

Locally convex hypersurfaces immersed in $\mathbb{H}^n \times \mathbb{R}$

Inês S. de Oliveira Padilha¹ · S. J. Paul A. Schweitzer²

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Abstract We prove a theorem of Hadamard–Stoker type: a connected locally convex complete hypersurface immersed in $\mathbb{H}^n \times \mathbb{R}$ ($n \geq 2$), where \mathbb{H}^n is n -dimensional hyperbolic space, is embedded and homeomorphic either to the n -sphere or to \mathbb{R}^n . In the latter case it is either a vertical graph over a convex domain in \mathbb{H}^n or has what we call a simple end.

Keywords Hypersurfaces · Convexity · Space product · Simple end · Geometry and topology

Mathematics Subject Classification 53-02 · 57R42

1 Introduction

Let Σ be a complete hypersurface immersed in $\mathbb{H}^n \times \mathbb{R}$, the product of n -dimensional hyperbolic space \mathbb{H}^n with the line, and let $\pi : \mathbb{H}^n \times \mathbb{R} \rightarrow \mathbb{H}^n$ be the projection on the first factor. A point θ in the sphere at infinity $\mathbb{S}_{\infty}^{n-1}$ of \mathbb{H}^n is a **simple end** of Σ if θ is the unique point of accumulation of $\pi(\Sigma)$ in $\mathbb{S}_{\infty}^{n-1}$ and every complete totally geodesic hyperplane $Q \subset \mathbb{H}^n$ either has θ as an accumulation point or else meets $\pi(\Sigma)$ in a compact set. We prove the following.

Main Theorem 1 *Let Σ be a complete, connected hypersurface immersed in $\mathbb{H}^n \times \mathbb{R}$ with positive definite second fundamental form, $n \geq 2$. Then Σ is embedded and homeomorphic*

✉ Inês S. de Oliveira Padilha
in.math@hotmail.com
S. J. Paul A. Schweitzer
paul37sj@gmail.com

¹ Departamento de Matemática, Universidade Federal do Amazonas - UFAM, Av. General Rodrigo Octávio Jordão Ramos 3000 Campus Universitário Senador Arthur Virgílio Filho, Manaus, AM 69077-000, Brazil

² Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro - PUC-Rio, Rua Marquês de São Vicente, 225, Rio de Janeiro, RJ 22453-900, Brazil

either to the n -sphere or to Euclidean space \mathbb{R}^n . In the latter case, it is either a vertical graph over an open convex set or has a simple end. When $n = 1$ the theorem holds under the additional hypothesis that Σ is embedded.

The case $n = 2$ was proven by Espinar et al. [8], so our paper is an extension of their work to higher dimensions. We prove the theorem by induction on n , beginning with the easy case $n = 1$, when Σ is assumed to be embedded. Our paper reproves the case $n = 2$. We use many ideas of [8]; most of our proof does not depend on their result, but in one part of the proof for $n = 2$ (Case 3 of Sect. 4) we reproduce part of their proof.

Historical background In 1897, Hadamard [11] proved a theorem about compact, locally strictly convex surfaces in the Euclidean space \mathbb{R}^3 , showing that such surfaces are embedded and homeomorphic to the sphere. Since then many generalizations have adapted the assumptions about the curvature and considered new spaces in which these surfaces could be immersed in order to obtain analogous results.

Important contributions were made by Stoker [15], Chern and Lashof [2], Sacksteder [14], do Carmo and Lima [6], do Carmo and Warner [7], Currier [3], Alexander [1], and Tribuzy [16].

In 2009, Espinar et al. [8] extended the Hadamard–Stoker theorem for surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ assuming that such a surface is connected and complete with all principal curvatures positive. They proved that such a surface is properly embedded and homeomorphic to the sphere if it is closed or to the plane \mathbb{R}^2 if not. In the second case, Σ is a graph over a convex domain in $\mathbb{H}^2 \times \{0\}$ or Σ has a simple end.

Espinar and Rosenberg [9] proved in 2010 that if Σ is a locally strictly convex, connected hypersurface properly immersed in $M^n \times \mathbb{R}$, where M^n is a $1/4$ -pinched manifold, then Σ is properly embedded and homeomorphic to S^n or \mathbb{R}^n . In the second case, Σ has a *top* or *bottom* end, where Σ has a *top* (respectively, *bottom*) end E if for any divergent sequence $\{p_n\} \subset E$ the height function $h : \Sigma \rightarrow E$ goes to $+\infty$ (respectively $-\infty$).

Also in 2010, Espinar and the first author [10] showed a Hadamard–Stoker–Currier type theorem for surfaces immersed in a 3-dimensional Riemannian manifold $\mathcal{M}(\kappa, \tau)$ that fibers over a 2-dimensional Riemannian manifold \mathbb{M}^2 so that the fibers are the trajectories of a unit Killing vector field. More precisely, \mathbb{M}^2 is required to be a strict Hadamard surface, i.e., \mathbb{M}^2 has Gaussian curvature κ less than a negative constant, and τ is the curvature of the bundle.

Open questions There are other spaces to be studied, for example: $S^n \times \mathbb{R}$, $\mathbb{M} \times \mathbb{R}$, where \mathbb{M} is a Hadamard manifold of dimension n , Heisenberg spaces of dimension n , among others. It possible that results of Hadamard–Stoker type are valid in these spaces.

Preliminary results and the case $n = 1$ are treated in Sect. 2. The proof of the Main Theorem in the case of a vertical graph is given in Sect. 3 and the remaining cases of the theorem are proven in Sect. 4. For Case 3 in Sect. 4 we reproduce the argument of [8]. Examples of hypersurfaces with simple ends are given in Sect. 5.

This paper is based on the doctoral thesis [4] of the first author at the Pontifical Catholic University of Rio de Janeiro under the direction of the second author.

2 Preliminary results and the case $n = 1$

Let \mathbb{H}^k be the k -dimensional hyperbolic space, with a Riemannian metric of constant curvature -1 (see [5, 13] for details of Riemannian geometry). In the Poincaré disk model on the open

unit ball $\mathbb{B}^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid |x| < 1\}$, where $|x|$ is the Euclidean norm given by $|x|^2 = \sum_{i=1}^k x_i^2$, the Riemannian metric is defined to be

$$ds^2 = \frac{4 \sum_{i=1}^k dx_i^2}{(1 - |x|^2)^2}.$$

In this model the sphere at infinity, \mathbb{S}_∞^{k-1} , is the unit sphere in \mathbb{R}^k , so that $\mathbb{H}^k \cup \mathbb{S}_\infty^{k-1}$ is a compact k -dimensional ball. If Q is a totally geodesic hyperplane in \mathbb{H}^k , we say that $P = Q \times \mathbb{R}$, which is also totally geodesic, is a **vertical hyperplane** in $\mathbb{H}^k \times \mathbb{R}$. If γ is a complete geodesic in $\mathbb{H}^k \times \{0\}$ parametrized by $t \in \mathbb{R}$, we let $P_\gamma^k(t)$ be the vertical hyperplane orthogonal to γ at the point $\gamma(t)$, and then these vertical hyperplanes $P_\gamma^k(t)$ form a foliation of $\mathbb{H}^k \times \mathbb{R}$. Let $\pi : \mathbb{H}^k \times \mathbb{R} \rightarrow \mathbb{H}^k$ denote the projection on the first factor.

