Geom Dedicata (2017) 190:103–133
DOI 10.1007/s10711-017-0230-4

ORIGINAL PAPER

Asymptotic behavior of Cauchy hypersurfaces in constant curvature space–times

Mehdi Belraouti[1]

Received: 17 June 2016 / Accepted: 21 February 2017 / Published online: 28 February 2017

Abstract We study the asymptotic behavior of convex Cauchy hypersurfaces on maximal globally hyperbolic spatially compact space–times of constant curvature. We generalise the result of Belraouti (Annales de l'institut Fourier 64(2):457–466, 2015) to the (2+1) de Sitter and anti de Sitter cases. We prove that in these cases the level sets of quasi-concave times converge in the Gromov equivariant topology, when time goes to 0, to a real tree. Moreover, this limit does not depend on the choice of the time function. We also consider the problem of asymptotic behavior in the flat $(n+1)$ dimensional case. We prove that the level sets of quasi-concave times converge in the Gromov equivariant topology, when time goes to 0, to a $CAT(0)$ metric space. Moreover, this limit does not depend on the choice of the time function.

Keywords Lorentzian geometry · Constant curvature space–time · Quasi-concave time function · Equivariant Gromov topology

Mathematics Subject Classification 53C50

1 Introduction

Space–times of constant curvature occupy an important place in Lorentzian geometry. Despite their trivial local geometry, these spaces have a very rich global geometry and constitute an important family of space–times in which we hope to understand many fundamental questions. The existence of time functions with levels of prescribed geometry constitutes one of these questions both from the geometrical and the physical point of view. We refer to these functions as geometric time functions. This question was amply studied in the literature in the works of Andersson, Barbot, Béguin, Benedetti, Bonsante, Fillastre, Galloway, Guadignini,

✉ Mehdi Belraouti
mehdi.belraouti@gmail.com

1 Pontificia Universidade Catolica do Rio de Janeiro, Rio de Janeiro, Brazil

Howard, Moncrief, Seppi, Zeghib (we cite for example [1,3–7,13–16]). The main object of this article is to study the asymptotic behavior of geometric time functions levels.

Recall that a Lorentzian manifold is a differentiable manifold endowed with a pseudo-Riemannian metric of signature $(-, +, \cdots, +)$. A space–time is an oriented and chronologically oriented Lorentzian manifold. A space–time is said to be globally hyperbolic (GH) if it possesses a function, called Cauchy time function, which is strictly increasing along causal curves (curves for which the norm of the tangent vectors are non positive) and surjective on inextensible causal curves. The levels of such function are called Cauchy hypersurfaces. If in addition the Cauchy time function is proper then we say that the space–time is globally hyperbolic spatially compact and we write GHC. By a classical result of Geroch [19], every GH space–time is diffeomorphic to the product of a Cauchy hypersurface S by an interval I of $\mathbb{R}$. A globally hyperbolic spatially compact space–time, solution of the Einstein equation, is said to be maximal if it doesn't extend to a constant curvature GHC space–time which is also solution of the Einstein equation. A maximal globally hyperbolic spatially compact space–time is denoted by $MGHC$. A space–time is said to be of constant curvature if it is endowed with a (G, X) structure where X is a constant model space and G his isometry group. Recall that the models of constant curvature space–times are:

(1) The Minkowski space $\mathbb{R}^{1,n}$. That is the vector space $\mathbb{R}^{n+1}$ endowed with the standard Lorentzian metric $q_{1,n} = -dx_0^2 + dx_1^2 + \cdots + dx_n^2$. It's a globally hyperbolic spactime whose group of isometry is the Poincaré group $O(1, n) \ltimes \mathbb{R}^{1,n}$;
(2) The de Sitter space $\mathbb{DS}_n$. That is the one sheeted hyperboloid $q_{1,n} = +1$ endowed with the Lorentzian metric induced by $q_{1,n}$. It is the positive curvature model space. It's a globally hyperbolic space–time whose group of isometry is $O(1, n)$.
(3) The anti de Sitter space $\mathbb{ADS}_n$. That is the quadric $q_{2,n-1} = -1$ endowed with the Lorentzian metric induced by $q_{2,n-1} = -1$, where $q_{2,n-1} = -dx_0^2 - dx_1^2 + \cdots + dx_n^2$ in the appropriate coordinates. It is the negative curvature model space. Unlike the Minkowski and the de Sitter space–times, the anti de Sitter space–time is not globally hyperbolic. His group of isometry is $O(2, n)$.

In [24], Mess gives a full classification of $MGHC$ space–times in the $2 + 1$ flat and anti de Sitter cases giving rise in the same time to a particular interest for $MGHC$ space–times of constant curvature. Following Mess work's Scannell, Barbot, Béguin, Bonsante and Zeghib ([3,6,14,32]) completed this classification in all constant curvature and all dimension cases. In the $2 + 1$ special case Mess [24], Benedetti and Bonsante [12] proved that there is a one to one correspondence between measured geodesic laminations on a given closed hyperbolic surface S and $MGHC$ constant curvature space–times admitting a Cauchy surface diffeomorphic to S.

Up to inversion of time orientation, the $MGHC$ space–times of constant curvature have the particularity to be geodesically complete in the future, but on the other hand often incomplete in the past; we say that they admit an initial singularity. These space–times have also the particularity to possess remarkable geometric time functions:

(1) The cosmological time, which is defined at a point p as the supremum of length of past causal curves starting at p. It gives a simple and important first example of quasi-concave time functions i.e those which the levels are convex, to which all other time functions can be compared (see [14,32]).
(2) The CMC time function i.e a time function where the levels have constant mean curvature. The existence and uniqueness of such a function in a given space–time was studied by Andersson, Barbot, Béguin and Zeghib in the flat, de Sitter and anti de Sitter cases

[3,8,9]. These functions define a regular foliation and play an important role in physics. In the flat case they have the particularity to be quasi-concave.

(3) The k-time (dimension $2+1$) i.e a time function where the levels have constant Gauss curvature. The existence and uniqueness of such a function in a given space–time was done by Barbot, Béguin and Zeghib [7]. They are by definition quasi-concaves.

Giving a mathematical sense to the notion of initial singularity constitutes an important problem in general relativity (see [20–23,27,31]). There are in the literature different ways to attach a boundary to a space–time; we cite for example the Penrose boundary [18], the b-boundary [33]. However, these constructions are not unique in general and all have disadvantages. We hope, through the study of asymptotic behavior of Cauchy hypersurfaces, to give a more intrinsic meaning to this notion of initial singularity.

Let M be a $MGHC$ space–time of constant curvature. A C^1 Cauchy time function $T : M \to \mathbb{R}$ defines naturally a 1-parameter family $(T^{-1}(a), g_a)_{a\in\mathbb{R}}$ of Riemannian manifolds or equivalently a 1-parameter family $(T^{-1}(a), d_a)_{a\in\mathbb{R}}$ of metric spaces. One can ask the natural important question of asymptotic behavior of this family with respect to the time in the following two cases: when time goes to 0 and when it goes towards infinity. In our case we consider the equivalent equivariant problem: the asymptotic behavior of the $\pi_1(M)$-equivariant family $(\pi_1(M), \tilde{T}^{-1}(a), \tilde{d}_a)_{a\in\mathbb{R}}$. Several notions of topology appear when we deal with the convergence of equivariant metric spaces. In this article our favorite convergences will be the compact open convergence and the Gromov equivariant convergence [29,30].

The study of such problem was first initiated by Benedetti–Guadagnini [13]. They noticed that the cosmological levels of $MGHC$ flat space–times of dimension $2+1$ converge, when time goes to 0, to the real tree dual to the measured geodesic lamination associated to M. This problem was finally treated by Bonsante, Benedetti in [12,14]. In the case of the CMC time Benedetti–Guadagnini [13] conjectured that in a flat globally hyperbolic spatially compact non elementary maximal space–time M of dimension $2+1$, the level sets of the CMC time converge when time goes to 0 to the real tree dual to the measured geodesic lamination associated to M and when time goes to the infinity to the hyperbolic structure associated to M. In Andersson [2] gives a positive answer to the Benedetti–Guadagnini conjecture in the case of simplicial flat space–time. A complete positive answer to this conjecture is given in [11]. Our goal here is to extend the result of [11] to the $2+1$ de Sitter and anti de Sitter cases as well as to the flat $n+1$ dimensional case.

In the $2+1$ case, one can formulate the asymptotic problem in the Teichmüller space. Let S be a closed hyperbolic surface and M be a constant curvature $MGHC$ space–time admitting a Cauchy surface diffeomorphic to S. A Cauchy time function $T : M \to]0, +\infty[$ defines naturally a curve $(S, g_a^T)_a$ in the space $\mathrm{Met}(S)$ of Riemannian metrics of S. This allows us to study the behavior of the projection curve $(S, [g_a^T])_a$ of $(S, g_a^T)_a$ in the Teichmüller space $\mathrm{Teich}(S)$ which is, as a topological space, much more pleasant to study than $\mathrm{Met}(S)$. In the flat case and thanks to the work of Benedetti and Bonsante [12], one can identify the curve $(S, [g_a^{T_{cos}}])_a$ in $\mathrm{Teich}(S)$ associated to the cosmological time T_{cos}. It corresponds to the grafting curve $(\mathrm{gra}_{\frac{\lambda}{a}}(S))_a$ defined by the measured geodesic lamination (λ, μ) associated to M. The curve $(\mathrm{gra}_{\frac{\lambda}{a}}(S))_a$ is real analytic and converges when time goes to $+\infty$, to the hyperbolic structure $\mathbb{H}^2/\pi_1(M)$.

In the case of the CMC time T_{cmc}, Moncrief [25] proved that the curve $(S, [g_a^T])_a$ is the projection in $\mathrm{Teich}(S)$ of a trajectory of an non-autonomous Hamiltonian flow on $T^*\,\mathrm{Teich}(S)$: we call this flow the Moncrief flow, and the curves the Moncrief lines. It is natural to ask whether the curve defined by the CMC time converges when time goes to 0 to the point, in the Thurston boundary of the Teichmüller space $\mathrm{Teich}(S)$, corresponding to the measured geodesic lamination, and when time goes to $+\infty$, to the hyperbolic structure $\mathbb{H}^2/\pi_1(M)$. One also can ask this question for the curve defined by the k-time. In this paper, we will be concerned with the behavior of such curves when time goes to infinity.

2 Statement of results

Let M be a future complete $MGHC$ space–time of constant curvature. Let $T : \widetilde{M} \to \mathbb{R}$ be a $\pi_1(M)$-invariant quasi-concave C^1 Cauchy time. Up to reparametrization we can suppose that T takes its values in $\mathbb{R}_+^*$. Consider the family of $\pi_1(M)$-inveriant metric spaces $(\pi_1(M), S_a^T, d_a^T)_{a\in\mathbb{R}_+^*}$ associated to T. Let $\gamma \in \pi_1(M)$ and $a > 0$, denote by $l_a^T(\gamma) := \inf_{x\in S_a^T} d_a^T(x, \gamma.x)$ the marked spectrum of d_a^T.

Benedetti and Guadignini [13] conjectured that:

Conjecture 1 *Let M be a future complete $MGHC$ non elementary flat space–time of dimension $2+1$ and let T_{cmc} be the associated CMC time. Then:*

- *(1) $\lim_{a\to 0} l_a^T(\gamma) = l_\Sigma(\gamma)$, where Σ is the real tree dual to the measured geodesic lamination associated to M.*
- *(2) $\lim_{a\to +\infty} a^{-1} l_a^T(\gamma) = l_{\mathbb{H}^2}(\gamma)$.*

Andersson [2] gives a positive answer to the first part of this conjecture in the case of simplicial space–times. In [11] we studied the past asymptotic behavior of quasi-concave Cauchy times in a $2+1$ flat space–times. We gave in particular a positive answer to the first part of the Benedetti–Guadignini conjecture.

Theorem 2.1 ([11, Theorem 1.1]) *Let M be a future complete $MGHC$ non elementary flat space–time of dimension $2+1$. Let T be a C^2 quasi-concave Cauchy time function on $\widetilde{M}$. Then the levels $(\pi_1(M), S_a^T, d_a^T)_{a\in\mathbb{R}_+^*}$ converge in the Gromov equivariant topology, when a goes to 0, to the real tree dual to the measured geodesic lamination associated to M. In particular this limit does not depend on the time function T.*

Our two first results concern the asymptotic behavior in the flat $n+1$ dimensional case. In dimension bigger than 3, the situation is more complicated. The initial singularity is no longer a real tree in general (see [14]). However, we have the following partial result which is a generalization of Theorem 2.1 to the $n+1$-dimensional flat case:

Theorem 2.2 *Let M be a future complete $MGHC$ flat non elementary space–time of dimension $n+1$. Let T be a C^2 quasi-concave Cauchy time on $\widetilde{M}$. Then the levels $(\pi_1(M), S_a^T, d_a^T)_{a\in\mathbb{R}_+^*}$ converge in the Gromov equivariant topology, when a goes to 0, to a $CAT(0)$ metric space. Moroever, the limit does not depend on the time function T.*

Near the infinity we obtain the following result:

Theorem 2.3 *Let M be a future complete standard flat space–time of dimension $n+1$* (See Sect. 3.1). *Then,*

- *(1) There is a constant C such that for every $C' > C$ and every C^1 quasi-concave Cauchy time T on $\widetilde{M}$, the renormalized T-levels $(\pi_1(M), S_a^T, (\sup_{S_a^T} T_{cos})^{-1} d_a^T)_{a\in\mathbb{R}_+^*}$ are, for a big enough, C'-quasi-isometric to $(\pi_1(M), \mathbb{H}^n, d_{\mathbb{H}^n})$. In particular all the limit points, for the Gromov equivariant topology, of $(\pi_1(M), S_a^T, (\sup_{S_a^T} T_{cos})^{-1} d_a^T)_{a\in\mathbb{R}_+^*}$ are C-bi-Lipschitz to $(\pi_1(M), \mathbb{H}^n, d_{\mathbb{H}^n})$;*

- *(2) In dimension* $2+1$*, the renormalized* CMC*-levels (respectively* k*-levels) converge in the Gromov equivariant topology, when time goes to* $+\infty$*, to* $(\pi_1(M), \mathbb{H}^n, d_{\mathbb{H}^n})$.

Remark 2.4 In fact Theorem 2.3 is the best result we can get in this generality. Indeed, in a static flat space–time $(\Gamma, \mathbf{C})$ (See Sect. 3.1 for the definition), consider a Γ-invariant complete convex surface S different than the cosmological ones. The family $(aS)_{a>0}$ constitutes a foliation of $\mathbf{C}$. The associated renormalized family of metric spaces converges in the Gromov equivariant topology, when a goes to $+\infty$, to (Γ, S, d_S).

Now focus on the $2+1$ dimensional case. In this article we obtain the analogue of Theorem 2.1 in the de Sitter and anti de Sitter cases. More precisely:

Theorem 2.5 *Let* M *be* $MGHC$ *de Sitter (or anti de Sitter) space–time of dimension* $2+1$*. Let* T *be a* C^2 *quasi-concave Cauchy time on* $\widetilde{M}$*. Then the levels* $(\pi_1(M), S_a^T, d_a^T)_{a\in\mathbb{R}_+^*}$ *converge in the Gromov equivariant topology, when* a *goes to* 0*, to the real tree dual to the measured geodesic lamination associated to* M*. In particular this limit does not depend on the time function* T*.*

Remark 2.6 (1) Theorem 2.2 and Theorem 2.5 are based essentially on Proposition 4.1 which was proven for C^2 quasi-concave time functions. However, one can hope to get a more general statement since Proposition 4.1 remains true in more general cases, as for the cosmological time.
(2) Theorem 2.5 is proven in the $2+1$ case. This is essentially due to our strategy of proof based on Wick rotations. However, we believe that it is possible to extend this result to the $n+1$ dimensional de Sitter and anti de Sitter cases.

