1, \dots, x_q, u)),\end{aligned} \tag{3}$$

where $k \in \{0, \dots, d-1\}$, $\underline{X}(\underline{z}) := X_1 \cap (X_2 + z_2) \cap \cdots \cap (X_q + z_q)$, and $h : \mathbb{R}^{qd} \times S^{d-1} \to [0, \infty]$ is an arbitrary nonnegative, Borel measurable function with compact support.

This iterated integral formula was first proved in the setting of convex geometry by Schneider and Weil [13] for $q = 2$ and by Weil [15] for $q \geq 2$ in a less general form, namely for a function h which is independent of the direction vector u. Subsequently, formula (3) was established in [8] for $q = 2$, and in [7] for general q, in the setting of sets with positive reach. An extension to relative curvature measures, that is, curvature measures defined with respect to a non-Euclidean metric, has been obtained in [6, Section 3].

For the mixed curvature measures of arbitrary sets with positive reach and $q \geq 3$, up to now only a representation was available which involves the notion of a rectifiable current (see [7]). In the special case of mixed curvature measures of two sets of positive reach (that is, for $q = 2$) an integral representation based on generalized curvature functions, defined on the normal bundles of the sets, has already been proved in [8,9], while the case of convex bodies and general q is covered in [6, Section 4]. In the present paper, we extend all these results by treating the case of a finite sequence of sets with positive reach. For convex polyhedra we obtain a simple description of the mixed curvature measures which has an intuitive geometric interpretation (see also [6, Section 4]) and extends the important special case considered in [16].

In Sect. 2 we introduce the notions and notation required in the following and provide two auxiliary results, one from multilinear algebra and the other from measure theory. In Sect. 3 we formulate our main result (Theorem 2) and provide sufficient conditions for the validity of its assumption. We also deal with some important particular cases such as that of convex polyhedra. The last section (Sect. 4) contains the proof of the main result.

2 Preliminaries

The basic setting for this paper will be the d-dimensional Euclidean space $\mathbb{R}^d$, $d \geq 2$, with scalar product $x \cdot y$ and norm $|x| = \sqrt{x \cdot x}$, $x, y \in \mathbb{R}^d$. The same notation will be adopted in any Euclidean space which will be considered, independent of its dimension. In particular, we shall investigate cartesian products such as $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$, with k factors, for which we also write $\mathbb{R}^{kd}$. In this case, we endow each factor with the same scalar product, and the cartesian product will carry the natural scalar product which is derived from its components by summation. Let $\mathcal{H}^s$, for $s \geq 0$, denote the s-dimensional Hausdorff measure. The Euclidean spaces where Hausdorff measures will be considered, will always be clear from the context. We write $\omega_n := 2\pi^{n/2}/\Gamma(n/2)$ for the $(n-1)$-dimensional Hausdorff measure of the $(n-1)$-dimensional unit sphere S^{n-1} in $\mathbb{R}^n$.

We shall use the standard notation of multilinear algebra as introduced in [2]. In particular, for $k \in \{0, \dots, d\}$ we denote by $\bigwedge_k V$ and $\bigwedge^k V$ the spaces of k-vectors and k-covectors, respectively, of a vector space V, and $\langle \alpha, \phi \rangle$ stands for the bilinear pairing, where $\alpha \in \bigwedge_k V$ and $\phi \in \bigwedge^k V$. We denote by $\Omega^d = e_1' \wedge \cdots \wedge e_d'$ the volume d-form in $\mathbb{R}^d$, where $\{e_1', \dots, e_d'\}$ is the basis which is dual to the canonical orthonormal basis $\{e_1, \dots, e_d\}$ of $\mathbb{R}^d$. The scalar product in $\mathbb{R}^d$ induces a natural linear isomorphism $v \mapsto v'$ from $\mathbb{R}^d$ to the dual space $\bigwedge^1 \mathbb{R}^d$ which in turn induces a natural linear isomorphism $\alpha \mapsto \alpha'$ from $\bigwedge_k \mathbb{R}^d$ to its dual $\bigwedge^k \mathbb{R}^d$. By means of this correspondence, the mapping $\alpha \mapsto *\alpha$ from $\bigwedge_k \mathbb{R}^d$ to $\bigwedge_{d-k} \mathbb{R}^d$ is defined (cf. [2]) by

$$*\alpha = (e_1 \wedge \cdots \wedge e_d) \llcorner \alpha',$$

where $\cdot \llcorner \cdot$ denotes the standard inner multiplication (see [2, §1.5.1 and §1.7.8]). It follows from the definition that

$$\langle \alpha \wedge *\alpha, \Omega^d \rangle = |\alpha|^2. \tag{4}$$

Let p be a natural number and let p multivectors $\alpha_1, \dots, \alpha_p$ in $\mathbb{R}^d$ be given such that the sum of their multiplicities equals $(p-1)d$. Then we define the *p-product* of $\alpha_1, \dots, \alpha_p$ as

$$[\alpha_1, \dots, \alpha_p] := \langle (*\alpha_1) \wedge \cdots \wedge (*\alpha_p), \Omega^d \rangle. \tag{5}$$

Note that this definition is consistent with that given in [7]. Moreover, if α_i is a unit simple multivector and L_i is the linear subspace corresponding to α_i, for $i = 1, \dots, p$, then the p-product $[\alpha_1, \dots, \alpha_p]$ coincides, up to sign, with the subspace determinant $[L_1, \dots, L_p]$ defined in [15] (see also [14, §14.1]).

Let $q \geq 1$, $d \geq 2$ and $r_1, \dots, r_q \in \{0, \dots, d\}$ be given with

$$(q-1)d \leq r_1 + \cdots + r_q \leq qd - 1.$$

We set $R_1 := r_1$, $R_2 := r_1 + r_2$, ..., $R_q := r_1 + \cdots + r_q$, $r_{q+1} := qd - 1 - R_q$ and $k := r_1 + \cdots + r_q - (q-1)d \in \{0, \dots, d-1\}$; hence $r_{q+1} = d - 1 - k$. Let $\mathrm{Sh}(r_1, \dots, r_{q+1})$ denote the set of all permutations of $\{1, \dots, qd-1\}$ which are increasing on each of the sets $\{1, \dots, R_1\}$, $\{R_1 + 1, \dots, R_2\}$, ..., $\{R_q + 1, \dots, qd - 1\}$.

We write $\varphi_{r_1,\dots,r_q} \in \mathcal{D}^{qd-1}(\mathbb{R}^{(q+1)d})$ for the differential form which is defined by

$$\left\langle \bigwedge_{i=1}^{qd-1} \left(a_i^1, \dots, a_i^{q+1}\right), \varphi_{r_1,\dots,r_q}(x_1, \dots, x_q, u) \right\rangle$$
$$= \frac{1}{\omega_{d-k}} (-1)^{c_1(d,r_1,\dots,r_q)} \sum_{\sigma \in \mathrm{Sh}(r_1,\dots,r_{q+1})} \mathrm{sgn}(\sigma)$$
$$\times \left[\bigwedge_{i=1}^{R_1} a_{\sigma(i)}^1, \bigwedge_{i=R_1+1}^{R_2} a_{\sigma(i)}^2, \dots, \bigwedge_{i=R_{q-1}+1}^{R_q} a_{\sigma(i)}^q, \bigwedge_{i=R_q+1}^{qd-1} a_{\sigma(i)}^{q+1} \wedge u \right],$$

where $a_j^i \in \mathbb{R}^d$, for $i \in \{1, \dots, q+1\}$ and $j \in \{1, \dots, qd-1\}$, is arbitrarily chosen and

$$c_1(d, r_1, \dots, r_q) = d \sum_{i=1}^{q} r_i + d \sum_{i=1}^{q} i r_i + \sum_{1 \le i < j \le q} r_i r_j.$$

Since $\varphi_{r_1,\dots,r_q}(x_1, \dots, x_q, u)$ depends only on the last vector component, we shall write briefly $\varphi_{r_1,\dots,r_q}(u)$.

In particular, for $q = 1$, $r_1 =: r \in \{0, \dots, d-1\}$, $r_2 = d - 1 - r_1$ and $k = r_1$, we have $(-1)^{c_1(d,r)} = 1$ and φ_r is the kth Lipschitz–Killing curvature form on $\mathbb{R}^{2d}$ involved in the definition of the kth curvature measure (see [8,17] for an alternative representation of this differential form). The sign determined by $c_1(d, r_1, \dots, r_q)$ differs from that given in [7], see the proof of Lemma 1 below for a correction of the last step of the proof of [7, Lemma 2].