The following proposition will be used in the inductive step of the proof of the Main Theorem when $M = \mathbb{H}^k \times \mathbb{R}$, but we state it in a more general context.

Proposition 2.1 *Let M be a $(k + 1)$ -dimensional Riemannian manifold and Σ^k a hypersurface immersed in M with strictly positive second fundamental form, $k \geq 2$, and let P be a totally geodesic hypersurface in M . If Σ^k and P intersect transversally, then every connected component Σ^{k-1} of $\Sigma^k \cap P$ is a $(k - 1)$ -dimensional hypersurface in P with second fundamental form $II(\Sigma^{k-1}) > 0$.*

Proof Let $\gamma(t)$ be a curve in Σ^{k-1} , where t is parametrized by arc length. As P is totally geodesic, we have $\bar{\nabla}_{\gamma'}(\gamma') = \nabla_{\gamma'}^P(\gamma')$, where ∇^P and $\bar{\nabla}$ are the connections in P and M respectively. Let $N_{\Sigma^{k-1}, P}$ and N_{Σ^k} be the unit normal vectors to Σ^{k-1} in P and to Σ^k in M , respectively. We want $\langle \bar{\nabla}_{\gamma'}(\gamma'), N_{\Sigma^{k-1}, P} \rangle$ to be positive. Note that $\langle \bar{\nabla}_{\gamma'}(\gamma'), N_{\Sigma^k} \rangle$ is positive, because by hypothesis the second fundamental form of Σ^n is strictly positive.

Writing $N_{\Sigma^k}(\gamma(t)) = \bar{N} + N^\perp$, where \bar{N} is tangent to P and N^\perp is orthogonal to P , and taking the inner product with $\bar{\nabla}_{\gamma'}(\gamma')$, we obtain

$$\begin{aligned} 0 < \langle \bar{\nabla}_{\gamma'}(\gamma'), N_{\Sigma^k} \rangle &= \langle \bar{\nabla}_{\gamma'}(\gamma'), \bar{N} + N^\perp \rangle \\ &= \langle \bar{\nabla}_{\gamma'}(\gamma'), \bar{N} \rangle. \end{aligned}$$

Note that \bar{N} is orthogonal to Σ^{k-1} and tangent to P . Hence \bar{N} is a multiple of $N_{\Sigma^{k-1}, P}$, so we can write

$$\begin{aligned} \langle \nabla_{\gamma'}^P(\gamma'), \bar{N} \rangle &= \langle \nabla_{\gamma'}^P(\gamma'), |\bar{N}| N_{\Sigma^{k-1}, P} \rangle \\ &= |\bar{N}| \langle \nabla_{\gamma'}^P(\gamma'), N_{\Sigma^{k-1}, P} \rangle, \end{aligned}$$

and therefore $\langle \nabla_{\gamma'}^P(\gamma'), N_{\Sigma^{k-1}, P} \rangle > 0$, as claimed. □

Next we begin the proof of the Main Theorem. The proof is by induction on the dimension n of Σ^n . The proof for the case $n = 1$ is given in this section, and the proof for $n = k > 1$, when the theorem is supposed true for $n = k - 1$, is given in the next two sections.

The Proof for $n = 1$. When $n = 1$, we assume the additional hypothesis that the curve Σ^1 is embedded in $\mathbb{H}^1 \times \mathbb{R}$, which is isometric to \mathbb{R}^2 . Then it is well-known that it is a circle or else a graph with respect to some vertical direction. If the embedded curve Σ^1 is not a vertical graph over an open interval in \mathbb{H}^1 , which is one of the possible conclusions of the theorem, then there must be a point $p_0 \in \Sigma^1$ where the tangent line is vertical. If there is a second such point, then Σ^1 closes up to a circle, and if p_0 is the only such point, then both ends of $\pi(\Sigma^1)$ must converge to one of the two points at infinity of \mathbb{H}^1 , which gives a simple end. This completes the proof of the theorem for $n = 1$. □

3 The case of a vertical graph

In the previous section the Main Theorem was proven for the case $n = 1$. We complete the proof in this section and the next one. In both sections we suppose that the theorem holds for $n = k - 1 \geq 1$ and we assume the hypotheses of the Main Theorem for the case $n = k$ in order to prove the inductive step. Thus we assume that Σ^k is a complete connected k -dimensional manifold immersed in $\mathbb{H}^k \times \mathbb{R}$, which has the product Riemannian metric, and we suppose that the second fundamental form of Σ^k is positive definite.

For the rest of this section we also assume the following.

Hypothesis 3.1 *No vertical hyperplane is tangent to Σ^k .*

We shall prove the Main Theorem under this additional Hypothesis by showing that in this case Σ^k is the graph of a smooth function $f : \Omega \rightarrow \mathbb{R}$ where Ω is an open convex domain in \mathbb{H}^k . Note that the Hypothesis implies that every vertical hyperplane P is transverse to Σ^k .

Lemma 3.1 *Hypothesis 3.1 implies that if P is a vertical hyperplane in $\mathbb{H}^k \times \mathbb{R}$, then each connected component Σ^{k-1} of the intersection $\Sigma^k \cap P$ is a graph over a convex domain in $\pi(P)$.*

Proof By Proposition 2.1, Σ^{k-1} is a $(k - 1)$ -dimensional surface with strictly positive second fundamental form in P . If $k = 2$, Σ^1 is a curve in $\mathbb{H}^1 \times \mathbb{R}$ with no vertical tangent, so beginning at any point (x_0, y_0) in Σ^1 , by continuation in both directions we find that Σ^1 is a graph over an open interval in \mathbb{H}^1 . Therefore Σ^1 is embedded. Hence we can apply the induction hypothesis for $k = 1$ as well as for $k > 2$. Thus Σ^{k-1} must be homeomorphic to \mathbb{S}^{k-1} or \mathbb{R}^{k-1} , and in the latter case it must have a simple end or be a vertical graph over a convex domain.