Now look to the asymptotic behavior in the Teichmüller space. Our fourth result concern the future behavior of the curve associated to the k-time and the CMC time. Let S be a closed hyperbolic surface and let (λ, μ) be a measured geodesic lamination on S. Let M be the $MGHC$ space–time of constant curvature associated to (λ, μ).

Theorem 2.7 *Let* T_k *and* T_{cmc} *be respectively the* k*-time and the* CMC *time of* M*. Then,*

- *In the flat case: the curves* $([g_a^{T_k}])_{a>0}$ *and* $([g_a^{T_{cmc}}])_{a>0}$ *in the Teichmüller space* $\mathrm{Teich}(S)$ *of* S *converge, when time goes to* $+\infty$*, to the hyperbolic structure of* S*.*
- *In the de Sitter case: The curve* $([g_a^{T_k}])_{a>0}$ *in the Teichmüller space* $\mathrm{Teich}(S)$ *of* S *stays at a bounded Teichmüller distance, when time goes to* $+\infty$*, from the grafting metric* $\mathrm{gra}_\lambda(S)$.

3 Backgrounds on constant curvature space–times

In all this paper and for the sake of simplicity we will denote by $\langle ., . \rangle$ (respectively by $|.|^2$) the scalar product (respectively the quadratic form) associated to the Lorentzian metric under consideration.

3.1 Flat space–times, initial Singularity and Horizon

Let $\mathbb{R}^{1,n}$ be the Minkowski space. An hyperplane P is said to be lightlike if it is orthogonal to a lightlike direction. Let $\mathfrak{P}$ be the space of all lightlike hyperplanes in $\mathbb{R}^{1,n}$. Let Λ be a closed subset of $\mathfrak{P}$ and consider $\Omega := \bigcap_{P\in\Lambda} I^+(P)$. By [6], the subset Ω is an open convex domain of $\mathbb{R}^{1,n}$. It is non empty as soon as Λ is compact. If Λ contains more than two elements, then the open convex domain Ω, if not empty, is called a future complete regular domain. In the same way one can define a past complete regular domain.

Let Ω be a future complete regular domain. By [14], the boundary $\partial\Omega$ of Ω is the graph of a 1-Lipschitz convex function $f : \mathbb{R}^n \to \mathbb{R}$.

Let $\mathfrak{L}$ be the set of Lipschitz curves contained in $\partial\Omega$. For every $\alpha \in \mathfrak{L}$, consider $l(\alpha) := \int \sqrt{|\dot{\alpha}(t)|^2} dt$ the Lorentzian length of α. Let d be the pseudo-distance defined on $\partial\Omega$ by:

$$d_{\partial\Omega}(p, q) = \inf \{l(\alpha), \text{ where } \alpha \text{ is a curve in } \mathfrak{L} \text{ joining } p \text{ and } q\}.$$

The cleaning $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ i.e the quotient of the pseudo metric space $(\partial\Omega, d_{\partial\Omega})$ by the equivalence relation $p \sim q$ if and only if $d_{\partial\Omega}(p, q) = 0$, is a length metric space (see for instance [10, Corollaire 2.2.14]).

Definition 3.1 The metric space $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ is the Horizon associated to Ω.

An hyperplane P is a support hyperplane of Ω if $\Omega \subset J^+(P)$. Note that if Ω admits two lightlike support hyperplanes then it admits a spacelike support hyperplane. Let Σ be the set of points $p \in \partial\Omega$ such that Ω have a spacelike support hyperplane passing through p. By a result of Bonsante (see [14, Proposition 7.8]), the restriction of the pseudo-distance $d_{\partial\Omega}$ to Σ is a distance denoted by d_Σ.

Definition 3.2 The metric space (Σ, d_Σ) is the initial Singularity associated to Ω.

Example 3.1 The future cone of the origin $\mathbf{C}$ is a typical example of regular domain. In this case the metric spaces $(\partial\mathbf{C}/\sim, \bar{d}_{\partial\mathbf{C}})$ and (Σ, d_Σ) are identified with the trivial metric space $(\{0\}, d = 0)$.

In, Bonsante [14] shows that to each point p in Ω corresponds a unique point $r(p)$ in $\partial\Omega$ realizing the cosmological time i.e such that $T_{cos}(p) = \sqrt{-|p - r(p)|^2}$. He proved also that the application $r : \Omega \to \partial\Omega$, called retraction map, is continuous and that $r(\Omega) = \Sigma$. Moreover, the cosmological time T_{cos} of Ω is a $C^{1,1}$ regular Cauchy time whose Lorentzian gradient is given by $N_p = -\nabla_p T_{cos} = \frac{1}{T_{cos}(p)}(p - r(p))$. Every point p in Ω can be decomposed as

$$p = r(p) + T_{cos}(p) N_p.$$

Actually all this remain true in any future complete convex domain of $\mathbb{R}^{1,n}$.

Let Γ be a torsion free uniform lattice of $SO^+(1, n)$. A cocycle of Γ is an application $\tau : \Gamma \to \mathbb{R}^{1,n}$ such that $\tau(\gamma_1.\gamma_2) = \gamma_1\tau(\gamma_2) + \tau(\gamma_1)$. An affine deformation of Γ associated to τ is the morphism $\rho_\tau : \Gamma \to SO^+(1, n) \ltimes \mathbb{R}^{1,n}$ defined by $\rho_\tau(\gamma).x = \gamma.x + \tau(\gamma)$ for every $\gamma \in \Gamma$ and $x \in \mathbb{R}^{1,n}$. By a result of Bonsante [14], to every affine deformation of Γ corresponds a unique future complete maximal flat regular domain Ω on which $\Gamma_\tau = \rho_\tau(\Gamma)$ acts freely properly discontinuously. In this case, the cosmological normal application N and the retraction map r of Ω are equivariant under the action of Γ. This means that $N_{\gamma_\tau.p} = \gamma.N_p$ and $r(\gamma_\tau.p) = \gamma_\tau.r(p)$ for every p in Ω and γ in Γ. By [14, Lemma 4.15] and [14, Lemma 3.12, Corollary 4.5], the normal application $N : \Omega \to \mathbb{H}^n$, when restricted to each cosmological level $S_a^{T_{cos}}$, is a surjective proper function. The $MGHC$ space–time $M_{[\tau]} := \Omega/\Gamma_\tau$ is called a standard flat space–time. In the special case of the trivial cocycle the space–time $M_{[0]} := \mathbf{C}/\Gamma$ is the static flat space–time.

A future complete $MGHC$ flat space–time M is said to be non elementary if $L(\pi_1(M))$ is a non elementary subgroup of $SO^+(1, n)$, where $L : \pi_1(M) \to SO^+(1, n)$ is the linear part of the holonomy morphism $\rho : \pi_1(M) \to SO^+(1, n) \ltimes \mathbb{R}^{1,n}$ of M. The following theorem gives a full classification of $MGHC$ flat non elementary space–times.

Theorem 3.1 ([6, Theorem 4.11]) *Every future complete $MGHC$ flat non elementary space–time M is up to finite cover the quotient of a future complete regular domain by a discrete subgroup of $SO^+(1, n) \ltimes \mathbb{R}^{1,n}$.*

3.2 De Sitter space–times

Let S be a simply connected Möbius manifold. That is a manifold equipped with a (G, X)-structure, where $G = O^+(1, n)$ and $X = \mathbb{S}^n$ is the Riemannian sphere. A Möbius manifold is elliptic (respectively parabolic) if it is conformally equivalent to $\mathbb{S}^n$ (respectively $\mathbb{S}^n$ minus a point). A non elliptic neither parabolic Möbius manifold is called hyperbolic Möbius manifold.

Let $d : S \to \mathbb{S}^n$ be a developing map of S. A round ball of S is an open convex set U of S on which d is an homeomorphism. It is said to be proper if $d(\bar{U})$ is a closed round balls of $\mathbb{S}^n$. Let $B(S)$ be the space of proper round ball of S. By a result of [3], there is a natural topology on $B(S)$ making it locally homeomorphic to $\mathbb{DS}_{n+1}$. By [3], the space $B(S)$ endowed with the pull back metric of $\mathbb{DS}_{n+1}$ is a simply connected future complete globally hyperbolic locally de Sitter space–time called dS-standard space–time.

In general $B(S)$ is not isometric to a part of $\mathbb{DS}_{n+1}$. However, there are some regions in $B(S)$ which embedd isometrically in $\mathbb{DS}_{n+1}$. Indeed, let x in S and let $U(x)$ be the union of all round ball containing x. Then by [3], the dS–standard spacetime $B(U(x))$ is isometric to an open domain of $\mathbb{DS}_{n+1}$. Moreover, for every proper round ball V containing x, the causal past of V in $B(S)$ is contained in $B(U(x))$.

In the case of dS–standard space–time of hyperbolic type, the cosmological time is regular (see [3]). One can attach to each hyperbolic type dS–standard space–time $B(S)$ a past boundary $\partial B(S)$, which can be seen locally as a convex hypersurface of $\mathbb{DS}_{n+1}$. Moreover, to every point p in $B(S)$ corresponds a unique point $r(p)$ on $\partial B(S)$ realizing the cosmological time. Actually the point $r(p)$ is the limit point in $B(S) \cup \partial B(S)$ of the past timelike geodesique starting at p with initial velocity $-N_p$, where N_p is the future oriented cosmological normal vector at p. The application N is the cosmological normal application and r is the retraction map.

Let p be a point in $B(S)$. The causal past of p is contained in a domain of $B(S)$ isometric to an open domain of $\mathbb{DS}_{n+1}$. So, after identification of $\mathbb{DS}_{n+1}$ with the pseudo-sphere in $\mathbb{R}^{1,n+1}$, the point p can be decomposed as:

$$p = \cosh(T_{cos}(p))\, r(p) + \sinh(T_{cos}(p))\, N_{r(p)};$$
$$N_p = \sinh(T_{cos}(p))\, r(p) + \cosh(T_{cos}(p))\, N_{r(p)}.$$

We have the following classification theorem:

Theorem 3.2 ([32, Theorem 1.1]) *Every $MGHC$ de Sitter space–time is the quotient of a standard dS space–time by a free torsion discret subgroup of $SO^+(1, n+1)$.*

3.3 Anti de Sitter space–times

Let M be a $MGHC$ anti de Sitter space–time of dimension $n + 1$. By [3,24], the universal cover $\widetilde{M}$ of M is isometric to an open convex domain, called regular domain, of the anti de Sitter space. Denote by $\widetilde{M}_-$ the tight past of $\widetilde{M}$ i.e the strict past in $\widetilde{M}$ of the cosmological level $S^{T_{cos}}_{\frac{\pi}{2}}$.

By [3], the cosmological time of a $\widetilde{M}_-$ is regular. One can attach a past boundary $\partial\widetilde{M}_-$ to $\widetilde{M}_-$ which can be seen as a convex hypersurface of $\mathbb{ADS}_{n+1}$. Moreover, to every point p in $\widetilde{M}_-$ corresponds a unique point $r(p)$ on $\partial\widetilde{M}_-$ realizing the cosmological time. The point $r(p)$ is the limit point in $\widetilde{M}_- \cup \partial\widetilde{M}_-$ of the past timelike geodesique starting at p with initial velocity $-N_p$, where N_p is the future oriented cosmological normal vector at p. The application N is the cosmological normal application and r is the retraction map. After identification of $\mathbb{ADS}_{n+1}$ with the pseudo-sphere in $\mathbb{R}^{2,n}$, we get that every point p in $\widetilde{M}_-$ can be decomposed as:

$$p = \cos(T_{cos}(p))\, r(p) + \sin(T_{cos}(p))\, N_{r(p)};$$
$$N_p = -\sin(T_{cos}(p))\, r(p) + \cos(T_{cos}(p))\, N_{r(p)}.$$

4 Quasi-concave times and their expansive character

Let M be a $MGHC$ space–time of constant curvature. Let S be a C^2 complete $\pi_1(M)$-invariant spacelike hypersurface of $\widetilde{M}$. Let Π_S be its second fundamental form defined by $\Pi_S(X, Y) = \langle\nabla_X \mathbf{n}, Y\rangle$, where $\mathbf{n}$ is the future oriented normal vector field. Recall that the mean curvature H_S at a point p of S is defined by $H_S = \frac{tr(\Pi)}{n}$ i.e $H_S = \frac{\lambda_1+\lambda_2+\ldots+\lambda_n}{n}$, where λ_i are the principal curvatures of S. Recall that in the case of dimension 2, the Gauss curvature k_S at a point p of S is defined by $k_S = -det(\Pi)$ i.e $k_S = -\lambda_1\lambda_2$.

Definition 4.1 The hypersurface S is said to be convex if its second fundamental form is negative-definite. In this case, the principal curvatures are negative.

The convexity of S is equivalent to the geodesic convexity of $J^+(S)$. Thus using this last characterisation one can generalise the notion of convexity to non smooth hypersurfaces.

A $\pi_1(M)$-invariant Cauchy time function $T : \widetilde{M} \to \mathbb{R}^*_+$ is quasi-concave if its levels are convex. The cosmological time, the CMC time and the k time provide us important examples of quasi-concave times.

Definition 4.2 The cosmological time T_{cos} is defined by: $T_{cos}(p) = \sup_\alpha \int \sqrt{-|\dot{\alpha}(s)|^2}$ where the supremum is taken over all the past causal curves starting at p.

In the flat case the cosmological time is a concave (and hence quasi-concave) Cauchy time (see [14]). By [3,32] the cosmological time is a regular quasi-concave time in the de Sitter case. In the anti de Sitter case it fails to be quasi-concave. However, by [3] the cosmological levels are convex near the initial singularity.

Definition 4.3 The CMC time is a $\pi_1(M)$-invariant Cauchy time $T : \widetilde{M} \to \mathbb{R}$ such that every level $T^{-1}(t)$, if not empty, is of constant mean curvature t.

The existence and uniqueness of such time was studied in [1,3,5,8,9]. In the flat case and by a result of Treibergs [34] the CMC time is quasi-concave. It is no more true in the anti de Sitter case. Unfortunately we don't now if it is the case in the de Sitter case.

In the flat case, the CMC time takes its values over $\mathbb{R}^*_-$. Up to the reparametrization $b \mapsto -\frac{1}{b}$, we will consider that the CMC time takes its values in $\mathbb{R}^*_+$. In other words: for every $b > 0$, the CMC level $S_b^{T_{cmc}}$ is of constant mean curvature $-\frac{1}{b}$.

Definition 4.4 Suppose that M is of the dimension $2 + 1$. The k time is a Cauchy time $T : \widetilde{M} \to \mathbb{R}$ such that every level $T^{-1}(t)$, if not empty, is of constant Gauss curvature t.

Barbot, Béguin and Zeghib [7] proved the existence and uniqueness of such time in the flat and de Sitter case. In the anti de Sitter case there is no globally defined k-time. However, the two connected components of the convex core admit a unique k-time. By definition, the k-time is quasi-concave.

In the flat and the anti de Sitter cases, the k-time is defined over $\mathbb{R}_-^*$. Up to the reparametrization $b \mapsto \sqrt{-b^{-1}}$, we will consider that the k-time takes its values over $\mathbb{R}_+^*$. In the de Sitter case, the k-time is defined over $]-\infty, -1[$. So we will consider it defined over $\mathbb{R}_+^*$ up to the reparametrization $b \mapsto \sqrt{-(b+1)^{-1}}$.