Let G_i, π be the projections defined on $(\mathbb{R}^d)^{q+1}$ by

$$G_i(x_1, \dots, x_q, u) := x_1 - x_i, \quad \pi(x_1, \dots, x_q, u) := (x_1, u),$$

$i = 2, \dots, q$.

Lemma 1 ([7, Lemma 2]) *For any* $q \ge 2$ *and* $0 \le k \le d-1$, *we have*

$$G_2^\# \Omega^d \wedge \dots \wedge G_q^\# \Omega^d \wedge \pi^\# \varphi_k = \sum_{\substack{0 \le r_1, \dots, r_q \le d \\ r_1 + \dots + r_q = (q-1)d + k}} \varphi_{r_1,\dots,r_q}.$$

Proof This result was shown in [7, Lemma 2]. In the last but one line of the proof, the sign was still correct and given by $(-1)^{c_1}$ with

$$c_1 = (k-1)(q-1)d + \sum_{i=2}^{q} (d - r_i)(k_i - (i-2)d - 1),$$

with $k_i = R_i - (i-1)d$. Using the symbol $m \sim n$ whenever two integers m, n differ by an even number, we have

$$c_1(d, r_1, \dots, r_q)$$
$$\sim (q-1)dR_q + \sum_{i=1}^{q} (d - r_i)(R_i - d - 1)$$
$$\sim (q-1)dR_q + d \sum_{i=1}^{q} R_i + \sum_{i=1}^{q} r_i R_i + (d-1) \sum_{i=1}^{q} r_i$$

$$\sim (q-1)dR_q + d\sum_{i=1}^{q}(q+1-i)r_i + \sum_{1\le i\le j\le q} r_i r_j + (d-1)\sum_{i=1}^{q} r_i$$

$$\sim ((q-1)d + d(q+1) + 1 + (d-1))\sum_{i=1}^{q} r_i + d\sum_{i=1}^{q} i r_i + \sum_{1\le i<j\le q} r_i r_j$$

$$\sim d\sum_{i=1}^{q} r_i + d\sum_{i=1}^{q} i r_i + \sum_{1\le i<j\le q} r_i r_j,$$

which agrees with the value given in the definition above. □

Let $X \subset \mathbb{R}^d$ have positive reach, and let nor X be its unit normal bundle, as defined in the introduction (cf. [1]). Then nor X is locally $(d-1)$-rectifiable, and for $\mathcal{H}^{d-1}$-almost all $(x,u) \in$ nor X, the tangent cone of nor X at (x,u) is the linear subspace spanned by the vectors

$$\frac{1}{\sqrt{1+k_i(x,u)^2}}\big(a_i(x,u), k_i(x,u)a_i(x,u)\big), \quad i = 1,\dots,d-1, \tag{6}$$

where $k_1(x,u),\dots,k_{d-1}(x,u) \in (-\infty,\infty]$ are the (generalized) principal curvatures and where $a_1(x,u),\dots,a_{d-1}(x,u)$ are the corresponding principal directions at (x,u) (cf. [17]). In the case of infinite principal curvatures, we use the conventions $\frac{1}{\sqrt{1+\infty^2}} = 0$ and $\frac{\infty}{\sqrt{1+\infty^2}} = 1$. The unit normal bundle is oriented by a unit simple $(d-1)$-vector field $a_X(x,u)$ which can be given as the wedge product of the vectors from (6) which are oriented in such a way that

$$\langle a_1(x,u)\wedge\cdots\wedge a_{d-1}(x,u)\wedge u, \Omega^d\rangle = 1.$$

Then the normal cycle of X is the integer rectifiable current

$$N_X = (\mathcal{H}^{d-1} \llcorner \text{nor } X)\wedge a_X$$

and the kth curvature measure of X, for $k \in \{0,\dots,d-1\}$, can be represented as

$$C_k(X;A) = (N_X \llcorner \mathbf{1}_A)(\varphi_k),$$

where A is a bounded Borel subset of $\mathbb{R}^d \times S^{d-1}$.

2.1 Mixed curvature measures and the translative integral formula

Let $q,d \ge 2$, and let $X_1,\dots,X_q \subset \mathbb{R}^d$ be sets with positive reach. For unit vectors $u_1,\dots,u_q \in S^{d-1}$ we set

$$\text{cone}\{u_1,\dots,u_q\} := \left\{\sum_{i=1}^{q}\lambda_i u_i : \lambda_i \ge 0 \text{ for } i=1,\dots,q, \sum_{i=1}^{q}\lambda_i^2 > 0\right\}.$$

Note that cone$\{u_1,\dots,u_q\}$ contains a line if and only if it contains the origin, or if and only if $o \in \text{conv}\{u_1,\dots,u_q\}$ (the origin is contained in the convex hull of $u_1,\dots,u_q$). If one of these equivalent conditions is violated, then cone$\{u_1,\dots,u_q\}$ is a proper convex cone. Next we introduce the *joint unit normal bundle*

$$\text{nor}\,(X_1,\dots,X_q) := \{(x_1,\dots,x_q,u) \in \mathbb{R}^{qd}\times S^{d-1} : u \in \text{cone}\{u_1,\dots,u_q\} \text{ for some } (x_i,u_i) \in \text{nor } X_i,\ i=1,\dots,q,\ o \notin \text{cone}\{u_1,\dots,u_q\}\};$$

compare [7]. Note that the open cone was used in [7]. However, in order that [7, Lemma 3] and further results hold, the definition of the cone given here should be applied. Further, we define the Borel sets

$$R^c := \{(x_1, u_1, \ldots, x_q, u_q) \in (\mathbb{R}^d \times S^{d-1})^q : o \notin \text{cone}\{u_1, \ldots, u_q\}\},$$
$$\underline{\mathcal{N}(X)} := (\text{nor}\, X_1 \times \cdots \times \text{nor}\, X_q) \cap R^c$$

and

$$S_+^{q-1} := \{(t_1, \ldots, t_q) \in S^{q-1} : t_i \geq 0 \text{ for } i = 1, \ldots, q\}.$$

The map

$$T : \underline{\mathcal{N}(X)} \times S_+^{q-1} \to \text{nor}\,(X_1, \ldots, X_q)$$

is defined by

$$T(x_1, u_1, \ldots, x_q, u_q, t) := \left(x_1, \ldots, x_q, \frac{\sum_{i=1}^q t_i u_i}{|\sum_{i=1}^q t_i u_i|} \right).$$

It is easy to see that T is well-defined, locally Lipschitz and onto. Although T is not injective, the following lemma (proved for the case $q = 2$ in [18]) is sufficient for our purposes.

Lemma 2 *For $\mathcal{H}^{qd-1}$-almost all elements of* im(T)*, the pre-image under T is a single point.*

Proof We write

$$\triangle(q-1) := \left\{ (t_1, \ldots, t_q) \in [0,1]^q : \sum_{i=1}^q t_i = 1 \right\}$$

for the $(q-1)$-dimensional simplex embedded in $\mathbb{R}^q$. Clearly, to prove the lemma it is sufficient to show that the map

$$G : \underline{\mathcal{N}(X)} \times \triangle(q-1) \to \mathbb{R}^{qd} \times S^{d-1},$$
$$(x_1, u_1, \ldots, x_q, u_q, t_1, \ldots, t_q) \mapsto \left(x_2 - x_1, \ldots, x_q - x_1, x_1, \frac{\sum_{i=1}^q t_i u_i}{|\sum_{i=1}^q t_i u_i|} \right),$$

has a unique pre-image for $\mathcal{H}^{qd-1}$-almost all elements of im(G). Excluding a set of $\mathcal{H}^{qd-1}$ measure zero from im(T), we see that it is sufficient to consider the restriction $\tilde{G}$ of G to the subset $\underline{\mathcal{N}(X)} \times \tilde{\triangle}(q-1)$ with $\tilde{\triangle}(q-1) = \triangle(q-1) \setminus \{(0, \ldots, 0, 1)\}$.