Now if Σ^{k-1} is homeomorphic to \mathbb{S}^{k-1} , then there exists a point in \mathbb{S}^{k-1} with a $(k - 1)$ -dimensional vertical tangent plane, which means that Σ^k has a vertical tangent k -plane at this point, contradicting Hypothesis 3.1, so this case is excluded. If Σ^{k-1} is homeomorphic to \mathbb{R}^{k-1} and has a simple end θ in \mathbb{S}_∞^{k-1} , the sphere at infinity of $\pi(P)$, let $\beta(t)$ be a complete horizontal geodesic in $\pi(P) \times \{0\} \subset P$ which converges to the point $(\theta, 0)$ as $t \rightarrow \infty$. Consider the foliation of P by vertical $(k - 1)$ -hyperplanes $P_\beta^{k-1}(t)$ orthogonal to β , where $P_\beta^{k-1}(t)$ meets β at $\beta(t)$. If \bar{t} is the smallest value of t such that $\Sigma^{k-1} \cap P_\beta^{k-1}(t)$ is non-empty, then the vertical hyperplane $P_\beta^{k-1}(\bar{t})$ will be tangent to Σ^{k-1} , so Σ^k will also have a tangent vertical hyperplane, contradicting Hypothesis 3.1. Thus Σ^{k-1} must be a vertical graph over a convex domain in $\pi(P)$. □

Lemma 3.2 *Let $\{P_\gamma^k(t)\}$ be the foliation of $\mathbb{H}^k \times \mathbb{R}$ by vertical hyperplanes orthogonal to a complete geodesic γ in $\mathbb{H}^k \times \{0\}$ and let $\Sigma^{k-1}(0)$ be a component of $\Sigma^k \cap P_\gamma(0)$. Let $\Sigma^{k-1}(t) \subset \Sigma^k \cap P_\gamma(t)$ be the continuous variation of $\Sigma^{k-1}(0)$ as t varies. Then $\Sigma^{k-1}(t)$ cannot become disconnected for any t .*

Proof Note that $\Sigma^{k-1}(t)$ is well defined by transversality. By Lemma 3.1, every component of $\Sigma^{k-1}(t)$ is the graph of a function defined on a convex open domain in $\pi(P_\gamma(0))$, which is isometric to \mathbb{H}^{k-1} . By transversality the set of t with $\Sigma^{k-1}(t)$ connected is open. Suppose, to find a contradiction, that $\Sigma^{k-1}(t)$ is not connected for some $t > 0$. Then there will be a smallest positive \bar{t} such that $\Sigma^{k-1}(\bar{t})$ is not connected. Take $x_{\bar{t}}$ and $y_{\bar{t}}$ to be points in distinct components of $\Sigma^{k-1}(\bar{t})$. By transversality there exist $\delta > 0$ and continuous curves $t \mapsto x_t$ and $t \mapsto y_t$ defined for $t \in [\bar{t} - \delta, \bar{t}]$ with $x_t, y_t \in \Sigma^{k-1}(t)$. For $t \in [\bar{t} - \delta, \bar{t})$, $\Sigma^{k-1}(t)$ is

connected and thus a graph over a convex open domain in $\pi(P_\gamma(t))$. Let α_t be the geodesic in $\pi(P_\gamma(t))$ joining $\pi(x_t)$ and $\pi(y_t)$ and take $A_t = \pi^{-1}(\alpha_t) \cap \Sigma^{k-1}(t)$, which is a graph over α_t and a curve joining x_t to y_t in $\Sigma^{k-1}(t)$. The limit of A_t as t tends to \bar{t} is a curve $A_{\bar{t}}$ in $\Sigma^{k-1}(\bar{t})$, which is complete, since Σ^k is complete, but $A_{\bar{t}}$ cannot be connected since it contains $x_{\bar{t}}$ and $y_{\bar{t}}$ which are in different components. It follows that $A_{\bar{t}}$ must diverge vertically to $-\infty$ or $+\infty$, but since $A_{\bar{t}}$ has strictly positive curvature that is impossible. \square

In view of the previous lemma, the union $\cup \Sigma^{k-1}(t)$ is an open and closed set in Σ^k , so it must be the whole connected set Σ^k . Consequently Σ^k is a vertical graph over a set Ω in \mathbb{H}^k . Since Ω is diffeomorphic to the k -dimensional manifold Σ^k under the projection π , it is open in \mathbb{H}^k . Now given any two points $x, y \in \Omega$, let P be a vertical hyperplane containing both $(x, 0)$ and $(y, 0)$, and apply the argument of Lemma 3.2 to see that $\Sigma^k \cap P$ must be connected. Then $\Sigma^k \cap P$ is a vertical graph over a convex domain Ω_P in $\pi(P)$. The geodesic from x to y lies in $\Omega_P \subset \Omega$, so Ω is convex. Now the convex open set Ω in \mathbb{H}^k is homeomorphic to \mathbb{R}^k , so Σ^k is also homeomorphic to \mathbb{R}^k . This completes the proof of the Main Theorem under Hypothesis 3.1.

4 The proof for the remaining cases

In this section we complete the proof of the Main Theorem by proving it in the case that the previous Hypothesis 3.1 does not hold, i.e., under the following assumption.

Hypothesis 4.1 *There is a vertical hyperplane P_0 that is tangent to Σ^k at a point $p_0 \in \Sigma^k$.*

Without loss of generality, we suppose that $p_0 \in \mathbb{H}^k \times \{0\}$. In view of the hypothesis that the hypersurface Σ^k has strictly positive curvature, there is a neighborhood \mathcal{U}_0 of p_0 in Σ^k that lies entirely on one side of P_0 , except at the point p_0 ; we shall call this the external side (see Fig. 1). Let γ be the geodesic in $\mathbb{H}^k \times \{0\}$ orthogonal to P_0 (and therefore also to Σ^k) at p_0 , parametrized by arclength, with $\gamma(0) = p_0$, oriented so that $\gamma(0, \infty)$ is on the external side of P_0 . Let $P_\gamma(t)$ be the vertical k -plane orthogonal to γ at $\gamma(t)$, and note that these vertical planes for all $t \in \mathbb{R}$ form a smooth foliation of $\mathbb{H}^k \times \mathbb{R}$. Denote by $\Sigma^{k-1}(t)$ the connected component of $\Sigma^k \cap P_\gamma(t)$ that is the continuation of $\Sigma^{k-1}(0) = \{p_0\} \subset \Sigma^k \cap P_\gamma(0)$. We do not exclude the possibility that there may be other components of $\Sigma^k \cap P_\gamma(t)$, but we only