Let $T : \widetilde{M} \to \mathbb{R}_+^*$ be a $\pi_1(M)$-invariant C^2 quasi-concave Cauchy time. Denote by $\xi_T = \frac{\nabla T}{|\nabla T|^2}$, where ∇T is the Lorentzian gradient of T and let Φ_T^t be the corresponding flow generated by ξ_T. Denote by S_1^T the level set $T^{-1}(1)$ of T.

Proposition 4.1 *Let $\alpha : [a, b] \to \widetilde{M}$ be a spacelike curve contained in the past of S_1^T. Then the length of α is less than the length of α_1 where $\alpha_1(s) = \Phi_T^{1-T(\alpha(s))}(\alpha(s))$ is the projection of α on S_1^T along the lines of Φ_T.*

Proof We proved this proposition in the 2+1 flat case [11, Proposition 4.2]. The proof does not use the fact that space–time is flat of dimension $2+1$ and remains true in our case (see [11, Remark 1.2]). □

Remark 4.2 Even if in Proposition 4.1 we restrict ourselves to C^2 quasi-concave times, one can prove analogue Propositions for the cosmological time, which is just $C^{1,1}$, in the Sitter and anti de Sitter cases (see Remark 6.11 and Remark 6.15).

5 Quasi-concave times versus cosmological time

Let M be a non negative constant curvature $MGHC$ space–time of dimension $n+1$ and let T_{cos} be the cosmological time of $\widetilde{M}$. The purpose of this section is to highlight the comparability between the cosmological time and the other quasi-concave times.

5.1 The flat case

Let us start with the following proposition which gives an estimate on the cosmological barriers in the flat $n+1$ dimensional case.

Proposition 5.1 *Let $M \simeq \Omega/\Gamma_\tau$ be a standard flat space–time, where Ω is a future complete flat regular domain, Γ a torsion free uniform lattice of $SO^+(1, n)$ and Γ_τ its affine deformation in $SO^+(1, n) \ltimes \mathbb{R}^{1,n}$. Let S be a convex complete Γ-invariant Cauchy hypersurface of Ω. There is a constant C depending only on Γ such that for every $C' > C$*

$$\frac{\sup_S T_{cos}}{\inf_S T_{cos}} \leq C',$$

for $\sup_S T_{cos}$ big enough.

Proof Fix an origin of the Minkowski space $\mathbb{R}^{1,n}$. Let N and r be respectively the normal application and the retraction map of Ω.

For simplicity denote by $a = \sup_S T_{cos}$ and by $b = \inf_S T_{cos}$. Let $F \subset \mathbb{H}^n$ be a compact fundamental domain for the action of Γ on $\mathbb{H}^n$. Note that $F' = r\left(N^{-1}(F)\right)$ is a fundamental domain for the action of Γ_τ on Σ. The closure of F' in $\mathbb{R}^{1,n}$ is compact. Denote then by $C_1 = \sup_{F'\times F'\times F} |\langle r_1 - r_2, n\rangle|$.

Now let $p \in S$ such that $T_{cos}(p) = a$. Up to isometry we can suppose that $N_p \in F$ and $r(p) \in F'$. The convexity of S implies that the tangent hyperplane P_p to S at p is the tangent hyperplane to $S_a^{T_{cos}}$ at p. Thus for every γ in Γ, $\gamma_\tau.P_p$ is the tangent hyperplane of S and $S_a^{T_{cos}}$ at $\gamma_\tau.p$. Hence, we obtain that for every x in S and every γ in Γ:

$$\left\langle \gamma_\tau p - x, \gamma.N_p \right\rangle \geq 0.$$

But $\gamma_\tau p = \gamma.p + \tau(\gamma)$, $x = r(x) + T_{cos}(x)N_x$ and $p = r(p) + T_{cos}(p)N_p$, so

$$T_{cos}(x)\left\langle N_x, \gamma.N_p\right\rangle \leq \left\langle p, N_p\right\rangle - \left\langle \gamma^{-1}r(x) + \tau(\gamma^{-1}), N_p\right\rangle.$$

Therefore

$$T_{cos}(x)\left\langle N_x, \gamma.N_p\right\rangle \leq -T_{cos}(p) + \left\langle r(p) - r(\gamma_\tau^{-1}x), N_p\right\rangle.$$

Thus

$$\left|\frac{T_{cos}(p)}{\left\langle N_x, \gamma.N_p\right\rangle}\right| - \left|\frac{\left\langle r(p) - r(\gamma_\tau^{-1}x), N_p\right\rangle}{\left\langle N_x, \gamma.N_p\right\rangle}\right| \leq T_{cos}(x),$$

and hence

$$\left|\frac{1}{\left\langle \gamma^{-1}.N_x, N_p\right\rangle}\right| - \frac{1}{a}\frac{\left\langle r(p) - r(\gamma_\tau^{-1}x), N_p\right\rangle}{\left|\left\langle \gamma^{-1}.N_x, N_p\right\rangle\right|} \leq \frac{T_{cos}(x)}{a}.$$

On the other hand, for every x in S, there exists a γ_x in Γ such that $\gamma_x^{-1}.N_x \in F$ and $r((\gamma_x)_\tau^{-1}x) \in F'$. Thus,

$$\frac{1}{C} - \frac{1}{a}C_1 \leq \frac{T_{cos}(x)}{a},$$

where $C = \sup_{F\times F} \left|\left\langle n, n'\right\rangle\right|$

Since the last inequality is true for every x in S, we obtain that

$$\frac{1}{C} - \frac{1}{a}C_1 \leq \frac{b}{a} = \frac{\inf_S T_{cos}}{\sup_S T_{cos}}.$$

When a goes to infinity, $\frac{1}{a}C_1$ goes to 0 and this finishes the proof. □

As a direct consequence of this proposition we obtain:

Corollary 5.2 *Let $T : \Omega \to]0, +\infty[$ be a Γ_τ-invariant quasi-concave Cauchy time. Then*

$$\lim_{t\to\infty} \frac{\sup_{S_t^T} T_{cos}}{\inf_{S_t^T} T_{cos}} \leq C.$$

Proof When t goes to infinity, $a_t := \sup_{S_t^T} T_{cos}$ goes to infinity. Then we conclude using Proposition 5.1. □

Remark 5.3 By a result of Andersson, Barbot, Béguin and Zeghib [3] we have that in the particular case of the CMC time : $\frac{\sup_{S_t^T} T_{cos}}{\inf_{S_t^T} T_{cos}} \leq n$ for every $t > 0$. Moreover,

$$\frac{1}{n}\sup_{S_t^T} T_{cos} \leq t \leq \sup_{S_t^T} T_{cos}.$$

Proposition 5.4 *Let $M \simeq \Omega/\Gamma_\tau$ be a non elementary future complete MGHC flat space–time of dimension $2+1$ and let $T_k : \Omega \to]0,+\infty[$ be the k-time of Ω. The cosmological time and the k-time are comparable near the infinity. Moreover*

$$\lim_{t\to+\infty} \frac{\inf_{S_t^{T_k}} T_{cos}}{t} = \lim_{t\to+\infty} \frac{\sup_{S_t^{T_k}} T_{cos}}{t} = 1.$$

For the proof we need the following Maximum Principle.

Lemma 5.5 *Let S and S' two spacelike hypersurfaces in a space–time M such that S' is in the future of S and $S \cap S' \neq \varnothing$. For every $p \in S \cap S'$ we have that the principal curvatures of S at p are bigger than the principal curvatures of S' at p. In particular the Gauss curvature of S is bigger than the Gauss curvature of S'.*

Remark 5.6 This lemma remains true in the case of C^0 hypersurfaces with generalized principal curvatures (See for instance [3, Proposition 4.4]). Thus one can apply the Maximum Principale on the cosmological levels.

Proof of Proposition 5.4 Let $S_1^{T_k}$ be the k-level of constant Gauss curvature -1. Let $H_0 = \inf H_{S_1^{T_k}}$ and $H_1 = \sup H_{S_1^{T_k}}$, where $H_{S_1^{T_k}}$ is the mean curvature of $S_1^{T_k}$.

Consider the Γ_τ-invariant future complete convex domain $A := J^+(S_1^{T_k})$. Denote respectively by T'_{cos}, r' the associated cosmological time and retraction map. For every $t > 1$, the Γ_τ-invariant k-level $S_t^{T_k}$ is entirely contained in A. As the action of Γ_τ on $S_t^{T_k}$ is cocompact, the restriction of the cosmological time T'_{cos} of A achieves its minimum on $S_t^{T_k}$. Let $p \in S_t^{T_k}$ such that $\inf_{S_t^{T_k}} T'_{cos} = T'_{cos}(p) := a$. By applying the Maximum Principle to the hypersurfaces $S_a^{T'_{cos}}$ and $S_t^{T_k}$ we get

$$k_{S_a^{T'_{cos}}}(p) \geq -\frac{1}{t^2},$$

where $k_{S_a^{T'_{cos}}}$ is the Gauss curvature of $S_a^{T'_{cos}}$.

On the one hand we have

$$k_{S_a^{T'_{cos}}}(p) = -\frac{1}{1 - 2H_{S_1^{T_k}}(r'(p))a + a^2}.$$

Hence

$$a \geq H_0 + \sqrt{H_1^2 - 1 + t^2}.$$

But

$$\inf_{S_t^{T_k}} T_{cos} \geq \inf_{S_a^{T'_{cos}}} T_{cos} \geq a.$$

So

$$\inf_{S_t^{T_k}} T_{cos} \geq H_0 + \sqrt{H_1^2 - 1 + t^2}.$$

On the other hand and by applying the Maximum Principle to the hypersurfaces $S_t^{T_k}$ and $S^{T_{cos}}_{\sup_{S_t^{T_k}} T_{cos}}$ we get

$$\sup_{S_t^{T_k}} T_{cos} \leq t.$$

Thus

$$1 \geq \frac{\sup_{S_t^{T_k}} T_{cos}}{t} \geq \frac{\inf_{S_t^{T_k}} T_{cos}}{t} \geq \frac{H_0}{t} + \sqrt{\frac{H_1^2}{t^2} - \frac{1}{t^2} + 1}.$$

which concludes the proof. □

Corollary 5.7 *We have:*

$$\lim_{t\to+\infty} \frac{\inf_{S_t^{T_{cmc}}} T_{cos}}{t} = \lim_{t\to+\infty} \frac{\sup_{S_t^{T_{cmc}}} T_{cos}}{t} = 1.$$

Proof Let $S_t^{T_{cmc}}$ be a CMC level of constant mean curvature $-\frac{1}{t}$. For every $p \in S_t^{T_{cmc}}$ we have $\frac{\lambda_1(p)+\lambda_2(p)}{2} = -\frac{1}{t}$. So

$$k_{S_t^{T_{cmc}}}(p) = -\lambda_1(p)\lambda_2(p) = \left(\lambda_1(p) + \frac{2}{t}\right)\lambda_1(p).$$

The function $f(\lambda) = \left(\lambda + \frac{2}{t}\right)\lambda$ achieves its minimum at $-\frac{1}{t}$, thus $k_{S_b^{T_{cmc}}} \geq -\frac{1}{t^2}$.

Then by [7, Remark 10.3], $S_t^{T_{cmc}}$ is in the future of the k-level $S_t^{T_k}$. We conclude using Proposition 5.4 and Remark 5.3. □

For every $a > 0$, let Ω_a be the regular domain defined by $\Omega_a := \frac{1}{a}\Omega$. Note that Ω_a is the regular domain associated to the cocycle $\frac{\tau}{a}$. The regular domain Ω_a converge when a goes to ∞ to the cone $\mathbf{C}$. Denote by T^a_{cos}, T^a_k and T^a_{cmc} respectively the cosmological time, the k-time and the CMC time of Ω_a. It is not hard to see that $aT^a(x) = T^1(ax)$ for each of the three times.

Corollary 5.8 *The Cauchy times T^a_k (respectively T^a_{cmc}) converge in the compact open topology, when a goes to $+\infty$, to the cosmological time of* **C**. *That is for every compact F of* **C** *and for a big enough, the Cauchy time T^a_k (respectively T^a_{cmc}) converge uniformly on F to the cosmological time of* **C**.

Proof Let F be a compact set in the interior of **C**. Note that for a big enough $F \subset \Omega_a$. By [14, Proposition 6.2], the cosmological time T^a_{cos} converge uniformly on F to the cosmological time of **C**. So to proof that T^a_k (respectively T^a_{cmc}) converge unifomly on F to the cosmological time of **C**, it is sufficent to proof that $\sup_{x\in F} \left|T^a_k(x) - T^a_{cos}(x)\right|$ (respectively $\sup_{x\in F} \left|T^a_{cmc}(x) - T^a_{cos}(x)\right|$) goes to 0, when a goes to $+\infty$.

1) The k-time case. We have

$$\sup_{x\in F} \left|T^a_k(x) - T^a_{cos}(x)\right| \leq \left[1 - \inf_{x\in F} \frac{T^1_{cos}(ax)}{T^1_k(ax)}\right] \sup_{x\in F} T^a_k(x).$$

Using Proposition 5.4, one can see that $T^a_k(x)$ is bounded on F and $\inf_{x\in F} \frac{T^1_{cos}(ax)}{T^1_k(ax)}$ goes to 1 when a goes to $+\infty$. Thus we get that $\sup_{x\in F} \left|T^a_k(x) - T^a_{cos}(x)\right|$ goes to 0 when a goes to $+\infty$.

2) The CMC-time case. We have

$$\sup_{x\in F}\left|T^a_{cmc}(x)-T^a_{cos}(x)\right|\leq\left[\sup_{x\in F}\frac{T^1_{cos}(ax)}{T^1_{cmc}(ax)}-1\right]\sup_{x\in F}T^a_{cmc}(x).$$

Then by Corollary 5.7, we have that $\sup_{x\in F}\left|T^a_{cmc}(x)-T^a_{cos}(x)\right|$ goes to 0 when a goes to $+\infty$. □

5.2 The de Sitter case

Let $M\simeq B(S)/\Gamma$ be a $2+1$-dimensional $MGHC$ de Sitter space–time of hyperbolic type. Let T_k be the k-time of $B(S)$.

Proposition 5.9 *We have:*

- *(1) There exists a constant* $D>0$ *such that* $\lim_{b\to+\infty}\left[\sup_{S^{T_k}_b}T_{cos}-\inf_{S^{T_k}_b}T_{cos}\right]\leq D$;
- *(2)* $\lim_{b\to+\infty}\dfrac{\inf_{S^{T_k}_b}T_{cos}}{\operatorname{argcoth}\left(\sqrt{\frac{b^2+1}{b^2}}\right)}=\lim_{b\to+\infty}\dfrac{\sup_{S^{T_k}_b}T_{cos}}{\operatorname{argcoth}\left(\sqrt{\frac{b^2+1}{b^2}}\right)}=1.$

Proof The proof is similar to the flat case. The k-level $S^{T_k}_1$ is of constant Gauss curvature -2. Let $H_0=\inf H_{S^{T_k}_1}$ and $H_1=\sup H_{S^{T_k}_1}$, where $H_{S^{T_k}_1}$ is the mean curvature of $S^{T_k}_1$.

Denote respectively by T'_{cos}, r' the cosmological time and retraction map of the Γ-invariant future complete convex domain $A:=J^+(S^{T_k}_1)$ of $B(S)$. For every $b>1$, let $p\in S^{T_k}_b$ such that $\inf_{S^{T_k}_b}T'_{cos}=T'_{cos}(p):=a$.