For the proof we proceed by induction. The case $q = 2$ has been established in [18]. Now we assume that the assertion has already been proved for $q - 1$ convex bodies. Set

$$\bar{R}^c := \{(y_1, \ldots, y_{q-1}, u, y, v) \in \mathbb{R}^{(q-1)d} \times S^{d-1} \times \mathbb{R}^d \times S^{d-1} : o \notin \text{cone}\{u, v\}\}.$$

To establish the assertion for q sets $X_1, \ldots, X_q$ with positive reach, $q \geq 3$, we introduce the maps

$$\varphi_q : \mathbb{R}^{qd} \times \mathbb{R}^d \to \mathbb{R}^{qd} \times \mathbb{R}^d, \quad (x_1, \ldots, x_q, u) \mapsto (x_2 - x_1, \ldots, x_q - x_1, x_1, u),$$
$$G_2 : \underline{\mathcal{N}(X)} \times \tilde{\triangle}(q-1)$$
$$\to ([\varphi_{q-1}(\text{nor}\,(X_1, \ldots, X_{q-1})) \times \text{nor}\, X_q] \cap \bar{R}^c) \times (0, \infty),$$
$$(x_1, u_1, \ldots, x_q, u_q, t_1, \ldots, t_q)$$
$$\mapsto \left(x_2 - x_1, \ldots, x_{q-1} - x_1, x_1, \frac{\sum_{i=1}^{q-1} t_i u_i}{|\sum_{i=1}^{q-1} t_i u_i|}, x_q, u_q, \frac{t_q}{|\sum_{i=1}^{q-1} t_i u_i|} \right),$$

and

$$
\begin{aligned}
&G_1 : ([\varphi_{q-1}(\text{nor}\,(X_1, \ldots, X_{q-1})) \times \text{nor}\, X_q] \cap \bar{R}^c) \times [0, \infty) \\
&\quad \to \varphi_q(\text{nor}\,(X_1, \ldots, X_q)), \\
&(z_2, \ldots, z_{q-1}, x_1, v, x_q, u_q, s) \mapsto \left(z_2, \ldots, z_{q-1}, x_q - x_1, x_1, \frac{v + su_q}{|v + su_q|}\right);
\end{aligned}
$$

hence, $\tilde{G} = G_1 \circ G_2$. By the inductive hypothesis and since $(q-1)d - 1 + d = qd - 1$, it follows that, for $\mathcal{H}^{qd-1}$-almost all elements of $\text{im}(G_2)$, the map G_2 has a unique pre-image. Thus, since G_1 is locally Lipschitz, the image under G_1 of the set of all elements of $\text{im}(G_2)$ for which the pre-image under G_2 is not uniquely determined has $(qd-1)$-dimensional Hausdorff measure zero.

Furthermore, for $\mathcal{H}^{qd-1}$-almost all

$$
\begin{aligned}
&(z_2, \ldots, z_{q-1}, x_1, v, x_q, u_q, s) \\
&\qquad \in ([\varphi_{q-1}(\text{nor}\,(X_1, \ldots, X_{q-1})) \times \text{nor}\,(X_q)] \cap \bar{R}^c) \times (0, \infty)
\end{aligned}
$$

we have

$$
(x_1, v) \in \text{nor}\,(X_1 \cap (X_2 - z_2) \cap \cdots \cap (X_{q-1} - z_{q-1})),
$$

and therefore the result in [18] shows that $\mathcal{H}^{qd-1}$-almost all elements of $G_1(\text{im}(G_2))$ have a unique pre-image under G_1. In fact, here we use that

$$
\frac{v + su_q}{|v + su_q|} = \frac{\frac{1}{s+1}v + \frac{s}{s+1}u_q}{|\frac{1}{s+1}v + \frac{s}{s+1}u_q|}
$$

and that $[0, \infty) \to [0, 1)$, $s \mapsto (1+s)^{-1}s$, is locally bi-Lipschitz. Thus the assertion follows. □

We recall now the description of the mixed curvature measures from [7]. Since T is locally Lipschitz, nor $(X_1, \ldots, X_q)$ is countably $(qd-1)$-rectifiable. We equip nor $(X_1, \ldots, X_q)$ with the orientation given by the unit simple tangent $(qd-1)$ vector field $a_{X_1,\ldots,X_q}$ associated with nor $(X_1, \ldots, X_q)$ and fulfilling

$$
\langle a_{X_1,\ldots,X_q}, \psi_\varepsilon(u)\rangle > 0 \tag{7}
$$

for sufficiently small $\varepsilon > 0$, where

$$
\psi_\varepsilon(u) = \sum_{\substack{0 \le r_1,\ldots,r_q \le d \\ (q-1)d \le r_1+\cdots+r_q \le qd-1}} \varepsilon^{qd-1-r_1-\cdots-r_q} \varphi_{r_1,\ldots,r_q}(u).
$$

It follows from the proof of Theorem 2 below that condition (7) is satisfied if $\varepsilon > 0$ is small enough.

Let $0 \le r_1, \ldots, r_q \le d-1$ be integers with $(q-1)d \le r_1 + \cdots + r_q$. The *mixed curvature measure* of $X_1, \ldots, X_q$ of order $r_1, \ldots, r_q$ is a signed Radon measure on $\mathbb{R}^{qd} \times S^{d-1}$ defined by

$$
\begin{aligned}
&C_{r_1,\ldots,r_q}(X_1, \ldots, X_q; A) \\
&\quad := \left[\left(\mathcal{H}^{qd-1} \llcorner \text{nor}\,(X_1, \ldots, X_q)\right) \wedge a_{X_1,\ldots,X_q}\right](\mathbf{1}_A \varphi_{r_1,\ldots,r_q}),
\end{aligned} \tag{8}
$$

where $A \subset \mathbb{R}^{qd} \times S^{d-1}$ is a Borel measurable set, provided that the integral on the right-hand side is well defined. Note that since T is only locally Lipschitz, it may happen that nor $(X_1, \dots, X_q)$ has not locally finite $\mathcal{H}^{qd-1}$ measure. Therefore, in order that the mixed curvature measures are well defined as Radon measures, we shall assume that the total variation measure $\|C\|_{r_1,\dots,r_q}(X_1, \dots, X_q; \cdot)$ corresponding to (8) is locally finite [Eq. (9)].

The mixed curvature measures are symmetric in the sense that for any permutation σ of $\{1, \dots, q\}$, we have

$$\begin{aligned}&C_{r_{\sigma(1)},\dots,r_{\sigma(q)}}(X_{\sigma(1)}, \dots, X_{\sigma(q)}; A_{\sigma(1)} \times \cdots \times A_{\sigma(q)} \times B)\\&\quad = C_{r_1,\dots,r_q}(X_1, \dots, X_q; A_1 \times \cdots \times A_q \times B)\end{aligned}$$

(see [7, Proposition 1 (c)]). The definition of mixed curvature measures is extended to arbitrary indices $0 \le r_i \le d$ by setting

$$\begin{aligned}&C_{d,\dots,d,r_{m+1},\dots,r_q}(X_1, \dots, X_q; \cdot)\\&\quad := (\mathcal{H}^d \llcorner X_1) \otimes \cdots \otimes (\mathcal{H}^d \llcorner X_m) \otimes C_{r_{m+1},\dots,r_q}(X_{m+1}, \dots, X_q; \cdot)\end{aligned}$$

for $m \in \{1, \dots, q-1\}$ and $r_{m+1}, \dots, r_q \le d-1$, provided that $r_{m+1}+\cdots+r_q \ge (q-m-1)d$, and by applying the symmetry. Consequently, the mixed curvature measures are defined for all integers $0 \le r_1, \dots, r_q \le d$ with $(q-1)d \le r_1 + \cdots + r_q \le qd - 1$.

As already mentioned, we will assume that

$$\begin{aligned}&\|C\|_{r_1,\dots,r_q}(X_1, \dots, X_q; \cdot) \text{ is locally finite for all } 0 \le r_1, \dots, r_q \le d\\&\quad \text{with } (q-1)d \le r_1 + \cdots + r_q \le qd - 1.\end{aligned} \tag{9}$$

Due to the definition, this is equivalent to

$$\begin{aligned}&\|C\|_{r_{i_1},\dots,r_{i_m}}(X_{i_1}, \dots, X_{i_m}; \cdot) \text{ is locally finite whenever } 2 \le m \le q,\\&\quad 1 \le i_1 < \cdots < i_m \le q,\ 0 \le r_{i_1}, \dots, r_{i_m} \le d-1 \text{ and } r_{i_1} + \cdots + r_{i_m} \ge (m-1)d.\end{aligned} \tag{10}$$

Condition (10) can be also written in the form

$$\int_{\mathrm{nor}\,(X_{i_1},\dots,X_{i_m})} \mathbf{1}_A \left|\langle a_{X_{i_1},\dots,X_{i_m}}, \varphi_{r_{i_1},\dots,r_{i_m}}\rangle\right| \, d\mathcal{H}^{md-1} < \infty$$

for all bounded Borel sets $A \subset \mathbb{R}^{md} \times S^{d-1}$ and all $i_1, \dots, i_m, r_{i_1}, \dots, r_{i_m}$ as in (10). From Theorem 2 below we obtain, in particular, a more explicit description of these total variation measures [see Remark 1(a)].