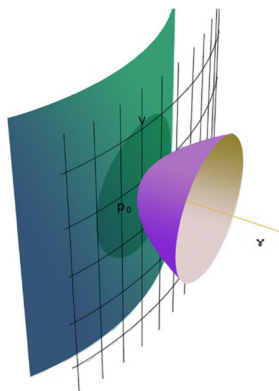


Fig. 1 Near p_0 , Σ^k lies on the external side of $P_\gamma(\bar{t})$

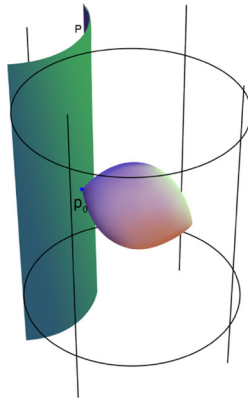


Fig. 2 Σ^k homeomorphic to the k -sphere \mathbb{S}^k

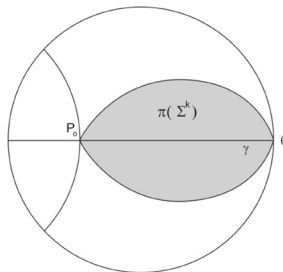


Fig. 3 Projection of Σ^k on \mathbb{H}^k in Case 2

consider $\Sigma^{k-1}(t)$. Note that for $t > 0$ close to 0, $\Sigma^{k-1}(t)$ is homeomorphic to the $(k - 1)$ -sphere \mathbb{S}^{k-1} . Furthermore, if $\Sigma^{k-1}(t)$ is compact and Σ^k is transverse to $P_\gamma(t)$ along $\Sigma^{k-1}(t)$ for all t in an interval $(0, t_0)$, the sets $\Sigma^{k-1}(t)$ vary continuously and are all diffeomorphic to \mathbb{S}^{k-1} (see [12], Theorem 3.1).

Now there are four cases to be considered.

Case 1. The set $\Sigma^{k-1}(t)$ is compact and Σ^k is transverse to $P_\gamma(t)$ along $\Sigma^{k-1}(t)$ for $0 < t < \bar{t}$, but Σ^k is not transverse to $P_\gamma(\bar{t})$ at some point $p_1 \in \Sigma^{k-1}(\bar{t})$.

As in the case of p_0 , the point p_1 has a neighborhood \mathcal{U}_1 in Σ^k such that $\mathcal{U}_1 \setminus \{p_1\}$ lies entirely on one side of $P_\gamma(\bar{t})$. Since $\Sigma^{k-1}(\bar{t})$ is the continuation of $\Sigma^{k-1}(t)$ with $t < \bar{t}$, $\mathcal{U}_1 \setminus \{p_1\}$ must lie on the side of $P_\gamma(\bar{t})$ with $t < \bar{t}$. The sets $\Sigma^{k-1}(t)$ for $0 < t < \bar{t}$ are all diffeomorphic to \mathbb{S}^{k-1} and their union with the two points p_0 and p_1 is homeomorphic to the sphere \mathbb{S}^k . This union is open and closed in Σ^k , so by connectedness it must coincide with Σ^k , which is therefore a topological k -sphere (see Fig. 2). This completes the proof in Case 1.

In the remaining cases we exclude Case 1, so if $\Sigma^{k-1}(t)$ is compact for all t in some interval $(0, t_0)$, then Σ^k is transverse to the planes $P_\gamma(t)$ at all points in these sets $\Sigma^{k-1}(t)$.

Case 2. The intersection $\Sigma^{k-1}(t)$ is compact and non-empty for all $t > 0$ (Fig. 3).

If $\Sigma^{k-1}(t) \subset \Sigma^k \cap P_\gamma(t)$ remains compact and non-empty for all $t > 0$, then the sets $\Sigma^{k-1}(t)$ for $t > 0$ are $(k - 1)$ -spheres that are embedded in $P_\gamma(t)$ by the inductive hypothesis. Hence the union of these sets with the point p_0 must be all of the connected hypersurface Σ^k ,

which will also be embedded and homeomorphic to \mathbb{R}^k . If we take θ in the sphere at infinity $\mathbb{S}_\infty^{k-1} = \partial\mathbb{H}^k$ such that $(\theta, 0)$ is the limit point of $\gamma(t)$ as $t \rightarrow \infty$, then $\partial_\infty\pi(\Sigma^k) = \{\theta\}$, since θ is the only point of \mathbb{S}_∞^{k-1} that is on the external side of all the planes $\pi(P_\gamma(t))$.

Furthermore, if Q is any complete totally geodesic hypersurface in \mathbb{H}^k which does not have θ as a point of accumulation, then $\pi(\Sigma^k) \cap Q$ cannot have any point of accumulation at infinity. Hence it must be a closed and bounded set in Q , so it is compact. Therefore in this case θ is a simple end of Σ^k .

Case 3. The intersection $\Sigma^{k-1}(t)$ remains compact and non-empty for $0 \leq t < \bar{t}$ but becomes empty for every $t > \bar{t}$, for some $\bar{t} > 0$.

Note first of all that Σ^k cannot intersect $P_\gamma(\bar{t})$ transversely at any point of $\Sigma^{k-1}(\bar{t})$, for otherwise $\Sigma^{k-1}(t)$ would be non-empty for values of t slightly greater than \bar{t} , contrary to the hypothesis of Case 3. Hence any point in $\Sigma^{k-1}(\bar{t})$ would have a vertical tangent plane, a situation which has already been treated in Case 1, so we may suppose that $\Sigma^{k-1}(\bar{t})$ is empty. Consequently $\cup_{t < \bar{t}} \Sigma^{k-1}(t)$ is both open and closed in Σ^k , so it must be all of Σ^k . As in Case 2, we see that Σ^k , which is the union of the point p_0 and the topological $(k - 1)$ -spheres $\Sigma^{k-1}(t)$ for $0 < t < \bar{t}$, is homeomorphic to \mathbb{R}^k .

Now take a sequence of points $p_n \in \Sigma^{k-1}(t_n)$ such that $t_n \rightarrow \bar{t}$. Since $\bar{\mathbb{H}}^k = \mathbb{H}^k \cup \partial\mathbb{H}^k$ is compact, there must be a subsequence of $\{\pi(p_n)\}$ that converges to a point θ in $\bar{\mathbb{H}}^k$. Since we are supposing that $\Sigma^{k-1}(\bar{t})$ is empty, θ must be in $\mathbb{S}_\infty^{k-2}(\bar{t}) = \partial\pi(P_\gamma(\bar{t}))$ and must be an accumulation point of $\pi(\Sigma^k)$ at infinity. To complete the proof in this case, we must show that θ is the only accumulation point and that it is a simple end.

The proof will be divided into two cases depending of the dimension k .

The Proof for $k = 2$. In this case we will use the arguments that can be found in [8].