By the Maximum Principle we have

$$k_{S^{T'_{cos}}_a}(p)\geq-\frac{1}{b^2}-1.$$

But

$$k_{S^{T'_{cos}}_a}(p)=-\left(\frac{\lambda_1(r'(p))-\tanh(a)}{1-\lambda_1(r'(p))\tanh(a)}\right)\left(\frac{\lambda_2(r'(p))-\tanh(a)}{1-\lambda_2(r'(p))\tanh(a)}\right),$$

where $\lambda_1(r'(p))$ and $\lambda_2(r'(p))$ are the principal curvatures of the k-level $S^{T_k}_1$ at $r'(p)$. Hence

$$k_{S^{T'_{cos}}_a}(p)=-\frac{2-2H_{S^{T_k}_1}(r'(p))\tanh(a)+\tanh^2(a)}{1-2H_{S^{T_k}_1}(r'(p))\tanh(a)+2\tanh^2(a)}.$$

Thus

$$\inf_{S^{T_k}_b}T_{cos}\geq\operatorname{argth}\left(\frac{H_0}{b^2+2}+\frac{1}{b^2+2}\sqrt{H_1^2+(b^2-1)(b^2+2)}\right).$$

On the other hand and by the Maximum Principle we have

$$\sup_{S^{T_k}_b}T_{cos}\leq\operatorname{argcoth}\left(\sqrt{\frac{b^2+1}{b^2}}\right).$$

Then a simple computation shows that:

- $\lim_{b\to+\infty}\left[\sup_{S_b^{T_k}} T_{cos} - \inf_{S_b^{T_k}} T_{cos}\right] \leq \frac{1}{2}\log(3 - H_0)$.
- $\lim_{b\to+\infty} \frac{\inf_{S_b^{T_k}} T_{cos}}{\operatorname{argcoth}\left(\sqrt{\frac{b^2+1}{b^2}}\right)} = \lim_{b\to+\infty} \frac{\sup_{S_b^{T_k}} T_{cos}}{\operatorname{argcoth}\left(\sqrt{\frac{b^2+1}{b^2}}\right)} = 1.$

□

6 Bilipschitz control of convex hypersurfaces

Let us consider M to be a $n+1$-dimensional

- future complete flat standard $MGHC$ space–time;
- or a future complete $MGHC$ de Sitter space–time of hyperbolic type;
- or the tight past of a $MGHC$ anti de Sitter space–time.

Our next proposition shows that the geometry of a convex spacelike surface can be compared uniformly to the cosmological one. More precisely:

Proposition 6.1 *Let $S \subset \widetilde{M}$ be a $\pi_1(M)$-invariant convex Cauchy hypersurface of $\widetilde{M}$ the universal cover of M. Let* $\mathbf{n}$ *its Gauss application and N the cosmological normal application. Then for every p in S we have,*

- $\left|\left\langle N_p, \mathbf{n}_p \right\rangle\right| \leq \left(\sup_S T_{cos}\right)(\inf_S T_{cos})^{-1}$ *if M is flat;*
- $\left|\left\langle N_p, \mathbf{n}_p \right\rangle\right| \leq \left(\sinh(\sup_S T_{cos})\right)(\sinh(\inf_S T_{cos}))^{-1}$ *if M is locally de Sitter;*
- $\left|\left\langle N_p, \mathbf{n}_p \right\rangle\right| \leq \left(\tan(\sup_S T_{cos})\right)(\tan(\inf_S T_{cos}))^{-1}$ *if M is locally anti de Sitter.*

For the proof we need the following lemma:

Lemma 6.2 *Let $S_a^{T_{cos}}$ et $S_b^{T_{cos}}$ be two cosmological levels of $\widetilde{M}$ the universal cover of M, with $b < a$. Then for every p in $S_b^{T_{cos}}$ and every unitary future oriented timelike tangent vector $x \in T_p\widetilde{M}$ such that $S_a^{T_{cos}} \subset J^+(P_p)$, where $P_p = x^\perp \subset T_p\widetilde{M}$, we have:*

- $\left|\left\langle N_p, x \right\rangle\right| \leq (a)(b)^{-1}$ *if M is flat;*
- $\left|\left\langle N_p, x \right\rangle\right| \leq (\sinh(a))(\sinh(b))^{-1}$ *if M is locally de Sitter;*
- $\left|\left\langle N_p, x \right\rangle\right| \leq (\tan(a))(\tan(b))^{-1}$ *if M is locally anti de Sitter.*

Proof of Lemma 6.2 in the flat case Fix an origin of $\mathbb{R}^{1,n}$ and suppose that $M \simeq \Omega/\Gamma_\tau$ is flat. Let p in $S_b^{T_{cos}} \subset \widetilde{M}$ and let $x \in \mathbb{H}^n$ such that $S_a^{T_{cos}} \subset J^+(p + x^\perp)$. For every y in $S_a^{T_{cos}}$ we have:

$$\langle y, x\rangle \leq \langle p, x\rangle .$$

Then

$$a\left\langle N_y, x\right\rangle \leq b\left\langle N_p, x\right\rangle + \langle r(p) - r(y), x\rangle .$$

The normal application $N : S_a^{T_{cos}} \to \mathbb{H}^n$ is surjective. So to conclude it is sufficient to take y in $S_a^{T_{cos}}$ such that $N_y = x$. □

Remark 6.3 We restrict ourselves to standard space–times to get the surjectivity of the normal cosmological application. However, it still true in any future regular domain. Indeed, consider two cosmological levels $S_a^{T_{cos}}$ and $S_b^{T_{cos}}$ with $b < a$. Let p in $S_b^{T_{cos}}$ and let S_a be the hyperboloid defined by: $S_a = \left\{y \in J^+(r(p)) \subset \mathbb{R}^{1,n} \text{ such that } |y - r(p)|^2 = -a^2\right\}$. Remark that S_a is in the future of $S_a^{T_{cos}}$. Thus for every y in S_a we have : $\langle y, x\rangle \leq \langle p, x\rangle$ and so $\langle y - r(p), x\rangle \leq b\left\langle N_p, x\right\rangle$. Then it is sufficient to take y such that $y - r(p) = ax$. □

Proof of Lemma 6.2 in the de Sitter case
Fix an origin of $\mathbb{R}^{1,n+1}$ and identify $\mathbb{DS}_{n+1}$ with the pseudo sphere in $\mathbb{R}^{1,n+1}$. Suppose that $M \simeq B(S)/\Gamma$ is locally de Sitter. Let p in $S_b^{T_{cos}}$ and let x be a unitary future oriented timelike tangent vector in $T_p\widetilde{M}$ such that $S_a^{T_{cos}} \subset J^+(P_p)$, where $P_p = x^\perp \subset T_pB(S)$. The proof is similar to the one of Remark 6.3 which depends only on $J^+(r(p))$. Note that $J^+(r(p))$ is isometric to a domain of $\mathbb{DS}_{n+1}$. So we can, without losing generality, restric ourselves and work in $\mathbb{DS}_{n+1}$.

For every y in the hypersurface $S_a = \left\{y \in J^+(r(p)) \subset \mathbb{DS}_{n+1} \text{ such that } d_{Lor}(y, r(p)) = a\right\}$ we have,

$$\langle x, p - y\rangle \geq 0,$$

where $\langle ., .\rangle$ is the scalar product of $\mathbb{R}^{1,n+1}$. Thus

$$0 = \langle x, p\rangle \geq \langle x, y\rangle .$$

Let us write:

- $r(p) = -\langle r(p), x\rangle\, x + u'$, where $u' \in x^\perp$;
- $p = \cosh(b)\, r(p) + \sinh(b)\, N_{r(p)}$, where $N_{r(p)} \in \mathbb{H}^{n+1} \cap T_{r(p)}\mathbb{DS}_{n+1}$ is the cosmological normal vector;
- $y = \cosh(a)\, r(p) + \sinh(a)\, v_y$, where $v_y \in \mathbb{H}^{n+1} \cap T_{r(p)}\mathbb{DS}_{n+1}$.

Now take $v_y = \left(\sqrt{1 + \langle r(p), x\rangle^2}\right) x - \frac{\langle r(p), x\rangle}{|u'|} u'$.

On the one hand $\langle x, p\rangle = 0$ and $N_p = \sinh(b)\, r(p) + \cosh(b)\, N_{r(p)}$ so,

$$\left\langle x, N_p\right\rangle = -\frac{1}{\sinh(b)} \langle x, r(p)\rangle .$$

On the other hand $\langle x, y\rangle \leq 0$ and hence,

$$\langle x, r(p)\rangle \leq \sinh(a) .$$

Thus

$$\left|\left\langle x, N_p\right\rangle\right| \leq \frac{\sinh(a)}{\sinh(b)}.$$

□

Proof of Lemma 6.2 in the anti de Sitter case
Fix an origin of $\mathbb{R}^{2,n}$ and identify $\mathbb{ADS}_{n+1}$ with the pseudo sphere in $\mathbb{R}^{2,n}$. Suppose that M is locally anti de Sitter. Note that $\widetilde{M}$ is isometric to a domain of $\mathbb{ADS}_{n+1}$. Let p in $S_b^{T_{cos}}$ and let $x \in \mathbb{ADS}_{n+1} \subset \mathbb{R}^{2,n}$ such that $P_p = x^\perp \subset T_p\widetilde{M}$, x is future oriented (with respect to the orientation of $\mathbb{ADS}_{n+1}$) and $S_a^{T_{cos}} \subset J^+(P_p)$. Let $S_a = \left\{y \in J^+(r(p)) \subset \mathbb{ADS}_{n+1} \text{ such that } d_{Lor}(y, r(p)) = a\right\}$. For every y in S_a we have,

$$\langle x, p - y\rangle \geq 0,$$

where $\langle ., . \rangle$ is the scalar product of $\mathbb{R}^{2,n}$. Thus

$$0 = \langle x, p \rangle \geq \langle x, y \rangle .$$

Let us write:

- $r(p) = -\langle r(p), x \rangle x - \langle p, r(p) \rangle p + u'$, where $u' \in Vect(x, p)^{\perp}$;
- $p = \cos(b)\, r(p) + \sin(b)\, N_{r(p)}$, where $N_{r(p)} \in \mathbb{ADS}_{n+1} \cap T_{r(p)}\mathbb{ADS}_{n+1}$ is the cosmological normal vector;
- $N_p = -\sin(b)\, r(p) + \cos(b)\, N_{r(p)}$;
- $y = \cos(a)\, r(p) + \sin(a)\, v_y$, where $v_y \in \mathbb{ADS}_{n+1} \cap T_{r(p)}\mathbb{ADS}_{n+1}$ is future oriented.

We get then:

- $\langle x, N_p \rangle = -\frac{1}{\sin(b)} \langle x, r(p) \rangle$;
- $\langle x, r(p) \rangle \leq -\tan(a) \langle x, v_y \rangle$.

For every $\beta \in \mathbb{R}$ let,

$$v(\beta) = (-\langle p, r(p) \rangle)\, x + \beta p + \frac{\sqrt{\langle p, r(p) \rangle^2 + \beta^2 - 1}}{|u'|} u'.$$

For every $\beta \in \mathbb{R}$ we have $|v(\beta)|^2 = -1$. A direct computation shows that there exists β_0 such that $\langle v(\beta_0), r(p) \rangle = 0$. In this case $v(\beta_0)$ is future oriented. Indeed, x is future oriented and $\langle v(\beta_0), x \rangle = \langle p, r(p) \rangle = -\cos(b) < 0$.

Thus the point $y := \cos(a)\, r(p) + \sin(a)\, v(\beta_0)$ belongs to S_a and hence

$$\left| \langle N_p, x \rangle \right| = \frac{1}{\sin(b)} \left| \langle x, r(p) \rangle \right| \leq \frac{1}{\sin(b)} \tan(a) \left| \langle x, v(\beta_0) \rangle \right| = (\tan(a))\, (\tan(b))^{-1} .$$

□

Proof of Proposition 6.1 Denote by $a = \sup_S T_{cos}$ and $b = \inf_S T_{cos}$. The hypersurface S is in the past of $S_a^{T_{cos}}$ and in the future of $S_b^{T_{cos}}$. Let p in S and let $P_p = \mathbf{n}_p^{\perp}$ the tangent hyperplane to S at p. As S is convex, we have that $S_a^{T_{cos}} \subset J^+(P_p)$. By Lemma 6.2 we have:

- $\left| \langle N_p, \mathbf{n}_p \rangle \right| \leq \frac{a}{T_{cos}(p)} \leq \frac{a}{b}$ in the flat case;
- $\left| \langle N_p, \mathbf{n}_p \rangle \right| \leq \frac{\sinh(a)}{\sinh(T_{cos}(p))} \leq \frac{\sinh(a)}{\sinh(b)}$ in the de Sitter case;
- $\left| \langle N_p, \mathbf{n}_p \rangle \right| \leq \frac{\tan(a)}{\tan(T_{cos}(p))} \leq \frac{\tan(a)}{\tan(b)}$ in the anti de Sitter case.

and this concludes the proof. □

6.1 The $(n+1)$-flat case

Let $M \simeq \Omega/\Gamma_\tau$ be a future complete $MGHC$ flat non elementary space–time of dimension $n+1$.

Proposition 6.4 *Let $S \subset \Omega$ be a C^2 convex Γ_τ invariant Cauchy hypersurface and let g_S be the Riemannian metric defined on S by the restriction of the ambient Lorentzian metric of the Minkowski space $\mathbb{R}^{1,n}$. Then (S, g_S) is K^4-bi-Lipschitz to $(S^{T_{cos}}_{\sup_S T_{cos}}, g^{T_{cos}}_{\sup_S T_{cos}})$, where $K = \frac{\sup_S T_{cos}}{\inf_S T_{cos}}$.*

Remark 6.5 The fact that (S, g_S) is bi-Lipschitz to $(S^{T_{cos}}_{\sup_S T_{cos}}, g^{T_{cos}}_{\sup_S T_{cos}})$ is a direct consequence of the cocompactness of the Γ_τ-action. What we are proving here is that the bi-Lipschitz constant K depend only on the cosmological barrier and not on the hypersurface S.

Let us start with the following proposition due to Bonsante:

Proposition 6.6 ([14, Lemme 7.4]). *The cosmological levels $S_a^{T_{cos}}$ and $S_b^{T_{cos}}$ with $b < a$ are $\left(\frac{a}{b}\right)^2$-bi-Lipschitz one to the other. More precisely,*

$$g_b \leq g_a \leq \left(\frac{a}{b}\right)^2 g_b.$$

Proof of Proposition 6.4 Let S be a convex Γ_τ invariant Cauchy hypersurface of Ω and let g_S its induced Riemannian metric. Denote by $a = \sup_S T_{cos}$ and by $b = \inf_S T_{cos}$.

Let $\alpha : [0, 1] \to S$ be a Lipschitz curve in S. For almost every s in $[0, 1]$, we have

$$\dot{\alpha}(s) = \dot{r}(s) + \dot{T}_{cos}(s)N_s + T_{cos}(s)\dot{N}(s)$$

and hence

$$|\dot{\alpha}(s)|^2 = \left|\dot{r}(s) + T_{cos}(s)\dot{N}(s)\right|^2 - \dot{T}_{cos}(s)^2.$$

For every $t > 0$ and every $s \in [0, 1]$ the vector $\dot{r}(s) + t\dot{N}(s)$ is tangent to the cosmological level $S_t^{T_{cos}}$. Thus by Proposition 6.6

$$|\dot{\alpha}(s)|^2 \geq \left|\dot{r}(s) + b\dot{N}(s)\right|^2 - \dot{T}_{cos}(s)^2.$$

Note that

$$\dot{T}_{cos}(s) = d_{\alpha(s)}T_{cos}.\dot{\alpha}(s) = -\langle N(\alpha(s)), \dot{\alpha}(s)\rangle .$$

Let us write $N(\alpha(s)) = h(s)\mathbf{n}(\alpha(s)) + v(s)$, where $\mathbf{n}$ is the normal map of S and $v(s)$ is in $\mathbf{n}(\alpha(s))^\perp$.