In the case $q = 2$, this condition has been considered in [10]; see also [9]. In Remark 1(b) below we explain why (9) is satisfied whenever $X_1, \dots, X_q$ are convex sets. Moreover, for sets $X_1, \dots, X_q \subset \mathbb{R}^d$ of positive reach, it is proved in Proposition 1 that (9) holds for $X_1, \rho_2 X_2, \dots, \rho_q X_q$ for almost all rotations $\rho_2, \dots, \rho_q \in SO(d)$.

We say that the sets $X_1, \dots, X_q$ of positive reach *osculate* if there exist $(x, u_i) \in \mathrm{nor}\, X_i$, $i = 1, \dots, q$, such that $o \in \mathrm{cone}\{u_1, \dots, u_q\}$ (equivalently, $o \in \mathrm{conv}\{u_1, \dots, u_q\}$).

As already mentioned in the Introduction, the mixed curvature measures appear in the translative integral formula for curvature measures of intersections.

Theorem 1 ([7, Theorem 1]) *Let $X_1, \dots, X_q$ be sets with positive reach in $\mathbb{R}^d$ (for $q \ge 2$) which satisfy (9) and are such that*

$$\mathcal{H}^{(q-1)d}(\{(z_2, \dots, z_q) : X_1, X_2 + z_2, \dots, X_q + z_q \text{ osculate}\}) = 0. \tag{11}$$

Then, for any $k \in \{0, 1, \dots, d-1\}$, the translative formula (3) holds.

For conditions sufficient for (11), see [7] and [8]. In particular, (11) is satisfied if all sets are convex, or if all sets are sufficiently smooth, or for arbitrary sets with positive reach in case $d = 2$. Moreover, if $X_1, \dots, X_q$ are arbitrary sets with positive reach in $\mathbb{R}^d$, then $X_1, \rho_2 X_2, \dots, \rho_q X_q$ satisfy (11) for $(\nu_d)^q$-almost all rotations $\rho_2, \dots, \rho_q \in SO(d)$ (see [9, Remark 3.2]). Here, ν_d denotes the normalized invariant measure on the group $SO(d)$ of proper rotations of $\mathbb{R}^d$.

In any case, it should be emphasized that condition (11) is not required for the definition of the mixed curvature measures and hence it is also not needed for the present study.

3 An integral representation of mixed curvature measures

In this section we derive a representation of mixed curvature measures as integrals over the product of unit normal bundles of the sets involved.

Let $q \geq 2$ and let $X_1, \dots, X_q$ be sets with positive reach. The principal curvatures and the principal directions of curvature of X_j at $(x_j, u_j) \in \operatorname{nor} X_j$ will be denoted by $k_i^{(j)}(x_j, u_j)$ and $a_i^{(j)}(x_j, u_j)$, respectively, for $i = 1, \dots, d-1$. In the following, we use the short notation

$$\mathbb{K}^{(j)}(x_j, u_j) := \prod_{i=1}^{d-1} \sqrt{1 + \left(k_i^{(j)}(x_j, u_j)\right)^2},$$

arguments of the curvature functions will often be omitted if these will be clear from the context. We shall also shortly write $\underline{x} := (x_1, \dots, x_q)$ and

$$\underline{(x, u)} := (x_1, u_1, \dots, x_q, u_q) \in \underline{\mathcal{N}(X)}.$$

For $\underline{(x, u)} \in R^c$, $s \in \mathbb{R}^q$ and $t \in S_+^{q-1}$, we set

$$\tilde{u}(s) := \sum_{i=1}^{q} s_i u_i \quad \text{and} \quad \underline{u}(t) := \frac{\sum_{i=1}^{q} t_i u_i}{|\sum_{i=1}^{q} t_i u_i|}.$$

Further, given $\underline{r} = (r_1, \dots, r_q)$ with $0 \leq r_1, \dots, r_q \leq d-1$, $r_1 + \cdots + r_q \geq (q-1)d$ and a Borel set $A \subset \mathbb{R}^{qd} \times S^{d-1}$, we set $k := r_1 + \cdots + r_q - (q-1)d$ and define

$$\mu_{\underline{r}}(\underline{(x, u)}; A) := \frac{1}{\omega_{d-k}} \int_{S_+^{q-1}} \mathbf{1}_A(\underline{x}, \underline{u}(t)) \prod_{i=1}^{q} t_i^{d-1-r_i} \, |\tilde{u}(t)|^{-(d-k)} \, \mathcal{H}^{q-1}(dt),$$

if $u_1, \dots, u_q \in S^{d-1}$ are linearly independent (and hence $o \notin \operatorname{cone}\{u_1, \dots, u_q\}$), and otherwise we define $\mu_{\underline{r}}(\underline{(x, u)}; A) := 0$. Note that the conditions $0 \leq r_i \leq d-1$, for $i = 1, \dots, q$, and $r_1 + \cdots + r_q \geq (q-1)d$ imply that $q \leq d$.

Now we can state our main result.

Theorem 2 *Let $q, d \geq 2$, let $X_1, \dots, X_q \subset \mathbb{R}^d$ be sets with positive reach satisfying (9), and let $r_1, \dots, r_q \in \{0, \dots, d-1\}$ with $r_1 + \cdots + r_q \geq (q-1)d$. Further, let $A \subset \mathbb{R}^{qd} \times S^{d-1}$ be Borel measurable and bounded. Then*

$$C_{r_1,\dots,r_q}(X_1,\dots,X_q;A)=\int_{\underline{\mathcal{N}(X)}}\mu_{\underline{r}}(\underline{(x,u)};A)\sum_{\substack{|I_j|=r_j\\ j=1,\dots,q}}\prod_{j=1}^{q}\frac{\prod_{i\in I_j^c}k_i^{(j)}}{\mathbb{K}^{(j)}}$$

$$\times\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}a_i^{(j)}\wedge u_1\wedge\dots\wedge u_q\right|^2\mathcal{H}^{q(d-1)}(d\underline{(x,u)}). \tag{12}$$

We postpone the proof of this theorem to the next section and first discuss assumption (9) and consider some special cases of Theorem 2.

Remark 1 (a) Theorem 2 and its proof show that condition (9) is equivalent to the requirement that for each bounded and Borel measurable set $A\subset\mathbb{R}^{qd}\times S^{d-1}$ we have

$$\int_{\underline{\mathcal{N}(X)}}\mu_{\underline{r}}(\underline{(x,u)};A)\left|\sum_{\substack{|I_j|=r_j\\ j=1,\dots,q}}\prod_{j=1}^{q}\frac{\prod_{i\in I_j^c}k_i^{(j)}}{\mathbb{K}^{(j)}}\right.$$

$$\left.\times\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}a_i^{(j)}\wedge u_1\wedge\dots\wedge u_q\right|^2\right|\mathcal{H}^{q(d-1)}(d\underline{(x,u)})<\infty.$$

In this case, the total variation measure $\|C\|_{r_1,\dots,r_q}(X_1,\dots,X_q;\cdot)$ of the mixed curvature measure $C_{r_1,\dots,r_q}(X_1,\dots,X_q;\cdot)$, evaluated at A, is given by this multiple integral.

(b) For convex bodies $K_1,\dots,K_q$, condition (9) is always satisfied for the following reason. All the mixed curvature measures are nonnegative in this case. Therefore they are always well defined, though possibly not finite. Nevertheless, the translative formula (3) must be true in this case and since its left-hand side is clearly locally bounded, the mixed curvature measures on the right-hand side will be locally bounded as well (cf. [10]).