We can assume that $P_\gamma(\bar{t}) = \alpha \times \mathbb{R}$, where α is the geodesic joining the points α^- and α^+ and we consider the 2-plane vertical $Q = \gamma \times \mathbb{R}$. We will take \tilde{C} the component of $Q \cap \Sigma^2$ containing p_0 . This component is compact, otherwise it would intersect the line $Q \cup P_\gamma(\bar{t})$ in two points, which may not occur. Based on this we consider the disk \tilde{D} bounded by \tilde{C} on Σ^2 and we denote $Q_\alpha(t)$ the foliation by vertical planes along α , with $Q_\alpha(0) = Q$. Without loss of generality we can assume that there is $t_0 < 0$ satisfying $Q_\alpha(t_0)$ touches \tilde{D} one side of \tilde{D} by compactness. Let $q \in \tilde{D} \cap Q_\alpha(t_0)$ be the point where they touch. We now analyse the variation of $\tilde{C}(t)$ of q on $\Sigma^2 \cap Q_\alpha(t)$ from $t = t_0$ to infinity. $\tilde{C}(t)$ is a convex embedded curve for t in a maximal interval (t_0, \bar{t}_0) with $0 < \bar{t}_0 \leq \infty$. Hence, Σ^2 is foliated by the $\tilde{C}(t)$, $\tilde{C} = \tilde{C}(0) = Q \cap \Sigma^2$ and $\alpha^- \notin \partial\pi(\Sigma^2)$ because Σ^2 is on side of $Q_\alpha(t_0)$. We now show that $\partial\pi(\Sigma^2) = \{\alpha^+\}$. For this we denote $\gamma(\theta)$ the complete horizontal geodesic starting in p_0 and making an angle θ with γ at p_0 . Assume $\gamma(\theta)$ enters the side of Q containing α^+ , for $0 < \theta < \pi/2$. Let $\tilde{\theta}$ be the value of θ such that $\gamma(\tilde{\theta})$ is asymptotic to α^+ . Consider $Q(\theta) = \gamma(\theta) \times \mathbb{R}$ and for each $0 \leq \theta < \tilde{\theta}$, we have $\Sigma^2 \cap Q(\theta)$ is one connected embedded compact curve $C'(\theta)$. the proof of this is the same as the previous one for \tilde{C} . Note that each $C'(\theta)$ is non empty since $p_0 \in C'(\theta)$ e $C'(\theta)$ can not be compact, otherwise Σ^2 could not be asymptotic to the plane $P_\gamma(\bar{t})$, a contradiction. Furthermore note that $C'(\theta)$ is compact, $\tilde{\theta} < \theta < \pi/2$ because $\Sigma^2 = \cup_{0 \leq t < \bar{t}} C(t)$ and $C'(\theta) \subset \tilde{D}$, $-\pi/2 < \theta < 0$, and \tilde{D} is compact. So we can conclude that Σ has a simple end.

The Proof for $k > 2$. Suppose there were another accumulation point of $\pi(\Sigma^k)$, say $\tilde{\theta} \in \mathbb{S}_\infty^{k-1}$. Then $\tilde{\theta}$, like θ and any other accumulation points of $\pi(\Sigma^k)$, must be in $\mathbb{S}_\infty^{k-2}(\bar{t})$, since there are no accumulation points for $t < \bar{t}$ and $\Sigma^k \cap P_\gamma(t)$ is empty for $t > \bar{t}$. Choose the parametrization of $\mathbb{S}_\infty^{k-2}(\bar{t})$ in the disk model of \mathbb{H}^k so that $\tilde{\theta}$ is the antipode of θ . Let μ be the totally geodesic 2-plane in $\mathbb{H}^k \times \{0\}$ that contains both p_0 and the geodesic from $(\theta, 0)$

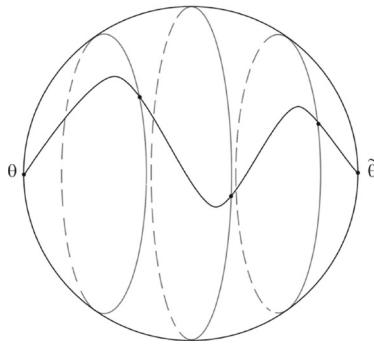


Fig. 4 The curve of accumulation points in $\partial P_\gamma(\bar{t})$, from θ to $\tilde{\theta}$

to $(\tilde{\theta}, 0)$. Take $\xi \subset \mathbb{H}^k \times \mathbb{R}$ to be the totally geodesic vertical $(k - 1)$ -plane orthogonal to μ that meets μ at the single point p_0 . Consider the 1-parameter family of vertical k -planes $\{P(\varphi) = Q(\varphi) \times \mathbb{R}\}$ containing ξ for $\varphi \in [-\pi/2, 3\pi/2]$, parametrized injectively so that θ is a point at infinity of $Q(0)$, $\tilde{\theta}$ is a point at infinity of $Q(\pi)$, and the tangent plane at p_0 is not among the planes $P(\varphi)$. Now the intersection

$$\Sigma^{k-1}(\varphi) = \Sigma^k \cap P(\varphi)$$

is a $(k - 1)$ -surface in $P(\varphi)$ which satisfies the hypotheses of the Main Theorem, so by the inductive hypothesis it must be homeomorphic to \mathbb{S}^{k-1} or \mathbb{R}^{k-1} and in the latter case $\Sigma^{k-1}(\varphi)$ it is either a vertical graph over a convex domain in \mathbb{H}^{k-1} or it has a simple end. However, $\Sigma^{k-1}(\varphi)$ can not be a vertical graph, because this $(k - 1)$ -surface contains the point p_0 where there is a vertical tangent hyperplane. Moreover, for each parameter φ with $\varphi < 0$ or $\varphi > \pi$ the intersection $\Sigma^{k-1}(\varphi)$ is a bounded complete $(k - 1)$ -surface contained in $\cup_{0 \leq t \leq \bar{t}_0} \Sigma^{k-1}(t)$ for some $t_0 < \bar{t}$ and is therefore compact, with no accumulation points at infinity.

If for some $\bar{\varphi}$ with $0 < \bar{\varphi} < \pi$, $\Sigma^{k-1}(\bar{\varphi})$ is compact, then by the inductive hypothesis it will be homeomorphic to the sphere \mathbb{S}^{k-1} and must bound a closed k -dimensional ball D in Σ^k , which is homeomorphic to \mathbb{R}^k . The ball D must coincide either with $\Sigma^k \cap \cup_{\varphi \leq \bar{\varphi}} Q(\varphi)$ or else with $\Sigma^k \cap \cup_{\varphi \geq \bar{\varphi}} Q(\varphi)$; in the first case, θ cannot be a limit point of Σ^k , and in the second case, $\tilde{\theta}$ cannot be a limit point of Σ^k , contradicting the choice of θ and $\tilde{\theta}$. Hence for $0 < \varphi < \pi$, $\Sigma^{k-1}(\varphi)$ must be noncompact and it must have a simple end, which we denote $\theta(\varphi)$.