By Proposition 6.1,

$$\left|\langle N_{\alpha(s)}, \mathbf{n}(\alpha(s))\rangle\right| \leq \frac{a}{b},$$

and hence

$$|v(s)|^2 \leq \left(\frac{a}{b}\right)^2 - 1.$$

But

$$\left|\dot{T}_{cos}(s)\right| = |\langle v(s), \dot{\alpha}(s)\rangle| \leq |v(s)|\,|\dot{\alpha}(s)|\,.$$

Thus

$$\dot{T}_{cos}(s)^2 \leq \left(\left(\frac{a}{b}\right)^2 - 1\right)|\dot{\alpha}(s)|^2\,.$$

Which proves that

$$\left(\frac{b}{a}\right)^2 \left|\dot{r}(s) + b\dot{N}(s)\right|^2 \leq |\dot{\alpha}(s)|^2\,.$$

On the other hand and by Proposition 6.6 we have

$$|\dot{\alpha}(s)|^2 \leq \left|\dot{r}(s) + a\dot{N}(s)\right|^2 \leq \left(\frac{a}{b}\right)^2 \left|\dot{r}(s) + b\dot{N}(s)\right|^2.$$

Thus

$$\left(\frac{b}{a}\right)^4 \left|\dot{r}(s) + a\dot{N}(s)\right|^2 \leq |\dot{\alpha}(s)|^2 \leq \left|\dot{r}(s) + a\dot{N}(s)\right|^2.$$

This proves that the cosmological flow induces a $\left(\frac{b}{a}\right)^4$-bi-Lipschitz identification between (S, g_S) and $(S_a^{T_{cos}}, g_a^{T_{cos}})$. □

Corollary 6.7 *Let M be a $MGHC$ flat future complete non elementary space–time. Let $T_{cmc} : \widetilde{M} \to \mathbb{R}_+$ its associated CMC time. Then for every $a > 0$, the hypersurface $(S_a^{T_{cmc}}, g_a^{T_{cmc}})$ is n^4-bi-Lipschitz to the hypersurface $(S_a^{T_{cos}}, g_a^{T_{cos}})$.*

Proof The corollary follows from Remark 5.3 and Proposition 6.4. □

6.2 The (2+1)-de Sitter case

Definition 6.1 Let M be a differentiable manifold endowed with two Lorentzian metrics g and $\mathfrak{g}$. Let ξ be a vector fields everywhere non zero. The Lorentzian metric $\mathfrak{g}$ is obtained by a Wick rotation from the Lorentzian metric g along the vector fields ξ if:

- (1) For every p in M, the sub-spaces g-orthogonal and $\mathfrak{g}$-orthogonal to ξ_p are the same;
- (2) there exists a positive function f such that $\mathfrak{g} = fg$ on the sub-space spanned by ξ_p;
- (3) There exists a positive function h such that : $\mathfrak{g} = hg$ on $\xi_p^{\perp}$.

Let Ω be a flat future complete regular domain of dimension $2+1$. Consider Ω_1 the past in Ω of the cosmological level $S_1^{T_{cos}}$ and g its induced Lorentzian metric. By [12], there exists a C^1 local diffeomorphism $\hat{D} : \Omega_1 \to \mathbb{DS}_3$ such that the pullback by $\hat{D}$ of the de Sitter metric is the Lorentzian metric $\mathfrak{g}$ obtained from g by a Wick rotation along the cosmological gradient with $\mathfrak{g} = \frac{1}{(1-T_{cos}^2)^2} g$ on $\mathbb{R}\xi_{T_{cos}}$ and $\mathfrak{g} = \frac{1}{1-T_{cos}^2} g$ on $\left\langle \xi_{T_{cos}} \right\rangle^{\perp}$. The space $(\Omega_1, \mathfrak{g})$ is a dS-standard spacetime of hyperbolic type i.e associated to some hyperbolic projective structure (given also by the canonical Wick rotation) on $S_1^{T_{cos}}$. In fact, this Wick rotation provides us a one to one correspondence between standard $2+1$ de Sitter space–times of hyperbolic type and flat future complete regular domains of dimension $2+1$. Moreover, this construction can be done in an equivariant way giving hence a one to one correspondence between future complete flat $MGHC$ non elementary space–times of dimension $2+1$ and future complete $MGHC$ de Sitter space–times of hyperbolic type of dimension $2+1$.

Proposition 6.8 ([12, Proposition 5.2.1]) *The cosmological time $\mathcal{T}_{cos}$ of $(\Omega_1, \mathfrak{g})$ is a Cauchy time. Moreover,*

$$\mathcal{T}_{cos} = \operatorname{argth}(T_{cos}),$$

where T_{cos} is the cosmological time of (Ω_1, g).

Suppose now that $M \simeq B(S)/\Gamma$ is a $MGHC$ de Sitter space–time of hyperbolic type and of dimension $2+1$. Let $(\Omega_1, \mathfrak{g})$ be the hyperbolic dS-standard space–time of dimension $2+1$ associated to M obtained by a Wick rotation from a flat regular domain (Ω, g). Let T_{cos} and $\mathcal{T}_{cos}$ be respectively the cosmological time of (Ω, g) and $(\Omega_1, \mathfrak{g})$.

Proposition 6.9 *The cosmological levels $S_a^{\mathcal{T}_{cos}}$ and $S_b^{\mathcal{T}_{cos}}$ of $B(S)$ with $b < a$ are $\left(\frac{\sinh(a)}{\sinh(b)}\right)^2$-bi-Lipschitz one to the other. More precisely,*

$$\mathfrak{g}_b^{\mathcal{T}_{cos}} \leq \mathfrak{g}_a^{\mathcal{T}_{cos}} \leq \left(\frac{\sinh(a)}{\sinh(b)}\right)^2 \mathfrak{g}_b^{\mathcal{T}_{cos}}.$$

Proof We have

$$\mathfrak{g}_a^{\mathcal{T}_{cos}} = \frac{1}{\left(1-\tanh^2(a)\right)} g_{\tanh(a)}^{\mathcal{T}_{cos}}.$$

But by Proposition 6.6

$$g_{\tanh(b)}^{\mathcal{T}_{cos}} \leq g_{\tanh(a)}^{\mathcal{T}_{cos}} \leq \left(\frac{\tanh\,(a)}{\tanh\,(b)}\right)^2 g_{\tanh(b)}^{\mathcal{T}_{cos}}$$

Thus

$$\mathfrak{g}_b^{\mathcal{T}_{cos}} \leq \mathfrak{g}_a^{\mathcal{T}_{cos}} \leq \left(\frac{\sinh\,(a)}{\sinh\,(b)}\right)^2 \mathfrak{g}_b^{\mathcal{T}_{cos}}.$$

□

Proposition 6.10 *Let $S \subset B(S)$ be a convex Γ invariant Cauchy hypersurface and let $\mathfrak{g}_S$ be the metric of S. Then, $(S, \mathfrak{g}_S)$ is K^4-bi-Lipschitz to $(S_{\sup_S \mathcal{T}_{cos}}^{\mathcal{T}_{cos}}, \mathfrak{g}_{\sup_S \mathcal{T}_{cos}}^{\mathcal{T}_{cos}})$, where $K = \frac{\sinh(\sup_S \mathcal{T}_{cos})}{\sinh(\inf_S \mathcal{T}_{cos})}$.*

Proof Let us denote for simplicity by $a = \sup_S \mathcal{T}_{cos}$, by $b = \inf_S \mathcal{T}_{cos}$ and by $|.|_1$ the de Sitter norm of Ω_1. Let $\alpha : [0, 1] \to S$ be a Lipschitz curve in Ω_1. For almost every s in $[0, 1]$ we have,

$$\dot{\alpha}(s) = \dot{r}(s) + T_{cos}(s)\dot{N}(s) + \dot{T}_{cos}(s).$$

Note that for every s in $[0, 1]$, the vector $\dot{r}(s) + T_{cos}(s)\dot{N}(s)$ is tangent to the cosmological level $S_{T_{cos}(s)}^{\mathcal{T}_{cos}}$. Using the Wick caracterisation of the de Sitter norm $|.|_1$ we get,

$$|\dot{\alpha}(s)|_1^2 = \frac{1}{1-T_{cos}^2(s)}\left|\dot{r}(s) + T_{cos}(s)\dot{N}(s)\right|^2 - \frac{1}{\left(1-T_{cos}^2(s)\right)^2}\dot{T}_{cos}^2(s).$$

Thus by Proposition 6.9,

$$\begin{aligned}\left|\dot{r}(s) + \tanh\,(b)\,\dot{N}(s)\right|_1^2 - \dot{\mathcal{T}}_{cos}^2(s) &\leq |\dot{\alpha}(s)|_1^2\\ &\leq \left|\dot{r}(s) + \tanh\,(a)\,\dot{N}(s)\right|_1^2.\end{aligned}$$

Using the same arguments as in the flat case we get that,

$$\dot{\mathcal{T}}_{cos}^2(s) \leq \left(\left(\frac{\sinh\,(a)}{\sinh\,(b)}\right)^2 - 1\right)|\dot{\alpha}(s)|_1^2.$$

Hence

$$|\dot{\alpha}(s)|_1^2 \geq \left(\frac{\sinh\,(b)}{\sinh\,(a)}\right)^2 \left|\dot{r}(s) + \tanh(b)\dot{N}(s)\right|_1^2.$$

Then by Proposition 6.9 we get

$$\left(\frac{\sinh(b)}{\sinh(a)}\right)^4 \left|\dot r(s)+\tanh(a)\,\dot N(s)\right|_1^2 \le |\dot\alpha(s)|_1^2 \le \left|\dot r(s)+\tanh(a)\,\dot N(s)\right|_1^2.$$

□

Remark 6.11 Actually in Proposition 6.10 we proved that if α is a spacelike curve contained in the past of the cosmological level $S_a^{T_{cos}}$, then the length $l(\alpha)$ of α is less than the length of $\Phi_{T_{cos}}^{a-T_{cos}(\alpha)}$, where $\Phi_{T_{cos}}$ is the cosmological flow.

6.3 The (2+1)-anti de Sitter case

Let Ω be a flat future complete regular domain of dimension $2+1$ and let g be its induced Lorentzian metric. By [12] there exists a C^1 local diffeomorphism $\hat D : \Omega \to \mathbb{ADS}_3$ such that the pullback by $\hat D$ of the anti de Sitter metric is the Lorentzian metric $\mathfrak{g}$ obtained from g by a Wick rotation along the cosmological gradient with $\mathfrak{g} = \frac{1}{(1+T_{cos}^2)^2} g$ on $\mathbb{R}\xi_{T_{cos}}$ and $\mathfrak{g} = \frac{1}{(1+T_{cos}^2)} g$ on $\left\langle \xi_{T_{cos}} \right\rangle^{\perp}$. In fact $(\Omega, \mathfrak{g})$ is the tight past region of its maximal anti de Sitter extension. Moreover, this Wick rotation provide us a one to one correspondence between 2+1 anti de Sitter regular domains and flat future complete regular domains of dimension $2+1$. This construction can be done in an equivariant way giving hence a one to one correspondence between future complete flat $MGHC$ non elementary space–times of dimension $2+1$ and future complete $MGHC$ anti de Sitter space–times of dimension $2+1$.

Proposition 6.12 ([12, Proposition 6.2.2]) *The cosmological time $\mathcal{T}_{cos}$ of $(\Omega, \mathfrak{g})$ is a Cauchy time. Moreover,*

$$\mathcal{T}_{cos} = \arctan(T_{cos}),$$

where T_{cos} is the cosmological time of (Ω, g).

Let M be the tight past of a $MGHC$ anti de Sitter space–time of dimension $2+1$. Recall that $\widetilde M \simeq (\Omega, \mathfrak{g})$, where $(\Omega, \mathfrak{g})$ is obtained by a Wick rotation from a flat regular domain (Ω, g). Let T_{cos} and $\mathcal{T}_{cos}$ be respectively the cosmological time of (Ω, g) and $(\Omega, \mathfrak{g})$.

Proposition 6.13 *The cosmological levels $S_a^{\mathcal{T}_{cos}}$ and $S_b^{\mathcal{T}_{cos}}$ of $\widetilde M$ with $b < a$ are $\left(\frac{\tan(a)}{\tan(b)}\right)^2$-bi-Lipschitz one to the other. More precisely,*

$$\left(\frac{\cos(a)}{\cos(b)}\right)^2 \mathfrak{g}_b^{\mathcal{T}_{cos}} \le \mathfrak{g}_a^{\mathcal{T}_{cos}} \le \left(\frac{\sin(a)}{\sin(b)}\right)^2 \mathfrak{g}_b^{\mathcal{T}_{cos}}.$$

Proof We have

$$\mathfrak{g}_a^{\mathcal{T}_{cos}} = \frac{1}{1+\tan^2(a)} g_{\tanh(a)}^{T_{cos}}.$$

But by Proposition 6.6

$$g_{\tan(b)}^{T_{cos}} \le g_{\tan(a)}^{T_{cos}} \le \left(\frac{\tan(a)}{\tan(b)}\right)^2 g_{\tan(b)}^{T_{cos}}.$$

Thus

$$\left(\frac{\cos(a)}{\cos(b)}\right)^2 \mathfrak{g}_b^{\mathcal{T}_{cos}} \le \mathfrak{g}_a^{\mathcal{T}_{cos}} \le \left(\frac{\sin(a)}{\sin(b)}\right)^2 \mathfrak{g}_b^{\mathcal{T}_{cos}}.$$

□

Proposition 6.14 *Let $S \subset \widetilde{M}$ be a convex Γ invariant Cauchy surface and let $\mathfrak{g}_S$ be the metric of S. Then $(S, \mathfrak{g}_S)$ is K^4-bi-Lipschitz to $(S^{\mathcal{T}_{cos}}_{\sup_S \mathcal{T}_{cos}}, \mathfrak{g}^{\mathcal{T}_{cos}}_{\sup_S \mathcal{T}_{cos}})$, where $K = \frac{\tan(\sup_S \mathcal{T}_{cos})}{\tan(\inf_S \mathcal{T}_{cos})}$.*

Proof Let us denote for simplicity by $a = \sup_S \mathcal{T}_{cos}$, by $b = \inf_S \mathcal{T}_{cos}$ and by $|.|_{-1}$ the anti de Sitter norm of Ω. Let $\alpha : [0, 1] \to S$ be a Lipschitz curve in S. For almost every s in $[0, 1]$ we have,

$$\dot{\alpha}(s) = \dot{r}(s) + T_{cos}(s)\dot{N}(s) + \dot{\mathcal{T}}_{cos}(s).$$

Note that for every s in $[0, 1]$, the vector $\dot{r}(s) + T_{cos}(s)\dot{N}(s)$ is tangent to the cosmological level $S^{T_{cos}}_{T_{cos}(s)}$. Using the Wick caracterisation of the anti de Sitter norm $|.|_{-1}$ we get,

$$|\dot{\alpha}(s)|^2_{-1} = \frac{1}{1 + T^2_{cos}(s)} \left|\dot{r}(s) + T_{cos}(s)\dot{N}(s)\right|^2 - \frac{1}{\left(1 + T^2_{cos}(s)\right)^2} \dot{T}^2_{cos}(s).$$

Thus by Proposition 6.13,

$$\left(\frac{\cos(a)}{\cos(b)}\right)^2 \left|\dot{r}(s) + \tan(b)\dot{N}(s)\right|^2_{-1} - \dot{\mathcal{T}}^2_{cos}(s) \leq |\dot{\alpha}(s)|^2_{-1}$$
$$\leq \left(\frac{\cos(b)}{\cos(a)}\right)^2 \left|\dot{r}(s) + \tan(a)\dot{N}(s)\right|^2_{-1}.$$

Using the same arguments as in the flat and the de Sitter case we get that,

$$\dot{\mathcal{T}}^2_{cos}(s) \leq \left(\left(\frac{\tan(a)}{\tan(b)}\right)^2 - 1\right) |\dot{\alpha}(s)|^2_{-1}.$$

Hence

$$|\dot{\alpha}(s)|^2_{-1} \geq \left(\frac{\sin(b)}{\sin(a)}\right)^2 \left|\dot{r}(s) + \tan(b)\dot{N}(s)\right|^2_{-1}.$$

Then by Proposition 6.13 we get

$$\left(\frac{\tan(b)}{\tan(a)}\right)^4 \left|\dot{r}(s) + \tan(a)\,\dot{N}(s)\right|^2_{-1} \leq |\dot{\alpha}(s)|^2_{-1} \leq \left(\frac{\tan(a)}{\tan(b)}\right)^4 \left|\dot{r}(s) + \tan(a)\,\dot{N}(s)\right|^2_{-1}.$$

□

Remark 6.15 Actually in Proposition 6.14 we proved that if α is a spacelike curve contained in the past of the cosmological level $S^{T_{cos}}_a$ and in the future of the cosmological level $S^{T_{cos}}_b$, then the length $l(\alpha)$ of α is less than $\frac{\cos(b)}{\cos(a)} l(\Phi^{a - T_{cos}(\alpha)}_{T_{cos}})$, where $\Phi_{T_{cos}}$ is the cosmological flow.