(c) For convex bodies $K_1,K_2\in\mathcal{K}^d$ and $\alpha\in\{1,\dots,d-1\}$, the relationship

$$\binom{d}{\alpha}V(K_1[\alpha],K_2[d-\alpha])=C_{\alpha,d-\alpha}(K_1,-K_2;\mathbb{R}^{2d}\times S^{d-1})$$

is well-known. A curvature based representation of general mixed volumes is provided in [6] and will be developed further in future work.

(d) By definition and using the preceding notation, we have

$$\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}a_i^{(j)}\wedge u_1\wedge\dots\wedge u_q\right|^2=\left[\operatorname{lin}\{a_i^{(1)}:i\in I_1\},\dots,\operatorname{lin}\{a_i^{(q)}:i\in I_q\}\right]^2,$$

where the bracket (subspace determinant) on the right-hand side was already defined in (5); see also the references after (5) and [16] or [14, p. 598].

To prepare the proof of a condition, stated in Proposition 1, which ensures that (9) is satisfied, we first provide the bounds given in the next lemma.

Lemma 3 *Let the assumptions of Theorem 2 (except for (9)) be satisfied with $A=B\times S^{d-1}$, for a Borel set $B\subset\mathbb{R}^{dq}$, and let $k=r_1+\dots+r_q-(q-1)d$. Then, $\mu_{\underline{r}}(\underline{(x,u)};A)=0$ if $u_1,\dots,u_q$ are linearly dependent or $\underline{x}\notin B$, and*

$$c\,\mu_{\underline{r}}(\underline{(x,u)};A) \le \begin{cases} (1+|\ln(|u_1\wedge\cdots\wedge u_q|)|)|u_1\wedge\cdots\wedge u_q|^{-1} & \text{if } k=d-q,\\ |u_1\wedge\cdots\wedge u_q|^{-(d-k-q+1)} & \text{if } k<d-q, \end{cases} \tag{13}$$

with some constants $c=c_{d,q,k}$, otherwise. In particular,

$$\begin{aligned} &\|C\|_{r_1,\dots,r_q}(X_1,\dots,X_q;B\times S^{d-1})\\ &\le \text{const}\int_{\underline{\mathcal{N}(X)}} \mathbf{1}_B(\underline{x})|u_1\wedge\cdots\wedge u_q|^{-(d-q)}\,\mathcal{H}^{q(d-1)}(d\underline{(x,u)}). \end{aligned} \tag{14}$$

Proof Assume that $u_1,\dots,u_q$ are linearly independent and $\underline{x}\in B$. Then, from the definition of $\mu_{\underline{r}}(\underline{(x,u)};A)$, we easily get

$$\mu_{\underline{r}}(\underline{(x,u)};A) \le \frac{1}{\omega_{d-k}}\int_{S^{q-1}_+} |\tilde{u}(t)|^{-(d-k)}\,\mathcal{H}^{q-1}(dt).$$

For the given linearly independent vectors, put $\underline{u}=(u_1,\dots,u_q)$ and let $\Delta_{\underline{u}}$ denote the convex hull of $u_1,\dots,u_q$. Further, let $w\in S^{q-1}$ be a unit vector in the linear hull of $u_1,\dots,u_q$ and perpendicular to $\Delta_{\underline{u}}$. Note that the origin o is not contained in the affine hull of $\{u_1,\dots,u_q\}$, the smallest (hence $(q-1)$-dimensional) affine subspace containing this set.

Consider the differentiable, one-to-one mapping

$$h:S^{q-1}_+\to\Delta_{\underline{u}},\quad t\mapsto(t_1+\cdots+t_q)^{-1}\tilde{u}(t).$$

In order to compute the Jacobian $J_{q-1}h(t)$ of h at $t\in S^{q-1}_+$, let $\{v_1,\dots,v_{q-1},t\}$ be an orthonormal basis of $\mathbb{R}^q$ and note that

$$Dh_t(v_j)=(t_1+\cdots+t_q)^{-1}(v_j^1u_1+\cdots+v_j^qu_q)+\alpha_j(t)\tilde{u}(t),\quad j=1,\dots,q-1,$$

for some $\alpha_j(t)\in\mathbb{R}$ and

$$\left|\bigwedge_{j=1}^{q-1}Dh_t(v_j)\wedge\tilde{u}(t)\right| = \frac{|u_1\wedge\cdots\wedge u_q|}{(t_1+\cdots+t_q)^{q-1}}.$$

Hence, for $t\in S^{q-1}_+$ we have

$$J_{q-1}h(t)=\left|\bigwedge_{i=1}^{q-1}Dh_t(v_j)\right| = \frac{|u_1\wedge\cdots\wedge u_q|}{(t_1+\cdots+t_q)^{q-1}|\tilde{u}(t)\cdot w|} \ge \frac{|u_1\wedge\cdots\wedge u_q|}{q^{(q-1)/2}|\tilde{u}(t)|}.$$

Since clearly $|\tilde{u}(t)|\ge|h(t)|$, $t\in S^{q-1}_+$, the area formula implies that

$$\begin{aligned} &q^{-(q-1)/2}\omega_{d-k}|u_1\wedge\cdots\wedge u_q|\mu_{\underline{r}}(\underline{(x,u)};S^{q-1}_+)\\ &\le\int_{\Delta_{\underline{u}}}|z|^{-(d-k-1)}\,\mathcal{H}^{q-1}(dz)\\ &\le\int_{B_1^{(q-1)}}(\rho^2+|x|^2)^{-\frac{d-k-1}{2}}\,\mathcal{H}^{q-1}(dx), \end{aligned}$$

where $\rho := \operatorname{dist}(o, \Delta_{\underline{u}}) \leq 1$ and $B_1^{(q-1)}$ denotes the unit ball in $\mathbb{R}^{q-1}$ with centre at the origin. The last integral can be bounded from above by

$$\begin{aligned}
&\omega_{q-1} \int_0^1 r^{q-2}(\rho^2 + r^2)^{-\frac{d-k-1}{2}}\, dr \\
&= \omega_{q-1}\rho^{-(d-k-1)} \int_0^1 \frac{r^{q-2}}{(1+(\frac{r}{\rho})^2)^{\frac{d-k-1}{2}}}\, dr \\
&= \omega_{q-1}\rho^{-(d-k-q)} \int_0^{\rho^{-1}} \frac{s^{q-2}}{(1+s^2)^{\frac{d-k-1}{2}}}\, ds \\
&\leq \omega_{q-1}\rho^{-(d-k-q)} \left(1 + \int_1^{\rho^{-1}} s^{-(d-k-q+1)}\right) ds,
\end{aligned}$$

and the last integral can be easily evaluated.

The proof of (13) will be finished by the following estimate. The norm $|u_1 \wedge \cdots \wedge u_q|$ is equal to the q-volume of the parallelepiped spanned by the vectors $u_1, \ldots, u_q$, and since these are unit vectors, we get

$$|u_1 \wedge \cdots \wedge u_q| \leq \kappa_{q-1} \frac{2}{q} \rho,$$

where κ_n is the volume of the n-dimensional unit ball. Applying now Remark 1 (a) and the fact that $|\xi \wedge u_1 \wedge \cdots \wedge u_q| \leq |u_1 \wedge \cdots \wedge u_q|$ for any simple unit multivector ξ, we obtain (14). □

Using Lemma 3 we now show that mixed curvature measures are defined for generic rotations of sets with positive reach (cf. [9, Proposition 4.6]). We emphasize that the upper bound in (14) is not finite for arbitrary sets with positive reach, but we will show that it is indeed finite for generic rotations of sets with positive reach. Recall the remarks after Theorem 1 where it is pointed out that the assumptions of Proposition 1 also imply that condition (11) is satisfied, although this follows from a different argument.