The sets $\overline{Q(\varphi)} \cap \mathbb{S}_\infty^{k-2}(\bar{t})$ form a singular foliation of $\mathbb{S}_\infty^{k-2}(\bar{t})$ by $(k - 3)$ -spheres for $0 < \varphi < \pi$ with two singular points $\theta = \theta(0)$ and $\tilde{\theta} = \theta(\pi)$. Thus, each leaf $\overline{Q(\varphi)} \cap \mathbb{S}_\infty^{k-2}(\bar{t})$ contains a single accumulation point $\theta(\varphi)$, the simple end if $0 < \varphi < \pi$, and the points θ and $\tilde{\theta}$ for $\varphi = 0$ or π , when the set $\overline{Q(\varphi)} \cap \mathbb{S}_\infty^{k-2}(\bar{t})$ consists of a single point.

The set formed by these points is the graph of a function $\theta : [0, \pi] \rightarrow \mathbb{S}_\infty^{k-2}(\bar{t})$, and it is a closed set in the sphere $\mathbb{S}_\infty^{k-2}(\bar{t}) = \partial_\infty \pi(P_\gamma(\bar{t}))$. By an elementary fact of general topology, it follows that the function θ is continuous. Therefore the set of all the accumulation points at infinity of $\pi(\Sigma^k)$ form a continuous curve in $\mathbb{S}_\infty^{k-2}(\bar{t})$ with extremities θ and $\tilde{\theta}$ (see Fig. 4).

Now taking another point of accumulation at infinity between θ and $\tilde{\theta}$, say θ' , we can consider the 2-plane μ' generated by p_0, θ and θ' and let ξ' be the $(k - 1)$ -plane orthogonal to μ' passing through p_0 . Repeating the previous argument for the new foliation by vertical hyperplanes containing the plane ξ' shows that θ and this other point θ' should be extremities

of another curve which also consists of all the points of accumulation at infinity of $\pi(\Sigma^k)$. This is absurd since $\tilde{\theta}$ and nearby limit points are excluded from this curve. The contradiction shows that θ is the only accumulation point at infinity. As in Case 2, every vertical hyperplane whose projection in \mathbb{H}^k does not have θ as a limit point must meet $\pi(\Sigma^k)$ in a compact set, so Σ^k has a simple end at θ .

Case 4. The intersection $\Sigma^{k-1}(t)$ becomes non-compact for some $\bar{t} > 0$.

First, note that this is the only remaining case to complete the proof of the Main Theorem.

By transversality the set of positive t 's for which the intersection of Σ^k with $P_\gamma(t)$ is compact along $\Sigma^{k-1}(t)$ is open in the line. (Recall that we are considering only the continuation of Σ^k starting from p_0 , without considering other possible components.) It follows that there is a first value of t , say $\bar{t} > 0$, such that $\Sigma^{k-1}(\bar{t})$ is not compact. Note that the union of the sets $\Sigma^{k-1}(t)$ for $t \leq \bar{t}$ is embedded. Then $\Sigma^{k-1}(\bar{t})$ is the limit of $\Sigma^{k-1}(t)$ as t approaches \bar{t} from below.

Lemma 4.1 *Let $\{Q_t\}$ be a foliation of $\mathbb{H}^k \times \mathbb{R}$ by vertical totally geodesic hyperplanes, $-\infty < t < \infty$. Suppose that $\Sigma^k \cap Q_t$ is empty for $t < 0$ and a single point q_0 for $t = 0$. Also suppose that Σ^k is transverse to Q_t for $0 < t < t'$. Let $\Sigma^{k-1}(t)'$ be the component of $\Sigma^k \cap Q_t$ obtained by continuation from q_0 and suppose that $\Sigma^{k-1}(t)'$ is compact for $0 \leq t < t'$ and non-compact for $t = t'$. Then $\Sigma^{k-1}(t)'$ is connected for $0 \leq t \leq t'$ and $\Sigma^{k-1}(t)'$ is not a vertical graph over a convex domain in \mathbb{H}^k .*

Proof By continuation using transversality and compactness from $t = 0$ where $\Sigma^{k-1}(0)'$ is the single point q_0 , it follows that $\Sigma^{k-1}(t)'$ is connected for $0 \leq t < t'$. If Σ^k has $P_\gamma(\bar{t})$ as a tangent plane, then either Σ^k closes up to a compact hypersurface with a single point in $P_\gamma(\bar{t})$, a case which has already been treated, or else there is a new component of Σ^k beginning at $t = \bar{t}$, and we are not considering such a new component. Hence we can suppose that Σ^k is transverse to $P_\gamma(\bar{t})$. Then the argument of Lemma 3.2 shows that $\Sigma^{k-1}(t)'$ is also connected. Since $\Sigma^{k-1}(t)'$ is not compact, the inductive hypothesis implies that $\Sigma^{k-1}(t)'$ is a $(k - 1)$ -surface homeomorphic to \mathbb{R}^{k-1} and that it is a vertical graph over a convex domain in \mathbb{H}^{k-1} or has a simple end.

Now suppose that $\Sigma^{k-1}(t)'$ is a vertical graph of a function f over a convex domain in $\pi(Q_{t'})$, which is isometric to \mathbb{H}^{k-1} , with $k - 1 \geq 1$, in order to obtain a contradiction. Let β be a geodesic segment contained in the domain of f . For each point q of β , the vertical line $r(q)$ passing through the point $(q, 0)$ meets $\Sigma^{k-1}(t)'$ in a unique point $(q, f(q))$, and these points form a curve $\tilde{\beta}$ lying over β , so that $\pi(\tilde{\beta}) = \beta$. If μ_q is the complete geodesic in \mathbb{H}^k containing the points $\pi(q_0)$ and $q \in \beta$, then the intersection of Σ^k with the 2-plane $\mu_q \times \mathbb{R}$ is a complete curve immersed in $\mu_q \times \mathbb{R}$ with strictly positive curvature and a vertical tangent at q_0 , and $\mu_q \times \mathbb{R}$ is isometric to \mathbb{R}^2 . Since $r(q)$ meets $\Sigma^{k-1}(t)'$ in a single point $(q, f(q))$, at that point two branches of the curve must cross each other. This holds for every point q in β , so the curve $\tilde{\beta}$ over β must have strictly positive curvature both above and below in the Euclidean 2-plane strip $\beta \times \mathbb{R}$, but that is absurd. \square

Applying the lemma to the foliation by the vertical planes $P_\gamma(t)$ with $t' = \bar{t}$ shows that $\Sigma^{k-1}(\bar{t})$ is connected and cannot be a vertical graph. Then, by the inductive hypothesis, $\Sigma^{k-1}(\bar{t})$ must have a simple end, which we denote θ and which belongs to $\mathbb{S}_\infty^{k-2} = \partial\pi(P_\gamma(\bar{t}))$. We shall show that θ is also a simple end of Σ^k . We need the following fact.