7 Asymptotic behavior in flat $(n + 1)$-space–times

The purpose of this section is to prove Theorem 2.2 and Theorem 2.3.

7.1 Generalities on geometric metric spaces

Let (X, d) be a metric space. The length $L_d(\alpha)$ of a path $\alpha : [a, b] \to X$ is defined to be the supremum, on finite subdivion of $[a, b]$, of $\sum d(\alpha(t_i), \alpha(t_{i+1}))$. The length distance $d_L(x, y)$ between two points x and y is the infimum of the length of paths joining x and y. The metric space (X, d_L) is then called a length metric space. A path α joining two points x and y is a geodesic of the length metric space (X, d_L) if $L_d(\alpha) = d_L(x, y)$. A length metric space such that every two points are joined by a geodesic is called geodesic metric space.

Let (X, d) be a geodesic metric space. Let $\Delta(x, y, z)$ be a geodesic triangle in X. A comparison triangle of $\Delta(x, y, z)$ in the model space $(\mathbb{R}^2, d_{\mathbb{R}^2})$ is the unique (up to isometry) triangle $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ of $(\mathbb{R}^2, d_{\mathbb{R}^2})$ such that $d(x, y) = d_{euc}(\bar{x}, \bar{y})$, $d(y, z) = d_{euc}(\bar{y}, \bar{z})$ and $d(x, z) = d_{euc}(\bar{x}, \bar{z})$. The comparison map from $\Delta(x, y, z)$ to $\bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$ is the unique map which sends the points x, y, z to the points $\bar{x}, \bar{y}, \bar{z}$ and the geodesic segments $[x, y]$, $[x, z]$, $[y, z]$ to the geodesic segments $[\bar{x}, \bar{y}]$, $[\bar{x}, \bar{z}]$, $[\bar{y}, \bar{z}]$.

Definition 7.1 A geodesic metric space (X, d) is $\mathrm{CAT}(0)$ if every comparison map is 1-Lipschitz.

A length metric space (X, d) is said to possess the *approximative midpoints property* if: for every x, y in X and $\epsilon > 0$ there exists z in X such that $d(x, z) \leq \frac{1}{2}d(x, y) + \epsilon$ and $d(y, z) \leq \frac{1}{2}d(x, y) + \epsilon$. The length metric space X satisfies the $\mathrm{CAT}(0)$ 4-*points condition* if for any 4-tuple of points (x_1, y_1, x_2, y_2) there exists a 4-tuple of points $(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$ in $\mathbb{R}^2$ such that: $d(x_i, y_j) = d(\bar{x}_i, \bar{y}_j)$ for $i, j \in \{1, 2\}$, and $d(x_1, x_2) \leq d(\bar{x}_1, \bar{x}_2)$, $d(y_1, y_2) \leq d(\bar{y}_1, \bar{y}_2)$. Note that a $\mathrm{CAT}(0)$ metric space satisfies the $\mathrm{CAT}(0)$ 4-*points condition* and have the *approximative midpoints property*. The converse is true in the complete case:

Proposition 7.1 ([17, Proposition II.1.11]) *Let (X, d) be a complete metric space. The following conditions are equivalent:*

- *(1) X is a* $\mathrm{CAT}(0)$ *metric space;*
- *(2) X possesses the approximative midpoints property and satisfies the* $\mathrm{CAT}(0)$ 4-*points condition.*

A geodesic metric space (X, d) is a real tree if any two points are joined by a unique path. Clearly a real tree is a $\mathrm{CAT}(0)$ metric space. An important example of real tree is the one given by a measured geodesic lamination (see for example [26,28]).

Let Γ be a finitely generated group. A metric space on which Γ acts by isometry is a Γ-metric space. Recall that a correspondence between two sets X_1 and X_2 is a subset $\mathcal{R}$ of $X_1 \times X_2$ such that the projections $\pi_1 : R \to X_1$ and $\pi_2 : R \to X_2$ are onto.

Definition 7.2 A sequence $(X_n, d_n, \Gamma)_{n\in\mathbb{N}}$ of Γ-metric spaces converge to a Γ-metric space (X, d, Γ) for the Gromov equivariant topology if and only if, for every finite set K of X, for every finite part P of Γ and for every $\epsilon > 0$, there exists N_0 such that for every $n \geq N_0$, there is a finite set K_n of X_n and a correspondence $\mathcal{R}_n$ between K and K_n satisfying: $\forall x, y \in K$, $\forall x_n, y_n \in K_n$, $\forall \gamma \in P$, if $x\mathcal{R}_n x_n$ and $y\mathcal{R}_n y_n$, then

$$|d(x, \gamma y) - d_n(x_n, \gamma y_n)| < \epsilon.$$

7.2 Geometric properties of the initial singularity

Let Ω be a flat future complete regular domain and let (Σ, d_Σ), $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ be the Initial Singularity and the Horizon associated to Ω. Denote by $(\Sigma^\bigstar, d_\Sigma^\bigstar)$ the completion of (Σ, d_Σ). By a result of Bonsante [14] the metric space (Σ, d_Σ) embed isometrically in $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$.

Proposition 7.2 *The Horizon* $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ *embeds isometrically in* $(\Sigma^\bigstar, d_\Sigma^\bigstar)$.

Proof Let Σ_∞ be the set of Cauchy sequences of (Σ, d_Σ) and let d_∞ the pseudo-distance defined by: if $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are two Cauchy sequences of (Σ, d_Σ), then $d_\infty((x_i)_i, (y_i)_i) = \lim_{i\to\infty} d_\Sigma(x_i, y_i)$. Denote by $\pi' : \Sigma_\infty \to \Sigma^\bigstar$ the projection of Σ_∞ in $\Sigma^\bigstar$.

Let x in $\partial\Omega \setminus \Sigma$ and let $(p_i)_{i\in\mathbb{N}}$ be a sequence of Ω converging to x and such that $T_{cos}(p_{i+1}) < T_{cos}(p_i)$, for every i in $\mathbb{N}$. Note that the sequence $r(p_i)$ stays in a compact of $\partial\Omega$. Thus extract a subsequence if necessary, we can suppose that $r(p_i)$ converges to y in $\partial\Omega$. The timelike vectors $p_i - r(p_i)$ converge to $x - y$. So the vector $x - y$ is a causal vector. But $\partial\Omega$ is achronal so $x - y$ is lightlike. Hence y should belong to the lightlike ray passing through x which is unique by Lemma [14, Lemma 4.11]. Thus for every x in $\partial\Omega$, there exists a sequence $(x_i)_{i\in\mathbb{N}}$ of Σ converging to a point $y \in \overline{\Sigma}$ such that $d_{\partial\Omega}(x, y) = 0$.

Now let $f : \partial\Omega \to \Sigma^\bigstar$ be the function which associates to each x in $\partial\Omega$ the image by π' of a sequence $(x_i)_{i\in\mathbb{N}}$ in Σ converging to a point y of $\overline{\Sigma}$ such that $d_{\partial\Omega}(x, y) = 0$. This function is well defined and induces an isometric embedding from $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ to $(\Sigma^\bigstar, d_\Sigma^\bigstar)$.

□

Proposition 7.3 *For every* x *and* y *in* Σ*, there exists a geodesic in* $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ *joining* x *and* y.

We will need the following lemma:

Lemma 7.4 *Consider the Lorentzian model* $\mathbb{H}^n$ *of the hyperbolic space. For every* $n_1 \neq n_2$ *in* $\mathbb{H}^n$*, the subset defined by* $F = \{v \in \mathbb{DS}_n \text{ such that } \langle v, n_1\rangle \geq 0 \text{ and } \langle v, n_2\rangle \leq 0\}$ *is precompact.*

Proof Fix an origin of the Minkowski space $\mathbb{R}^{1,n}$. Let n_1 and n_2 in $\mathbb{H}^n$ and v in $\mathbb{R}^{1,n}$ such that $|v|^2 = 1$, $\langle v, n_1\rangle \geq 0$ and $\langle v, n_2\rangle \leq 0$.

One can write $v = -\langle v, n_1\rangle n_1 + v_1$, where v_1 is in $n_1^\perp$. We have then

$$-\langle v, n_1\rangle^2 + |v_1|^2 = 1,$$

And hence

$$||v||^2 = 1 + 2\langle v, n_1\rangle^2,$$

where $||.||$ is the euclidean norm of $\mathbb{R}^{n+1}$.

Thus if we want to proof that v stays in a compact, we need to proof that $\langle v, n_1\rangle$ is bounded independently of v.

In the same way we can write $n_2 = -\langle n_1, n_2\rangle n_1 + u_1$, where u_1 is in $n_1^\perp$. Thus

$$-(\langle n_2, n_1\rangle)^2 + |u_1|^2 = -1.$$

But $\langle v, n_2\rangle \leq 0$, so

$$-\langle n_1, n_2\rangle \langle n_1, v\rangle + \langle v_1, u_1\rangle \leq 0,$$

Hence

$$0 \leq \langle v, n_1\rangle \leq \frac{\langle v_1, u_1\rangle}{\langle n_1, n_2\rangle}.$$

Let's write $v_1 = -\langle v_1, u_1\rangle u_1 + v_1'$, where v_1' is in $n_1^\perp \cap u_1^\perp$. Thus

$$-\langle v, n_1\rangle^2 + \langle v_1, u_1\rangle^2 + \left|v_1'\right|^2 = 1,$$

Then

$$\langle v_1, u_1\rangle^2 \leq \frac{\langle n_1, n_2\rangle^2}{\langle n_1, n_2\rangle^2 - 1}.$$

And this proves that

$$0 \leq \langle v, n_1\rangle \leq \frac{1}{\sqrt{\langle n_1, n_2\rangle^2 - 1}}.$$

□

Proposition 7.5 *Let* $\alpha : [0, l] \to S_a^{T_{cos}}$ *be the geodesic joining two point* p *and* q *of* $S_a^{T_{cos}}$. *Then for every* s *in* $[0, l]$ *we have*

$$\left\langle \dot\alpha(s), N_p\right\rangle \leq 0 \text{ and } \left\langle \dot\alpha(s), N_q\right\rangle \geq 0,$$

where N_p *and* N_q *are the normal vectors of* $S_a^{T_{cos}}$ *at* p *and* q *respectively.*

Proof Let $(x_0, x_1, \ldots, x_n)$ be a coordinate system of $\mathbb{R}^{1,n}$ such that $p = (0, \ldots, 0)$ and $N_p = (1, 0, \ldots, 0)$. The hypersurface $S_a^{T_{cos}}$ is the graph of 1-Lipschitz convex C^1 function $\phi : \mathbb{R}^n \to \mathbb{R}$. We have $\left\langle \dot\alpha(s), N_p\right\rangle = -\dot\phi(s)$. By [14, Lemma 7.7], ϕ is increasing, hence $\left\langle \dot\alpha(s), N_p\right\rangle \leq 0$. In the same way we prove that $\left\langle \dot\alpha(s), N_p\right\rangle \geq 0$. □

Let T_{cos} the cosmological time of Ω and consider $X_{T_{cos}}$ the space of gradient lines of T_{cos}. Note that the normal application and the retraction map of Ω can be seen as maps on $X_{T_{cos}}$.

Proof of Proposition 7.3 Let $\pi : \partial\Omega \to \partial\Omega/\sim$ be the projection of $\partial\Omega$ in $\partial\Omega/\sim$. Note that if F is a compact of $\partial\Omega \subset \mathbb{R}^{1,n}$, then $\pi(F)$ is a compact of $(\partial\Omega/\sim, \bar d_{\partial\Omega})$. Let d_{euc} be the euclidean metric of $\mathbb{R}^{n+1}$ and L_{euc} its associated euclidean length structure. Denote by L the length structure defined on $\partial\Omega/\sim$ by the distance $\bar d_{\partial\Omega}$ and by $\mathcal{L}$ the one induced by the Minkowski metric on $\partial\Omega$.

We want to prove that for every $\mathbf{p}$ and $\mathbf{q}$ in $X_{T_{cos}}$, there is a geodesic in $(\partial\Omega/\sim, \bar d_{\partial\Omega})$ joining $r(\mathbf{p})$ and $r(\mathbf{q})$. There are two distinct cases:

(1) If $N_{\mathbf{p}} = N_{\mathbf{q}}$. Then by Proposition [14, Proposition 4.14], $r(\mathbf{p}) + s(r(\mathbf{p}) - r(\mathbf{q}))$ is contained in $\partial\Omega$ for every s in $[0, 1]$. Clearly $r(\mathbf{p}) + s(r(\mathbf{p}) - r(\mathbf{q}))$ is a geodesic in $(\partial\Omega/\sim, \bar d_{\partial\Omega})$ joining $r(\mathbf{p})$ and $r(\mathbf{q})$.

(2) If $N_{\mathbf{p}} \neq N_{\mathbf{q}}$. For every $0 < a < 1$, let $\alpha_a : [0, l_a] \to S_a^{T_{cos}}$ be the geodesic joining $\mathbf{p}$ and $\mathbf{q}$ i.e joining the intersection point of $\mathbf{p}$ and $S_a^{T_{cos}}$ with the intersection point of $\mathbf{q}$ and $S_a^{T_{cos}}$. By Proposition 7.5 we have $\left\langle \dot\alpha_a(s), N_{\mathbf{p}}\right\rangle \leq 0$ and $\left\langle \dot\alpha_a(s), N_{\mathbf{q}}\right\rangle \geq 0$, for every s in $[0, l_a]$. Therefore, by Lemma 7.4, there is a compact $F \subset dS_n \subset \mathbb{R}^{n+1}$ such that $\dot\alpha_a(s) \in F$ for every $0 < a < 1$ and every s in $[0, l_a]$. There is hence a constant $C > 0$ such that $L_{euc}(\alpha_a) \leq C$ for every $0 < a < 1$. This means that the geodesics α_a are contained in a compact F' of $\overline{\Omega}$.

On the one hand, as $J^-(F') \cap \partial\Omega$ is compact in $\partial\Omega$, the curves $\pi \circ r \circ \alpha_a$ stay in a compact of $(\partial\Omega/\sim, \bar d_{\partial\Omega})$.