Proposition 1 *Let $q, d \geq 2$ and let $X_1, \ldots, X_q \subset \mathbb{R}^d$ be sets with positive reach. Then (9) is satisfied by $X_1, \rho_2 X_2, \ldots, \rho_q X_q$ for $(\nu_d)^{q-1}$-almost all rotations $\rho_2, \ldots, \rho_q \in SO(d)$.*

Proof It suffices to show that $\|C\|_{r_1,\ldots,r_m}(X_1, \rho_2 X_2, \ldots, \rho_m X_m; B \times S^{d-1}) < \infty$ for all $2 \leq m \leq q$, $0 \leq r_1, \ldots, r_m \leq d-1$ with $r_1 + \cdots + r_m \geq (m-1)d$, any ball $B \subset \mathbb{R}^{md}$ and almost all rotations $\rho_2, \ldots, \rho_m \in SO(d)$. Due to (14), and since $(x, u) \in \operatorname{nor}(\rho X_i)$ if and only if $(\rho^{-1}x, \rho^{-1}u) \in \operatorname{nor} X_i$, it is sufficient to show that

$$\int_{\underline{\mathcal{N}(X)}} \mathbf{1}_B(\underline{x}) |u_1 \wedge \rho_2 u_2 \wedge \cdots \wedge \rho_m u_m|^{-(d-m)}\, \mathcal{H}^{m(d-1)}(d(\underline{(x, u)})) < \infty$$

and almost all $\rho_2, \ldots, \rho_m \in SO(d)$. Clearly, for any given $u \in S^{d-1}$, the image of the normalized Haar measure on $SO(d)$ by the mapping $\rho \mapsto \rho u$ is the uniform distribution on S^{d-1}. Thus, the $(m-1)$-fold integral of the last integral expression over $\rho_2, \ldots, \rho_m$ equals

$$\int_{\underline{\mathcal{N}(X)}} \mathbf{1}_B(\underline{x})\, H(u_1)\, \mathcal{H}^{m(d-1)}(d(\underline{(x, u)})),$$

where

$$H(u_1) = \int_{S^{d-1}} \cdots \int_{S^{d-1}} |u_1 \wedge u_2 \wedge \cdots \wedge u_m|^{-(d-m)}\, \mathcal{H}^{d-1}(du_m) \ldots \mathcal{H}^{d-1}(du_2).$$

Thus, it will be sufficient to show that $H(u_1)$ is bounded from above by a constant. First, observe that

$$|u_1 \wedge \cdots \wedge u_m| = |u_1 \wedge \cdots \wedge u_{m-1}||p_V u_q|,$$

where V is the orthogonal complement to the linear hull of $u_1, \ldots, u_{m-1}$ (note that $\dim V = d - m + 1$) and p_V denotes the orthogonal projection to V. Moreover, a direct calculation shows that if L is an l-dimensional linear subspace of $\mathbb{R}^d$, $l \in \{1, \ldots, d-1\}$, and $p + l > 0$, then

$$\begin{aligned}&\int_{S^{d-1}} |p_L u|^p \,\mathcal{H}^{d-1}(du)\\ &= \int_{S^{d-1}\cap L} \int_{S^{d-1}\cap L^\perp} \int_0^{\frac{\pi}{2}} (\cos t)^{l-1} (\sin t)^{d-l-1} (\cos t)^p \,dt\, \mathcal{H}^{d-l-1}(dx)\, \mathcal{H}^{l-1}(dy)\\ &= \omega_l \omega_{d-l} \frac{\Gamma\left(\frac{d-l}{2}\right) \Gamma\left(\frac{p+l}{2}\right)}{2\,\Gamma\left(\frac{d+p}{2}\right)}. \end{aligned} \tag{15}$$

Applying (15) with $L = V$, $l = d - m + 1$ and $p = -d + m$ in a first step, it remains to be shown that

$$u_1 \mapsto \int_{S^{d-1}} \cdots \int_{S^{d-1}} |u_1 \wedge \cdots \wedge u_{m-1}|^{-(d-m)}\, \mathcal{H}^{d-1}(du_{m-1}) \ldots \mathcal{H}^{d-1}(du_2)$$

is bounded by a constant from above. Repetition of the preceding argument yields the assertion. □

Theorem 2 can be specified in various ways. First, let $X_1, \ldots, X_q$ be sets with positive reach, $A \subset \mathbb{R}^{qd} \times S^{d-1}$ a Borel set, and $r_1 = \cdots = r_q = d - 1$ with $q, d \geq 2$. Then $k = d - q$ and

$$\mu_{\underline{d-1}}((\underline{x}, \underline{u}); A) = \frac{1}{\omega_q} \int_{S_+^{q-1}} \mathbf{1}_A(\underline{x}, \underline{u}(t))\, |\tilde{u}(t)|^{-q}\, \mathcal{H}^{q-1}(dt),$$

if $u_1, \ldots, u_q \in S^{d-1}$ are linearly independent, and zero otherwise. Furthermore,

$$\begin{aligned}C_{d-1,\ldots,d-1}(X_1, \ldots, X_q; A) = 2^q \int \ldots \int \mu_{\underline{d-1}}((\underline{x}, \underline{u}); A)\, |u_1 \wedge \cdots \wedge u_q|^2\\ \times\, C_{d-1}(X_q; d(x_q, u_q)) \ldots C_{d-1}(X_1; d(x_1, u_1)).\end{aligned}$$

The special case where the sets $X_1, \ldots, X_q$ are convex polytopes, but $r_1, \ldots, r_q \in \{0, \ldots, d-1\}$ are arbitrary, is of particular interest, since it shows that the representation of mixed curvature measures given in Theorem 2 extends the defining relationship (3.1) in [16] in a natural way.

For a polytope $P \subset \mathbb{R}^d$ and $j \in \{0, \ldots, d-1\}$, we write $\mathcal{F}_j(P)$ for the set of all j-dimensional faces of P (see [12, p. 16]), and $N(P, F)$ for the normal cone of P at a face F of P (see [12, p. 83]). For faces $F_i \in \mathcal{F}_{r_i}(P_i)$, $i = 1, \ldots, q$, the bracket $[F_1, \ldots, F_q]$ is defined as in [16] or [14, p. 598].

Corollary 1 *Let $q, d \geq 2$, and let $P_1, \ldots, P_q$ be convex polytopes (or polyhedral sets). Let $r_1, \ldots, r_q \in \{0, \ldots, d-1\}$ with $r_1 + \cdots + r_q \geq (q-1)d$ and $k := r_1 + \cdots + r_q - (q-1)d$. Further, let $B \subset \mathbb{R}^{qd}$ and $C \subset S^{d-1}$ be Borel measurable sets. Then*

$$
\begin{aligned}
&C_{r_1,\ldots,r_q}(P_1,\ldots,P_q;B\times C)\\
&\quad=\sum_{F_1\in\mathcal{F}_{r_1}(P_1)}\cdots\sum_{F_q\in\mathcal{F}_{r_q}(P_q)}\frac{\mathcal{H}^{d-1-k}\left(\left(\sum_{i=1}^q N(P_i,F_i)\right)\cap C\right)}{\omega_{d-k}}\\
&\qquad\times[F_1,\ldots,F_q]\left(\otimes_{i=1}^q\left(\mathcal{H}^{r_i}\llcorner F_i\right)\right)(B).
\end{aligned}
$$

In particular, Corollary 1 is an extension of the defining relation in [16, (3.1)], since

$$
\gamma(F_1,\ldots,F_q;P_1,\ldots,P_q)=\frac{\mathcal{H}^{d-1-k}\left(\left(\sum_{i=1}^q N(P_i,F_i)\right)\cap S^{d-1}\right)}{\omega_{d-k}},
$$

provided that $\operatorname{lin} N(P_1,F_1),\ldots,\operatorname{lin} N(P_q,F_q)$ are linearly independent subspaces. Also note that if these subspaces are not linearly independent, then

$$
\dim\left(\left(\sum_{i=1}^q N(P_i,F_i)\right)\cap S^{d-1}\right)<d-r_1+\cdots+d-r_q-1=d-k-1,
$$

and hence $\gamma(F_1,\ldots,F_q;P_1,\ldots,P_q)=0$ in this case.

Proof We continue to use the previous notation. Under the present special assumptions, the formula of Theorem 2 yields

$$
\begin{aligned}
&C_{r_1,\ldots,r_q}(P_1,\ldots,P_q;B\times C)\\
&\quad=\sum_{F_1\in\mathcal{F}_{r_1}(P_1)}\cdots\sum_{F_q\in\mathcal{F}_{r_q}(P_q)}[F_1,\ldots,F_q]\left(\otimes_{i=1}^q\left(\mathcal{H}^{r_i}\llcorner F_i\right)\right)(B)\\
&\qquad\times\frac{1}{\omega_{d-k}}\int_{S_+^{q-1}}\int_{N(P_1,F_1)\cap S^{d-1}}\cdots\int_{N(P_q,F_q)\cap S^{d-1}}\left(\prod_{j=1}^q t_j^{d-1-r_j}\right)\\
&\qquad\times\left|\bigwedge_{j=1}^q\bigwedge_{i\in I_j^c}a_i^{(j)}\wedge u_1\wedge\ldots\wedge u_q\right|\mathbf{1}_C(\underline{u}(t))|\tilde{u}(t)|^{-(d-k)}\\
&\qquad\times\mathcal{H}^{d-1-r_q}(du_q)\ldots\mathcal{H}^{d-1-r_1}(du_1)\,\mathcal{H}^{q-1}(dt),
\end{aligned}
$$

where $\{a_i^{(j)}:i\in I_j^c\}$ is an orthonormal basis of $\operatorname{Tan}(N(P_j,F_j)\cap S^{d-1},u_j)$ and $\{a_i^{(j)}:i\in I_j\}$ spans $\operatorname{lin}(F_j-F_j)$, $j=1,\ldots,q$. Here we adopt the convention that the integrand is zero if $u_1,\ldots,u_q$ are linearly dependent.