Proposition 4.2 *Under Hypothesis 4.1, the set Σ^k is embedded in $\mathbb{H}^k \times \mathbb{R}$.*

Proof In Case 4, we are supposing that Σ^k is transverse to $P_\gamma(t)$ with compact intersection $\Sigma^{k-1}(t)$ for $0 < t < \bar{t}$. By continuation from p_0 we see that $\Sigma^{k-1}(t)$ is a $(k - 1)$ -sphere for these values of t , so $\Sigma^{k-1}(t)$ separates $P_\gamma(t)$ into two components, a compact component on the concave side, and a non-compact one on the convex side. Let us consider the set S of values of $t > 0$ such that

1. Σ^k is transverse to $P_\gamma(t)$;
2. $\Sigma^{k-1}(t)$ is connected and embedded in $P_\gamma(t)$;
3. $\Sigma^{k-1}(t)$ separates $P_\gamma(t)$ into two components, the 'outside' whose boundary is concave and the 'inside' whose boundary is convex.

Suppose that for some $t' \geq \bar{t}$, $(0, t') \subset S$. If Σ^k is not transverse to $P_\gamma(t')$ then at some point q_0 $P_\gamma(t)$ is tangent to Σ^k . Near to q_0 Σ^k must lie on one side of $P_\gamma(t')$. If it lies on the side $t > t'$, then it is a new component which is not contained in the continuation from p_0 , and $q_0 \notin \Sigma^{k-1}(t')$. If it lies on the side $t < t'$, then the point q_0 closes Σ^k up as an embedded hypersurface, for $\Sigma^{k-1}(t)$ must be a $(k - 1)$ -sphere for $t < t'$ near to t' . Hence we need only consider the case that Σ^k is transverse to $P_\gamma(t)$ at $t = t'$.

We claim that $\Sigma^{k-1}(t')$ is embedded. If not, since $\Sigma^{k-1}(t)$ is embedded for $t < t'$, there must be a double point x in $P_\gamma(t')$ where two branches of Σ^k are tangent to each other. Then there is loop in $\cup_{0 \leq t \leq t'} \Sigma^{k-1}(t)$ based at x that connects the two branches, and we may shrink it to a geodesic in this set. Note that this geodesic lies on the boundary of the outside of $\cup_{0 \leq t \leq t'} \Sigma^{k-1}(t)$ in $\cup_{0 \leq t \leq t'} P_\gamma(t)$. If the geodesic is contained in $P_\gamma(t')$ then every point in it is a saddle point, since it will curve away from the inside along the geodesic and towards the inside in orthogonal directions, but this contradicts the curvature hypothesis. On the other hand, if the geodesic passes into $P_\gamma(t)$ for some $t < t'$, then any point of the geodesic where t reaches its the minimum value will be also a saddle point, with the geodesic curving away from the inside in the direction of the geodesic and towards the inside in the remaining directions, again giving a contradiction. Hence $\Sigma^{k-1}(t')$ is embedded.

Let us suppose that $\Sigma^{k-1}(t')$ is not connected, to obtain a contradiction. Then the continuation of the inside in $P_\gamma(t')$ will not be connected, but the inside of $\cup_{0 \leq t \leq t'} \Sigma^{k-1}(t)$ will be connected since $\Sigma^{k-1}(t)$ is connected for $t < t'$. Take a path in $\cup_{0 \leq t \leq t'} \Sigma^{k-1}(t)$ between two points x to y in different connected components of $\Sigma^{k-1}(t')$ and shrink it to a geodesic α joining the two points in that set. Consider the point z in α with the smallest value t_0 of t . The curve α at z must curve in the positive direction of t , but $\Sigma^{k-1}(t)$ must curve downwards in the other directions at z , so z is a saddle point, again a contradiction. Thus we have shown that $(0, t'] \subset S$.

Next, suppose that we have $(0, t'] \subset S$. We want to show that for some $\delta > 0$ we have $[t', t' + \delta) \subset S$. Transversality of $P_\gamma(t)$ to $\Sigma^{k-1}(t)$ will hold in any compact domain in $\mathbb{H} \times [t', t' + \delta)$ for some $\delta > 0$, but as we move off toward infinity δ may have to tend to zero.

If $\Sigma^{k-1}(t)$ is not connected for some $t_1 > t'$, then as above we can take a path between two components and shrink it to a geodesic, to obtain a contradiction, as before. Similarly, if $\Sigma^{k-1}(t)$ is not embedded for some $t_1 > t'$, the above argument will produce a contradiction. Hence there is a number $\delta > 0$ such that $\Sigma^{k-1}(t)$ is connected and embedded for $t' \leq t \leq t' + \delta$. As before, we can exclude any case in which transversality fails in this interval. Hence $(0, t' + \delta) \subset S$. Then by continuation, it follows that $\Sigma^{k-1}(t)$ is connected and embedded for all $t > 0$, as claimed. □

Now let $\Omega(\gamma, \theta)$ be the hyperbolic 2-plane in $\mathbb{H}^k \times \{0\}$ that contains the complete geodesic γ orthogonal to Σ^k at p_0 and also has $(\theta, 0)$ as an accumulation point at infinity. Parametrize

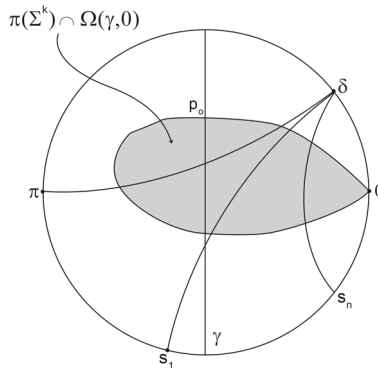


Fig. 5 Situation 1 of Case 4

the circle $\mathbb{S}^1_\infty = \partial_\infty(\Omega(\gamma, \theta))$ by the numbers 0 to 2π such that 0 is the parameter for $(\theta, 0)$ and π is the other point in $\mathbb{S}^1_\infty \cap \partial P_\gamma(\bar{t})$, with the orientation such that the points $0 < s < \pi$ are on the same side of $\partial P_\gamma(\bar{t})$ as the point p_0 .

Now for $\delta > 0$ near to 0 consider the complete geodesic $\{\delta, s\}$ from δ to s in $\Omega(\gamma, \theta)$, where $\delta < s < 2\pi$, and let $W(\delta, s)$ be the vertical k -plane in $\mathbb{H}^k \times \mathbb{R}$ that contains the geodesic $\{\delta, s\}$ and is orthogonal to the plane $\Omega(\gamma, \theta)$. Note that $\cup_{t \leq t_0} \Sigma^{k-1}(t)$ is compact for any $t_0 < \bar{t}$. It follows that for a fixed δ there is a number $s_\delta \in (\delta, \pi)$ such that $W(\delta, s)$ is tangent to Σ^k at a point q_δ , while $\cup_{t \leq t_0} \Sigma^{k-1}(t) \cap W(\delta, s)$ is empty for $\delta < s < s_\delta$.