On the other hand, for every $0 < a < 1$ and every s_1, s_2 in $[0, l_a]$ we have,

$$\bar d_{\partial\Omega}\left(\pi\left(r\left(\alpha_a(s_1)\right)\right), \pi\left(r\left(\alpha_a(s_2)\right)\right)\right) = d_\Sigma\left(r\left(\alpha_a(s_1)\right), r\left(\alpha_a(s_2)\right)\right).$$

But by [14, Lemma 7.4, Proposition 7.8],

$$d_\Sigma\left(r\left(\alpha_a(s_1)\right), r\left(\alpha_a(s_2)\right)\right) \le d_a^{T_{cos}}\left(\alpha_a(s_1), \alpha_a(s_2)\right).$$

And hence,

$$\bar{d}_{\partial\Omega}\left(\pi\left(r\left(\alpha_a(s_1)\right)\right), \pi\left(r\left(\alpha_a(s_2)\right)\right)\right) \le |s_1 - s_2|.$$

This proves that the family $(\pi \circ r \circ \alpha_a)_{0<a<1}$ is an equicontinuous family of curves. Thus by the Ascoli–Arzéla Theorem we deduce that $\pi \circ r \circ \alpha_a$ converges uniformly in $(\partial\Omega/\sim, \bar{d}_{\partial\Omega})$ to a curve α joining $r(\mathbf{p})$ and $r(\mathbf{q})$. Since $L(\pi \circ r \circ \alpha_a) \le \mathcal{L}(r \circ \alpha_a)$ and $\lim_{a\to 0}\mathcal{L}(r \circ \alpha_a) = d_\Sigma(r(\mathbf{p}), r(\mathbf{q})) = \bar{d}_{\partial\Omega}(\pi \circ r(\mathbf{p}), \pi \circ r(\mathbf{q}))$, we have that $\lim_{a\to 0} L(\pi \circ r \circ \alpha_a) = \bar{d}_{\partial\Omega}(r(\mathbf{p}), r(\mathbf{q}))$. But the length structure L is lower semi continuous, thus $L(\alpha) = \bar{d}_{\partial\Omega}(r(\mathbf{p}), r(\mathbf{q}))$. □

Proposition 7.6 *For every $a > 0$, the cosmological level $(S_a^{T_{cos}}, d_a^{T_{cos}})$ is a $CAT(0)$ metric space.*

Proof The cosmological hypersurface $S_a^{T_{cos}}$ is the graph of a C^1 convex function $\phi : \mathbb{R}^n \to \mathbb{R}$. Using convolution, one can get a uniform C^1 approximation of ϕ by smooth convex functions $\psi_i : \mathbb{R}^n \to \mathbb{R}$. Thus, on the one hand the hypersurface $S_a^{T_{cos}}$ is a geodesic metric space. On the other hand, by the Gauss's Theorema Egrugium and the Theorem [17, Theorem II.1A.6] we have that every smooth convex surface is $CAT(0)$. Hence the cosmological level $S_a^{T_{cos}}$ is $CAT(0)$. □

Proposition 7.7 *The completion $(\Sigma^\bigstar, d_\Sigma^\bigstar)$ of the initial singularity (Σ, d_Σ) is a $CAT(0)$ metric space.*

Proof We are first going to prove that $(\Sigma^\bigstar, d_\Sigma^\bigstar)$ possesses the approximative midpoints property. For that, it is sufficient to prove it for (Σ, d_Σ).

Let $\mathbf{p}$, $\mathbf{q}$ two points of $X_{T_{cos}}$ the space of gradient lines of the cosmological time T_{cos} and let $\epsilon > 0$. For every $a > 0$, denote by p_a (respectively q_a) the intersection point of $\mathbf{p}$ and $S_a^{T_{cos}}$ (respectively the intersection point of $\mathbf{q}$ and $S_a^{T_{cos}}$). Since every $(S_a^{T_{cos}}, d_a^{T_{cos}})$ is geodesic, it possesses the midpoints property. So for every $a > 0$, let z_a be the point in $S_a^{T_{cos}}$ such that $d_a(p_a, z_a) = d_a(q_a, z_a) = \frac{1}{2}d_a(p_a, q_a)$. For every $a > 0$, let us denote by $\mathbf{z}_a$ the cosmological gradient line passing through z_a.

By [14, Proposition 7.6, Proposition 7.8], the distances $d_a^{T_{cos}}(p_a, q_a)$ converge, when a goes to 0, to $d_\Sigma(r(\mathbf{p}), r(\mathbf{q}))$. Then let,

- $a_0 > 0$ such that for every $0 < a \le a_0$ we have $|d_\Sigma(r(\mathbf{p}), r(\mathbf{q})) - d_a(p_a, q_a)| < \epsilon$;
- $a_1 > 0$ so that for every $0 < a \le a_1$ we have $\left|d_\Sigma(r(\mathbf{p}), r(\mathbf{z}_{a_0})) - d_a(p_a, \mathbf{z}_{a_0})\right| < \frac{\epsilon}{2}$.

For every $0 < a < \min(a_0, a_1)$ we have,

$$d_\Sigma(r(\mathbf{p}), r(\mathbf{z}_{a_0})) \le d_a(p_a, \mathbf{z}_{a_0}) + \frac{\epsilon}{2}.$$

But $d_a(p_a, \mathbf{z}_{a_0}) \le d_{a_0}(p_{a_0}, z_{a_0})$, for $0 < a < \min(a_0, a_1)$. Hence

$$d_\Sigma(r(\mathbf{p}), r(\mathbf{z}_{a_0})) \le \frac{1}{2}d_{a_0}(p_{a_0}, q_{a_0}) + \frac{\epsilon}{2} \le \frac{1}{2}d_\Sigma(r(\mathbf{p}), r(\mathbf{q})) + \epsilon.$$

In the same way we show that

$$d_\Sigma(r(\mathbf{q}), r(\mathbf{z}_{a_0})) \le \frac{1}{2}d_\Sigma(r(\mathbf{p}), r(\mathbf{q})) + \epsilon.$$

We obtain in this way an ϵ-approximative midpoint $r(\mathbf{z}_{a_0})$.

By [14, Proposition 7.6, Proposition 7.8], the $CAT(0)$ metric spaces $(S_a^{T_{cos}}, d_a^{T_{cos}})$ converge in the compact open topology to (Σ, d_Σ). Thus the metric spaces (Σ, d_Σ) and $(\Sigma^\bigstar, d_\Sigma^\bigstar)$ satisfy the $CAT(0)$ 4-points condition. As $(\Sigma^\bigstar, d_\Sigma^\bigstar)$ is complete, by Proposition 7.1 it is $CAT(0)$. □

7.3 Asymptotic convergence in the past

In this part we will prove the last point of Theorem 2.2. Let $M \simeq \Omega/\Gamma_\tau$ be a future complete $MGHC$ flat non elementary space–time of dimension $n+1$, where Ω is a future complete regular domain and Γ_τ a discrete subgroup of $SO^+(1,n) \ltimes \mathbb{R}^{1,n}$.

Let T_{cos} be the cosmological time of Ω and let and T be a quasi-concave Γ_τ-invariant Cauchy time of Ω. Denote respectively by $X_{T_{cos}}, X_T$ the space of gradient lines of T_{cos} and the space of gradient lines of T. The gradient lines of T_{cos} (respectively T) being inextensible temporal curves, they intersect every level set of T_{cos} (respectively every level set of T), which are Cauchy hypersurfaces, at a unique point. It follows that every level set of T_{cos} and every level set of T is identified with the space $X_{T_{cos}}$ and the space X_T respectively. Denote by $d_a^{T_{cos}}$ (respectively $\delta_a^{T_{cos}}$) the distance of $S_a^{T_{cos}}$ transported on $X_{T_{cos}}$ (respectively on X_T). In the same way we define the distances d_a^T on X_T and δ_a^T on $X_{T_{cos}}$. Since the Cauchy hypersurfaces are homeomorphic one to each other, the distances $d_a^{T_{cos}}$ and δ_a^T (respectively d_a^T and $\delta_a^{T_{cos}}$) define the same topology on $X_{T_{cos}}$ (respectively on X_T).

The three following results were proved in [11] (see for instance [11, Remark 1.2]).

Proposition 7.8 *The distances d_a^T defined on X_T converge in the compact open topology to a pseudo-distance d_0^T.*

In the case of the cosmological time, the cleaning of the pseudo-metric space $(X_{T_{cos}}, d_0^{T_{cos}})$ is isometric to the Initial Singularity (Σ, d_Σ).

Proposition 7.9 *Up to a subsequence, the sequence $(\delta_{a_n}^{T_{cos}})_{n\geq 0}$ (respectively $(\delta_{a_n}^T)_{n\geq 0}$) converge in the compact open topology to a pseudo-distance $\delta_0^{T_{cos}}$ (respectively δ_0^T) when n goes to ∞. Moreover,*

$$\delta_0^{T_{cos}} \leq d_0^T;$$
$$\delta_0^T \leq d_0^{T_{cos}}.$$

Corollary 7.10 *The marked spectrum of $d_a^{T_{cos}}$, d_a^T, $\delta_0^{T_{cos}}$ and δ_0^T are two by two equals.*

The next proposition gives a more precise description of the behavior of the distances δ_a^T near the initial singularity.

Proposition 7.11 *The distances δ_a^T, converge in the compact open topology to the pseudo-distance $d_0^{T_{cos}}$.*

Proof By Proposition 7.9, it is sufficient to proof that every compact-open limit point δ_0^T of $(\delta_a^T)_{a>0}$ verifies $\delta_0^T \geq d_0^{T_{cos}}$.

Let $(\delta_{a_i}^T)_{i\in\mathbb{N}}$ a subsequence of $(\delta_a^T)_{a>0}$ converging to δ_0^T. Let $\mathbf{p}$ and $\mathbf{q}$ in $X_{T_{cos}}$. For every $i \in \mathbb{N}$, denote respectively by p_i, q_i the intersection points of $S_{a_i}^T$ and $\mathbf{p}$, $\mathbf{q}$. Note that $J^-(p_i)\cap\overline{\Omega}$ (respectively $J^-(q_i)\cap\overline{\Omega}$) is a decreasing sequence of compacts which converge to $r(\mathbf{p})$ (respectively $r(\mathbf{q})$).

Let $i \in \mathbb{N}$, there exists $f(a_i)$ such that the hypersurface $S^{T_{cos}}_{f(a_i)}$ is in the past of the hypersurface $S^T_{a_i}$. Denote respectively by $\mathbf{x_i}$, $\mathbf{y_i}$ the gradient lines of T passing through the points p_i, q_i of $S^T_{a_i}$. Let us denote again by $x_{f(a_i)}, y_{f(a_i)}$ respectively the intersection points of $\mathbf{x_i}$ and $\mathbf{y_i}$ with $S^{T_{cos}}_{f(a_i)}$. We get then:

$$d^{T_{cos}}_{f(a_i)}(\mathbf{p},\mathbf{q}) \leq \delta^{T_{cos}}_{f(a_i)}(\mathbf{x_i},\mathbf{y_i}) + d^{T_{cos}}_{f(a_i)}(\mathbf{p}, x_i) + d^{T_{cos}}_{f(a_i)}(\mathbf{q}, y_i).$$

But by Proposition 4.1, we have,

$$\delta^{T_{cos}}_{f(a_i)}(\mathbf{x_i},\mathbf{y_i}) \leq d^T_{a_i}(\mathbf{x_i},\mathbf{y_i}) = \delta^T_{a_i}(\mathbf{p},\mathbf{q}).$$

Hence

$$d^{T_{cos}}_{f(a_i)}(\mathbf{p},\mathbf{q}) \leq \delta^T_{a_i}(\mathbf{p},\mathbf{q}) + d^{T_{cos}}_{f(a_i)}(\mathbf{p}, x_i) + d^{T_{cos}}_{f(a_i)}(\mathbf{q}, y_i).$$

On the one hand we have that $d^{T_{cos}}_{f(a_i)}(\mathbf{p}, x_i)$ (respectively $d^{T_{cos}}_{f(a_i)}(\mathbf{q}, y_i)$) is bounded from above by $\left|\left|p_{f(a_i)} - x_i\right|\right|$ (respectively $\left|\left|q_{f(a_i)} - y_i\right|\right|$), where $||.||$ is the euclidean norm of $\mathbb{R}^{n+1}$.

But x_i, $p_{f(a_i)}$ (respectively y_i, $q_{f(a_i)}$) converge when i goes ton ∞ to the same point which is $r(\mathbf{p})$ (respectively $r(\mathbf{q})$). This proves that $d^{T_{cos}}_{f(a_i)}(\mathbf{p}, x_i)$ and $d^{T_{cos}}_{f(a_i)}(\mathbf{q}, y_i)$ converge to 0 when i goes toward ∞.

On the other hand, the distances $d^{T_{cos}}_{f(a_i)}$ and $\delta^T_{a_i}$ converge respectively, when i goes to ∞, to $d^{T_{cos}}_0$ and δ^T_0. Thus we have

$$d^{T_{cos}}_0 \leq \delta^T_0.$$

and hence $d^{T_{cos}}_0 = \delta^T_0$. □

This proposition proves that the Γ_τ-metric spaces $(\Gamma_\tau, S^T_a, d^T_a)_{a>0}$ converge in the compact open topology, when a goes to 0, to the initial singularity $(\Gamma_\tau, \Sigma, d_\Sigma)$. Thus the Γ_τ-metric spaces $(\Gamma_\tau, S^T_a, d^T_a)_{a>0}$ converge in the Gromov equivariant topology, when a goes to 0 to the initial singularity $(\Gamma_\tau, \Sigma, d_\Sigma)$ and hence to its completion $(\Gamma_\tau, \Sigma^\star, d^\star_\Sigma)$. This finishes the proof of Theorem 2.2.

7.4 Asymptotic convergence in the future

The object of this part is to prove Theorem 2.3. We use the same notation as in the previous part. Let $T : \Omega \to \mathbb{R}_+$ be a C^1 quasi-concave Γ_τ-invariant Cauchy time.

Proposition 7.12 *There exists a constant $C > 0$ (depending only on Γ) such that:*

- *(1) for every $C' > C$, the renormalized distances $\frac{\delta^T_a}{\sup_{S^T_a} T_{cos}}$ are, near the infinity, C'-quasi-isometric to the hyperbolic metric $d_{\mathbb{H}^n}$. In particular, the limit points, for the compact open topology, of the family $(\frac{\delta^T_a}{\sup_{S^T_a} T_{cos}})_a$ are all C-bi-Lipschitz to $d_{\mathbb{H}^n}$;*
- *(2) In the 2+1 case, the renormalized CMC distances (respectively k distances) converge for the compact open topology, when times goes to infinity, to the hyperbolic distance $d_{\mathbb{H}^2}$.*

Proof Let $a > 0$. Denote by $a_+ = \sup_{S^T_a} T_{cos}$ and by $a_- = \inf_{S^T_a} T_{cos}$. By Proposition 6.4 we have that for every x and y in $X_{T_{cos}}$,

$$\frac{a_-}{a_+} d^{T_{cos}}_{a_-}(x,y) \leq \delta^T_a(x,y) \leq d^{T_{cos}}_{a_+}(x,y).$$

So

$$\left(\frac{a_-}{a_+}\right)^2 \frac{d_{a_-}^{\mathcal{T}_{cos}}(x,y)}{a_-} \le \frac{\delta_a^T(x,y)}{a_+} \le \frac{d_{a_+}^{\mathcal{T}_{cos}}(x,y)}{a_+}.$$

(1) The general case: by Proposition 5.1, there exists a constant C' such that $\frac{a_+}{a_-} \le C'$ for a big enough. Together with Proposition [14, Proposition 7.1] we conclude that for a big enough the distance δ_a^T is C'-quasi-isometric to the hyperbolic hyperbolic $d_{\mathbb{H}^n}$. In particular, all the limit points (for the compact open topology) of the family $(\delta_a^T)_{a>0}$ are C-bi-Lipschitz to the hyperbolic distance $d_{\mathbb{H}^n}$ where C is the constant depending only on Γ given in Proposition 5.1.