Let $F_j\in\mathcal{F}_{r_j}(P_j)$, for $j=1,\ldots,q$, be fixed and assume that the linear subspaces $\operatorname{lin} N(P_1,F_1),\ldots,\operatorname{lin} N(P_q,F_q)$ are linearly independent. Consider the bijective map

$$
\begin{aligned}
T:&(N(P_1,F_1)\cap S^{d-1})\times\cdots\times(N(P_q,F_q)\cap S^{d-1})\times S_+^{q-1}\\
&\to\left(\sum_{i=1}^q N(P_i,F_i)\right)\cap S^{d-1},\\
&(u_1,\ldots,u_q,t)\mapsto\underline{u}(t).
\end{aligned}
$$

Then the required equality for any such summand follows by an application of the area formula once we have checked that

$$J_{d-k-1}T(\underline{u},t)=|\tilde{u}(t)|^{-(d-k)}\left(\prod_{j=1}^{q}t_j^{d-1-r_j}\right)\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}a_i^{(j)}\wedge u_1\wedge\cdots\wedge u_q\right|.$$

In fact, using the previous notation, we find that

$$\frac{\partial T}{\partial a_i^{(j)}}(\underline{u},t)=\frac{t_j a_i^{(j)}}{|\tilde{u}(t)|}+\lambda_i^{(j)}\underline{u}(t),$$

where $i\in I_j^c$, $j\in\{1,\dots,q\}$, and $\lambda_i^{(j)}\in\mathbb{R}$,

$$\frac{\partial T}{\partial f_l}(\underline{u},t)=\frac{\tilde{u}(f_l)}{|\tilde{u}(t)|}+\lambda_l\underline{u}(t),$$

where $l\in\{1,\dots,q-1\}$ and $\lambda_l\in\mathbb{R}$, and

$$\left\langle\frac{\partial T}{\partial a_i^{(j)}}(\underline{u},t),\underline{u}(t)\right\rangle=\left\langle\frac{\partial T}{\partial f_l}(\underline{u},t),\underline{u}(t)\right\rangle=0.$$

Here $f_1,\dots,f_{q-1},t$ is an orthonormal basis of $\mathbb{R}^q$. Thus

$$\begin{aligned}J_{d-k-1}T(\underline{u},t)&=\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}\left(\frac{t_j a_i^{(j)}}{|\tilde{u}(t)|}+\lambda_i^{(j)}\underline{u}(t)\right)\wedge\bigwedge_{i=1}^{q-1}\left(\frac{\tilde{u}(f_i)}{|\tilde{u}(t)|}+\lambda_i\underline{u}(t)\right)\right|\\&=\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}\left(\frac{t_j a_i^{(j)}}{|\tilde{u}(t)|}\right)\wedge\bigwedge_{i=1}^{q-1}\left(\frac{\tilde{u}(f_i)}{|\tilde{u}(t)|}\right)\wedge\underline{u}(t)\right|\\&=|\tilde{u}(t)|^{-(d-k)}\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_j^c}(t_j a_i^{(j)})\wedge\bigwedge_{i=1}^{q-1}\tilde{u}(f_i)\wedge\tilde{u}(t)\right|,\end{aligned}$$

from which the formula for the Jacobian immediately follows.

If $\operatorname{lin} N(P_1,F_1),\dots,\operatorname{lin} N(P_q,F_q)$ are not linearly independent, then $[F_1,\dots,F_q]=0$, and thus the requested equality is also true in this case. □

Further representation formulas, which are needed for the analysis of Boolean models in stochastic geometry can be derived from Theorem 2 and Corollary 1. Various examples of such results and their applications are provided in [3], [4, Section 3] and [5].

4 Proof of Theorem 2

For given $t\in S_+^{q-1}$ we denote by $f_1,\dots,f_{q-1},t$ an orthonormal basis of $\mathbb{R}^q$ whose orientation is chosen in such a way that

$$\det(f_1,\dots,f_{q-1},t)=(-1)^{(d-1)\binom{q}{2}}.$$

Denote

$$\tilde{a}(\underline{(x,u)},t) = (\wedge_{d-1}\Pi_1)\, a_{X_1}(x_1,u_1) \wedge \cdots \wedge \left(\wedge_{d-1}\Pi_q\right) a_{X_q}(x_q,u_q) \\ \wedge \left(\wedge_{q-1}\Pi_{q+1}\right)(f_1 \wedge \cdots \wedge f_{q-1}),$$

where Π_i, $i = 1,\ldots,q+1$, are the canonical embeddings into $(\mathbb{R}^{2d})^q \times \mathbb{R}^q$ such that

$$(a_1,\ldots,a_q,a_{q+1}) = \sum_{i=1}^{q+1} \Pi_i a_i;$$

note that $\tilde{a}(\underline{(x,u)},t)$ is a $(qd-1)$ vector field tangent to $\underline{\mathcal{N}(X)} \times S^{q-1}$. Using the area formula for currents [2, §4.1.30] and Lemma 2, we obtain

$$\begin{aligned} &C_{r_1,\ldots,r_q}(X_1,\ldots,X_q;A) \\ &= \int_{\underline{\mathcal{N}(X)}} \int_{S_+^{q-1}} \left\langle \wedge_{qd-1}\mathrm{ap}\; DT(\underline{(x,u)},t)\tilde{a}(\underline{(x,u)},t), \varphi_{r_1,\ldots,r_q}(\underline{u}(t)) \right\rangle \\ &\quad \times \mathbf{1}_A(\underline{x},\underline{u}(t))\, \mathcal{H}^{q-1}(dt)\, \mathcal{H}^{q(d-1)}(d\underline{(x,u)}), \end{aligned}$$

provided that the orientation is chosen properly, i.e., such that

$$\wedge_{qd-1}\mathrm{ap}DT(\underline{(x,u)},t)\tilde{a}(\underline{(x,u)},t)$$

is a positive multiple of $a_{X_1,\ldots,X_q}$; this will be verified later.

A direct calculation shows that, for $\mathcal{H}^{qd-1}$-almost all $(\underline{(x,u)},t) \in \underline{\mathcal{N}(X)} \times S_+^{q-1}$,

$$\begin{aligned} \wedge_{qd-1}\mathrm{ap}\; DT\tilde{a}(\underline{(x,u)},t) = &\frac{1}{\mathbb{K}_1} \bigwedge_{i=1}^{d-1} \left(a_i^{(1)}, o, \ldots, o, \frac{t_1 k_i^{(1)}}{|\tilde{u}(t)|} a_i^{(1)} + \lambda_i^{(1)} \tilde{u}(t) \right) \wedge \\ &\vdots \\ &\wedge \frac{1}{\mathbb{K}_q} \bigwedge_{i=1}^{d-1} \left(o, \ldots, o, a_i^{(q)}, \frac{t_q k_i^{(q)}}{|\tilde{u}(t)|} a_i^{(q)} + \lambda_i^{(q)} \tilde{u}(t) \right) \\ &\wedge \bigwedge_{j=1}^{q-1} \left(o, \ldots, o, \frac{\tilde{u}(f_j)}{|\tilde{u}(t)|} + \lambda_j \tilde{u}(t) \right), \end{aligned}$$

where $\lambda_i^{(j)}$, $i \in \{1,\ldots,d-1\}$ and $j \in \{1,\ldots,q\}$, and λ_j, $j \in \{1,\ldots,q-1\}$, are suitably chosen. We write $\mathrm{Sh}^*(r_1,\ldots,r_{q+1})$ for the set of all $\sigma \in \mathrm{Sh}(r_1,\ldots,r_{q+1})$ which satisfy