(2) In the $2+1$ case: if T is the CMC time or the k-time then by Proposition 5.4 and the Corollary 5.7, the constant C is equal to one and hence the family $(\delta_a^T)_{a>0}$ converges in the compact open topology, when a goes to infinity, to $d_{\mathbb{H}^2}$.

□

This last proposition together with the fact that compact open convergence of Γ-metric spaces is stronger than the Gromov equivariant one conclude the proof of Theorem 2.3.

8 Past convergence in (2+1)-de Sitter space–times

In this section we will proof Theorem 2.5 in de Sitter case. Let $M \simeq B(S)/\Gamma$ be a $2+1$ dimensional $MGHC$ future complete de Sitter space–time of hyperbolic type, where $B(S) \simeq (\Omega_1, \mathfrak{g})$ is the associated hyperbolic dS-standard spacetime of dimension obtained by a Wick rotation from a flat regular domain (Ω, g). Let (λ, μ) be the measured geodesic lamination on $\mathbb{H}^2$ associated to (Ω, g). Let's denote respectively by T_{cos} and $\mathcal{T}_{cos}$ the cosmological time of (Ω, g) and $(\Omega_1, \mathfrak{g})$.

Proposition 8.1 *The cosmological level* $(\Gamma, S_a^{\mathcal{T}_{cos}}, d_a^{\mathcal{T}_{cos}})_{a>0}$ *converge in the compact open topology, when a goes to* 0, *to* $(\Gamma, \Sigma, d_\Sigma)$ *the real tree dual to the measured geodesic lamination* (λ, μ).

Proof Note that the space of cosmological gradient lines of $(\Omega_1, \mathfrak{g})$ is the same as the space of cosmological gradient lines of (Ω_1, g). Let's denote it by X_{cos}. For every $a > 0$, the distance $d_a^{\mathcal{T}_{cos}}$ of $S_a^{\mathcal{T}_{cos}}$ transported to X_{cos} is also denoted by $d_a^{\mathcal{T}_{cos}}$.

On the one hand, by Proposition 7.8 the distances $d_a^{T_{cos}}$ (respectively $d_{\tanh(a)}^{T_{cos}}$) converge in the compact open topology to the pseudo-distance $d_0^{\mathcal{T}_{cos}}$ (respectively $d_0^{T_{cos}}$) on X_{cos}.

On the other hand and for every $a > 0$ we have: $d_a^{\mathcal{T}_{cos}}(x,y) = \frac{1}{1-\tanh^2(a)} d_{\tanh(a)}^{T_{cos}}(x,y)$. Thus, the distances $d_a^{\mathcal{T}_{cos}}$ converge in the compact open topology, when a goes to 0, to the pseudo-distances $d_0^{T_{cos}}$. But the cleaning of $(X_{T_{cos}}, d_0^{T_{cos}})$ is isometric to (Σ, d_Σ), which is by [12, Proposition 3.7.2] isometric to the real tree dual to the measured geodesic lamination (λ, μ). So the Γ metric spaces $(\Gamma, S_a^{\mathcal{T}_{cos}}, d_a^{\mathcal{T}_{cos}})_{a>0}$ converge, when a goes to 0, in the compact open topology to the real tree $(\Gamma, \Sigma, d_\Sigma)$. Then the Γ metric spaces $(\Gamma, S_a^{\mathcal{T}_{cos}}, d_a^{\mathcal{T}_{cos}})_{a>0}$ converge, when a goes to 0, in the Gromov equivariant topology to the real tree $(\Gamma, \Sigma, d_\Sigma)$. □

Proof of Theorem 2.5 in the de Sitter case Thanks to Proposition 4.1, Proposition 8.1 and Remark 6.11, one can reproduce the proof of Theorem 2.1 without any modification and proves Theorem 2.5 in the de Sitter case. □

9 Past convergence in (2+1)-anti de Sitter space–times

In this section we will proof Theorem 2.5 in the anti de Sitter case. Let $M \simeq \widetilde{M}/\Gamma$ be the tight past of a $2+1$ dimensional $MGHC$ anti de Sitter space–time, where $\widetilde{M} \simeq (\Omega, \mathfrak{g})$ is obtained by a Wick rotation from a flat regular domain (Ω, g). Let (λ, μ) be the measured geodesic lamination on $\mathbb{H}^2$ associated to (Ω, g). Let's denote respectively by T_{cos} and $\mathcal{T}_{cos}$ the cosmological time of (Ω, g) and $(\Omega, \mathfrak{g})$.

Proposition 9.1 *The cosmological level* $(\Gamma, S_a^{\mathcal{T}_{cos}}, d_a^{\mathcal{T}_{cos}})_{a>0}$ *converge in the compact open topology, when a goes to* 0*, to* $(\Gamma, \Sigma, d_\Sigma)$ *the real tree dual to the measured geodesic lamination* (λ, μ)*.*

Proof Note that the space of cosmological gradient lines of $(\Omega_1, \mathfrak{g})$ is the same as the space of cosmological gradient lines of (Ω_1, g). Let's denote it by X_{cos}. For every $a > 0$, the distance $d_a^{\mathcal{T}_{cos}}$ of $S_a^{\mathcal{T}_{cos}}$ transported to X_{cos} is also denoted by $d_a^{\mathcal{T}_{cos}}$.

On the one hand, by Proposition 7.8 the distances $d_a^{\mathcal{T}_{cos}}$ (respectively $d_{\tan(a)}^{T_{cos}}$) converge in the compact open topology to the pseudo-distance $d_0^{\mathcal{T}_{cos}}$ (respectively $d_0^{T_{cos}}$) on X_{cos}.

On the other hand and for every $a > 0$ we have: $d_a^{\mathcal{T}_{cos}}(x, y) = \frac{1}{1+\tan^2(a)} d_{\tan(a)}^{T_{cos}}(x, y)$. Thus, the distances $d_a^{\mathcal{T}_{cos}}$ converge in the compact open topology, when a goes to 0, to the pseudo-distances $d_0^{\mathcal{T}_{cos}}$. So the Γ metric spaces $(\Gamma, S_a^{\mathcal{T}_{cos}}, d_a^{\mathcal{T}_{cos}})_{a>0}$ converge, when a goes to 0, in the Gromov equivariant topology to the real tree $(\Gamma, \Sigma, d_\Sigma)$. □

Proof of Theorem 2.5 in the anti Sitter case Thanks to Proposition 4.1, Proposition 9.1 and Remark 6.15, one can reproduce the proof of Theorem 2.1 without any modification and proves Theorem 2.5 in the anti de Sitter case. □

10 Asymptotic behavior in the Teichmüller space

The aim object of this part is to proof Theorem 2.7. Let $S \simeq \mathbb{H}^2/\Gamma$ be a closed hyperbolic surface. Denote by Teich(S) the Teichmüller space of S. On Teich(S) consider the Teichmüller metric d_{Teich}. As a K-bilipschitz diffeomorphism is K^2-quasiconformal we have the following result:

Proposition 10.1 *Let* g_1 *and* g_2 *two Riemmannian metric on* S *such that* (S, g_1) *is* K*-bilipchitz to* (S, g_2)*. Then* $d_{\text{Teich}}([g_1], [g_2]) \leq \log K$*.*

Let (λ, μ) be a measured geodesic lamination on S. Let M be the unique flat (or de Sitter, or the tight past of anti de Sitter) $MGHC$ space–time of dimension $2+1$ associated to (λ, μ). Let T_{cmc} and T_k be respectively the CMC time and the k time of $\widetilde{M}$. For each of the cosmological time, the k time and the CMC time, let us consider respectively the associated curves $a \mapsto \left[g_a^{T_{cos}}\right]$, $a \mapsto \left[g_a^{T_k}\right]$ and $a \mapsto \left[g_a^{T_{cmc}}\right]$ in the Teichmüller space Teich(S) of S.

Proposition 10.2 ***The flat case****. The curves* $a \mapsto \left[g_a^{T_k}\right]$ *and* $a \mapsto \left[g_a^{T_{cmc}}\right]$ *converge when a goes to infinity to the hyperbolic structure* $\mathbb{H}^2/\Gamma$*.*

Proof On the one hand and by Proposition 6.4, $g_a^{T_k}$ (respectively $g_a^{T_{cmc}}$) is C_a^4 bi-Lipschitz to $g_a^{T_{cos}}$ for every $a > 0$. Moreover by Proposition 5.4 and the Corollary 5.7, C_a goes to one when

a goes to ∞. Thus by Proposition 10.1 we have that $d_{\text{Teich}}\left(\left[g_a^{T_k}\right], \left[g_a^{T_{cos}}\right]\right)$ (respectively $d_{\text{Teich}}\left(\left[g_a^{T_{cmc}}\right], \left[g_a^{T_{cos}}\right]\right)$ goes to 0 when a goes to ∞.

On the other hand, by a result of Bonsante–Benedetti [12], the cosmological curve $a \mapsto \left[g_a^{T_{cos}}\right]$ corresponds to the grafting associated to the measured geodesic lamination (λ, μ). The grafting curve converges when times goes to $+\infty$, to the hyperbolic structure $\mathbb{H}^2/\Gamma$. Hence $\left[g_a^{T_k}\right]$ (respectively $\left[g_a^{T_{cmc}}\right]$) converges when a goes to infinity to to the hyperbolic structure $\mathbb{H}^2/\Gamma$. □

Proposition 10.3 *The de Sitter case. The limit points, when time goes to $+\infty$, of the curve $a \mapsto \left[g_a^{T_k}\right]$ are at bounded Teichmüller distance from the hyperbolic structure $\mathbb{H}^2/\Gamma$.*

Proof On the one hand and by Propositions 6.10, 5.9 we have that $d_{\text{Teich}}(\left[g_a^{T_k}\right], \left[g_a^{T_{cos}}\right]) \leq \log(3 - H_0)$ where H_0 is the constant given in the proof of Proposition 5.9.

On the other hand $\left[g_a^{T_{cos}}\right]$, goes to the grafting metric $\text{gra}_\lambda(S)$ when time goes to $+\infty$. Hence the limit points, when a goes to infinity, of $\left[g_a^{T_k}\right]$ stay at $\log(3 - H_0)$ Teichmüller distance from the grafting metric $\text{gra}_\lambda(S)$. □

Acknowledgements I want to thank Thierry Barbot for his considerable support and for many helpful discussions.

References

1. Andersson, L.: Constant mean curvature foliations of flat space–times. Comm. Anal. Geom. **10**(5), 1125–1150 (2002)
2. Andersson, L.: Constant mean curvature foliations of simplicial flat spacetimes. Comm. Anal. Geom. **13**(5), 963–979 (2005)
3. Andersson, L., Barbot, T., Béguin, F., Zeghib, A.: Cosmological time versus CMC time in spacetimes of constant curvature. Asian J. Math. **1**, 37–88 (2012)
4. Andersson, L., Galloway, G.J., Howard, R.: The cosmological time function. Class. Quantum Gravity **15**(2), 309–322 (1998)
5. Andersson, L., Moncrief, V., Tromba, A.J.: On the global evolution problem in 2 + 1 gravity. J. Geom. Phys. **23**(3–4), 191–205 (1997)
6. Barbot, T.: Globally hyperbolic flat space–times. J. Geom. Phys. **53**(2), 123–165 (2005)
7. Barbot, T., Béguin, F., Zeghib, A.: Prescribing Gauss curvature of surfaces in 3-dimensional spacetimes, application to the Minkowski problem in the Minkowski space. Ann. Inst. Fourier **61**(2), 511–591 (2011)
8. Barbot, T., Béguin, F., Zeghib, A.: Feuilletages des espaces temps globalement hyperboliques par des hypersurfaces à courbure moyenne constante. C. R. Math. Acad. Sci. Paris **336**(3), 245–250 (2003)
9. Barbot, T., Béguin, F., Zeghib, A.: Constant mean curvature foliations of globally hyperbolic spacetimes locally modelled on AdS_3. Geom. Dedicata **126**, 71–129 (2007)
10. Belraouti, M.: Convergence asymptotique des niveaux de temps quasi-concave dans un espace temps à courbure constante. Ph.D. thesis, (2013)
11. Belraouti, M.: Sur la géométrie de la singularité initiale des espaces-temps plats globalement hyperboliques. Annales de l'institut Fourier **64**(2), 457–466 (2015)
12. Benedetti, R., Bonsante, F.: Canonical Wick rotations in 3-dimensional gravity. Mem. Amer. Math. Soc., 198(926), viii+164, (2009)
13. Benedetti, R., Guadagnini, E.: Cosmological time in (2 + 1)-gravity. Nuclear Phys. B **613**(1–2), 330–352 (2001)
14. Bonsante, F.: Flat spacetimes with compact hyperbolic Cauchy surfaces. J. Differ. Geom. **69**(3), 441–521 (2005)
15. Bonsante, F., Fillastre, F.: The equivariant Minkowski problem in Minkowski space. arxiv:1405.4376

16. Bonsante, F., Seppi, A.: On Codazzi tensors on a hyperbolic surface and flat Lorentzian geometry. arxiv:1501.04922
17. Bridson, M.R., Haefliger, A.: Metric Spaces of Non-Positive Curvature, Volume 319 of Grundlehren der Mathematischen Wissenschaften. Springer, Berlin (1999)
18. Geroch, R., Kronheimer, E.H., Penrose, R.: Ideal points in space–time. Proc. R. Soc. Lond. Ser. A **327**, 545–567 (1972)
19. Geroch, R.: Domain of dependence. J. Math. Phys. **11**, 437–449 (1970)
20. Hawking, S.W.: The occurrence of singularities in cosmology. I. Proc. R. Soc. Ser. A **294**, 511–521 (1966)
21. Hawking, S.W.: The occurrence of singularities in cosmology. II. Proc. R. Soc. Ser. A **295**, 490–493 (1966)
22. Hawking, S.W., Ellis, G.F.R., The large scale structure of space-time. Cambridge University Press, London, : Cambridge Monographs on Mathematical. Physics, No 1. (1973)
23. Hawking, S.W., Penrose, R.: The singularities of gravitational collapse and cosmology. Proc. R. Soc. Lond. Ser. A **314**, 529–548 (1970)
24. Mess, G.: Lorentz spacetimes of constant curvature. Geom. Dedicata **126**, 3–45 (2007)
25. Moncrief, V.: Reduction of the Einstein equations in $2 + 1$ dimensions to a Hamiltonian system over Teichmüller space. J. Math. Phys. **30**(12), 2907–2914 (1989)
26. Morgan, J.W., Shalen, P.B.: Valuations, trees, and degenerations of hyperbolic structures. I. Ann. Math. (2) **120**(3), 401–476 (1984)
27. O'Neill, B.: Semi-Riemannian Geometry: With Applications to Relativity. Academic Press, London (1983)
28. Otal, J. P.: Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3. Astérisque, (235):x+159, (1996)
29. Paulin, F.: Topologie de Gromov équivariante, structures hyperboliques et arbres réels. Invent. Math. **94**(1), 53–80 (1988)
30. Paulin, F.: The Gromov topology on R-trees. Topol. Appl. **32**(3), 197–221 (1989)
31. Penrose, R.: Gravitational collapse and space–time singularities. Phys. Rev. Lett. **14**, 57–59 (1965)
32. Scannell, K.P.: Flat conformal structures and the classification of de Sitter manifolds. Comm. Anal. Geom. **7**(2), 325–345 (1999)
33. Schmidt, B.G.: A new definition of singular points in general relativity. Gen. Relat. Gravit, **1**(3):269–280, (1970/71)
34. Treibergs, A.E.: Entire spacelike hypersurfaces of constant mean curvature in Minkowski space. Invent. Math. **66**, 39–56 (1982)