$$\begin{aligned} \sigma(\{1,\ldots,R_1\}) &\subset \{1,\ldots,d-1\} \\ &\vdots \\ \sigma(\{R_{q-1}+1,\ldots,R_q\}) &\subset \{(q-1)(d-1)+1,\ldots,q(d-1)\}, \end{aligned}$$

and then we define

$$I_\sigma(i) = \sigma(\{R_{i-1}+1,\ldots,R_i\}) - (i-1)(d-1)$$

and

$$I_\sigma(i)^c = \{1,\ldots,d-1\} \setminus I_\sigma(i), \quad i = 1,\ldots,q,$$

for $\sigma \in \mathrm{Sh}^*(r_1, \ldots, r_{q+1})$. By $I_\sigma(j) I_\sigma(j)^c$ we shall denote the permutation of $\{1, \ldots, d-1\}$ mapping the first $r = |I_\sigma(j)|$ elements increasingly on $I_\sigma(j)$ and the remaining $d-1-r$ elements increasingly on $I_\sigma(j)^c$. Thus we arrive at

$$\begin{aligned} &\left\langle \wedge_{qd-1} \mathrm{ap}\, DT\tilde{a}(\underline{(x,u)}, t), \varphi_{r_1,\ldots,r_q}(\underline{u}(t)) \right\rangle = \frac{1}{\omega_{d-k}} (-1)^{c_1(d,r_1,\ldots,r_q)} \\ &\quad \times \sum_{\sigma \in \mathrm{Sh}^*(r_1,\ldots,r_{q+1})} \mathrm{sgn}(\sigma) \left(\prod_{j=1}^q t_j^{d-1-r_j} \right) \prod_{j=1}^q \frac{\prod_{i \in I_\sigma(j)^c} k_i^{(j)}}{\mathbb{K}_j} |\tilde{u}(t)|^{-(d-k)} \\ &\quad \times \left[\bigwedge_{i \in I_\sigma(1)} a_i^{(1)}, \ldots, \bigwedge_{i \in I_\sigma(q)} a_i^{(q)}, \bigwedge_{i \in I_\sigma(1)^c} a_i^{(1)} \wedge \cdots \wedge \bigwedge_{i \in I_\sigma(q)^c} a_i^{(q)} \wedge \bigwedge_{i=1}^{q-1} \tilde{u}(f_i) \wedge \tilde{u}(t) \right]. \end{aligned} \tag{16}$$

Observe that

$$\bigwedge_{i=1}^{q-1} \tilde{u}(f_i) \wedge \tilde{u}(t) = \det(f_1, \ldots, f_{q-1}, t)\, u_1 \wedge \cdots \wedge u_q, \tag{17}$$

$$\mathrm{sgn}(\sigma) = \left(\prod_{j=1}^q \mathrm{sgn}(I_\sigma(j) I_\sigma(j)^c) \right) (-1)^{c_2(d,r_1,\ldots,r_q)} \tag{18}$$

with

$$\begin{aligned} &c_2(d, r_1, \ldots, r_q) \\ &\quad = \sum_{j=1}^q (d-1-r_j)(R_q - R_j) \\ &\quad \sim q(d-1)R_q + (d-1)\sum_{i=1}^q R_i + R_q \sum_{i=1}^q r_i + \sum_{i=1}^q r_i R_i \\ &\quad \sim q(d-1)R_q + (d-1)\left((q+1)\sum_{i=1}^q r_i - \sum_{i=1}^q i r_i \right) + R_q + \sum_{i=1}^q r_i + \sum_{1 \le i < j \le q} r_i r_j \\ &\quad \sim (d-1)\sum_{i=1}^q r_i + (d-1)\sum_{i=1}^q i r_i + \sum_{1 \le i < j \le q} r_i r_j, \\ &\qquad * \left(\bigwedge_{i \in I_\sigma(j)} a_i^{(j)} \right) = \mathrm{sgn}(I_\sigma(j) I_\sigma(j)^c) \bigwedge_{i \in I_\sigma(j)^c} a_i^{(j)} \wedge u_j \end{aligned}$$

and

$$\bigwedge_{j=1}^q \left(\bigwedge_{i \in I_\sigma(j)^c} a_i^{(j)} \wedge u_j \right) = (-1)^{c_3} \bigwedge_{j=1}^q \bigwedge_{i \in I_\sigma(q)^c} a_i^{(j)} \wedge \bigwedge_{i=1}^q u_i$$

with

$$c_3 = \sum_{j=2}^q (j-1)(d-1-r_j) \sim (d-1)\binom{q}{2} + \sum_i i r_i + \sum_i r_i.$$

Using (5), we thus have

$$
\begin{aligned}
&\operatorname{sgn}\sigma\left[\bigwedge_{i\in I_\sigma(1)} a_i^{(1)},\dots,\bigwedge_{i\in I_\sigma(q)} a_i^{(q)},\bigwedge_{i\in I_\sigma(1)^c} a_i^{(1)}\wedge\cdots\wedge\bigwedge_{i\in I_\sigma(q)^c} a_i^{(q)}\wedge\bigwedge_{i=1}^{q-1}\tilde u(f_i)\wedge\tilde u(t)\right]\\
&\quad=(-1)^{c_2}\left\langle\bigwedge_{j=1}^{q}\left(\bigwedge_{i\in I_\sigma(j)^c} a_i^{(j)}\wedge u_j\right)\wedge *\left(\bigwedge_{j=1}^{q}\bigwedge_{i\in I_\sigma(j)^c} a_i^{(j)}\wedge\bigwedge_{i=1}^{q-1}\tilde u(f_i)\wedge\tilde u(t)\right),\Omega^d\right\rangle\\
&\quad=(-1)^{c^*}\left\langle\bigwedge_{j=1}^{q}\bigwedge_{i\in I_\sigma(j)^c} a_i^{(j)}\wedge\bigwedge_{j=1}^{q}u_j\wedge *\left(\bigwedge_{j=1}^{q}\bigwedge_{i\in I_\sigma(q)^c} a_i^{(j)}\wedge\bigwedge_{i=1}^{q}u_j\right),\Omega^d\right\rangle\\
&\quad=(-1)^{c^*}\left|\bigwedge_{j=1}^{q}\bigwedge_{i\in I_\sigma(j)^c} a_i^{(j)}\wedge\bigwedge_{j=1}^{q}u_j\right|,
\end{aligned}
\tag{19}
$$

where $c^* := c_2 + c_3 + (d-1)\binom{q}{2}$ and we have used (4) in the last step. Combining (16) and (19) and observing that

$$
c_1 + c_2 + c_3 + (d-1)\binom{q}{2} \sim 0,
$$

we finally obtain the required formula.

It remains to verify that the orientation of the joint unit normal bundle has been chosen appropriately, i.e., that

$$
\left\langle \wedge_{qd-1}\mathrm{ap}DT(\underline{(x,u)},t)\tilde a(\underline{(x,u)},t), \sum \varepsilon^{qd-1-r_1-\cdots-r_q}\varphi_{r_1,\dots,r_q}(\underline{u}(t))\right\rangle > 0, \tag{20}
$$

where the sum extends over all $r_1,\dots,r_q\in\{0,\dots,d\}$ such that $(q-1)d \le r_1+\cdots+r_q \le qd-1$ for sufficiently small $\varepsilon>0$. Consider first the case when $X_1,\dots,X_q$ have $C^{1,1}$ smooth boundaries. Since all curvatures are finite in this case, we get from (16) and (19) that

$$
\langle \wedge_{qd-1}\mathrm{ap}DT(\underline{(x,u)},t)\tilde a(\underline{(x,u)},t), \varphi_{d-1,\dots,d-1}(\underline{u}(t))\rangle > 0.
$$

But since $\varphi_{r_1,\dots,r_q}$ vanishes over $\operatorname{nor}(X_1,\dots,X_q)$ if $r_j=d$ for some $j\in\{1,\dots,q\}$, this is the leading term in the polynomial expression of (20) which therefore will be positive for small $\varepsilon>0$. General sets $X_1,\dots,X_q$ of positive reach can be approximated by parallel bodies with $C^{1,1}$ smooth boundaries so that the corresponding unit normal cycles are arbitrarily close in the flat norm (see [9]). Thus, the expression in (20) can be approximated by the corresponding one for the parallel bodies and since it can never be zero for sufficiently small $\varepsilon>0$, it will remain positive.

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