Let us analyze the intersection $\Sigma^{k-1}(\delta, s)$ of Σ^k and $W(\delta, s)$ for $\pi < s < 2\pi$, where as before we only consider the continuation of Σ^k from q_δ passing through the leaves of the foliation $W(\delta, s)$ as s increases (even if, a priori, there may be other parts of Σ^k .) Repeating the analyses of Cases 1, 2, and 3, we see that the only remaining situation to be treated occurs when Σ^k is transverse to $W(\delta, s)$ with compact intersection $\Sigma^{k-1}(\delta, s)$ for $\delta < s < \bar{s}_\delta$, where we take $\bar{s}_\delta \leq 2\pi$ to be the largest possible s for which this holds.

There are two possible situations:

1. The hypersurfaces $\Sigma^{k-1}(\delta, s)$ are always compact for all $\pi \leq s < 2\pi$ and all $\delta > 0$ sufficiently near to zero. (Thus $\bar{s}_\delta = 2\pi$ for all small $\delta > 0$.) In this case, as $\delta \rightarrow 0$ we have a situation similar to Case 2, so the argument there shows that θ is a simple end of Σ^k (see Fig. 5).
2. There are numbers $\delta > 0$ arbitrarily near to zero such that $\Sigma^{k-1}(\delta, s)$ becomes non-compact for some s , $\pi < s < 2\pi$, i.e., $\bar{s}_\delta < 2\pi$ (Fig. 6).

In this second situation, $\Sigma^{k-1}(\delta, \bar{s}_\delta)$ is not compact. It is easy to see that as δ decreases \bar{s}_δ cannot increase. Since the vertical hyperplane $W(\delta, s_\delta)$ is tangent to Σ^k at q_δ , we can apply Lemma 4.1 using the foliation with leaves $W(\delta, s)$ and conclude that $\Sigma^{k-1}(\delta, \bar{s}_\delta)$ cannot be a vertical graph. Therefore it must be a $(k - 1)$ -surface with a simple end, which we denote by $\tilde{\phi}_\delta$. Note that $\tilde{\phi}_\delta$ is not necessarily in the circle $\partial_\infty(\Omega(\gamma, 0))$, but it is in S^{k-1}_∞ . Let \bar{s} be the limit of \bar{s}_δ as δ decreases to 0. Then there is a decreasing sequence δ_n such that $\tilde{\phi}_{\delta_n}$ converges to a point $\bar{\phi} \in S^{k-1}_\infty$. Note that $\bar{s} \geq \pi$ and suppose $\bar{s} < 2\pi$ to arrive at a contradiction. Let T_n be the vertical hyperbolic $(k - 1)$ -plane that contains the geodesic $\{\delta_n, \bar{s}_{\delta_n}\}$ and is orthogonal to the 2-plane $\Omega(\gamma, \theta)$. Then the limit T of the $(k - 1)$ -planes T_n is the hyperbolic $(k - 1)$ -plane that contains the geodesic $\{0, \bar{s}\}$ and is orthogonal to $\Omega(\gamma, \theta)$. Note that $\bar{\phi}_{\delta_n} \in \partial_\infty T_n$ and $\bar{\phi} \in \partial_\infty T$. To see that $\bar{\phi}$ is an accumulation point of $\Sigma^k \cap (T \times \mathbb{R})$, we use the following proposition.

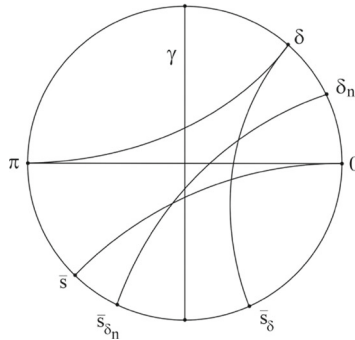


Fig. 6 Situation 2 of Case 4

Proposition 4.3 *Under the hypotheses of the Main Theorem and Hypothesis 4.1, let $\theta \in \partial\mathbb{H}^k$ be an accumulation point both of $\pi(\Sigma^k)$ and of a totally geodesic hyperplane Q in \mathbb{H}^k . If there exists a neighborhood V of θ in $\mathbb{H}^k = \mathbb{H}^k \cup \partial\mathbb{H}^k$ such that $V \cap \pi(\Sigma^k) \cap Q$ is empty, then θ is the only accumulation point of $\pi(\Sigma^k)$ in the sphere at infinity $\partial\mathbb{H}^k$.*

Supposing this Proposition for the moment, note that both θ and $\bar{\phi}$ are accumulation points at infinity of T and also of $\pi(\Sigma^k)$. If θ and $\bar{\phi}$ are distinct, then the Proposition shows that both of them must be accumulation points of $\pi(\Sigma^k) \cap T$. This is impossible since by the inductive hypothesis $\pi(\Sigma^k) \cap T$ must have a simple end and thus it has only one accumulation point at infinity. Consequently θ and $\bar{\phi}$ must coincide, so $\pi(\Sigma^k)$ has only one accumulation point θ at infinity. As in Case 2, θ must be a simple end of $\pi(\Sigma^k)$. This will complete the proof of the Main Theorem, once we have proven the Proposition. □

We shall use the following lemma in the proof of Proposition 4.3.

Lemma 4.2 *In the conditions of Proposition 4.3, the image $\pi(\Sigma^k)$ is a closed connected set embedded in \mathbb{H}^k and its frontier $\partial\pi(\Sigma^k)$ is a complete smooth hypersurface embedded in \mathbb{H}^k with strictly positive curvature at every point.*

Proof Here we use the fact that Σ^k is embedded in $\mathbb{H}^k \times \mathbb{R}$ (see Proposition 4.2). First, we show that $\pi(\Sigma^k)$ is closed in \mathbb{H}^k . Let $q_0 = \pi(p_0) \in \mathbb{H}^k$ be the projection of the point p_0 where we are assuming there is a vertical tangent plane and consider a geodesic β in \mathbb{H}^k beginning at $q_0 = \beta(0)$, with $s \geq 0$. If there exists some $s > 0$ such that $\beta(s) \notin \pi(\Sigma^k)$, let $\bar{s} > 0$ be the largest number such that $z = \beta(s) \in \pi(\Sigma^k)$ for all $s \in [0, \bar{s})$. Consider the intersection of the plane $\beta \times \mathbb{R}$ with Σ^k . This is a complete convex embedded curve with a vertical point at p_0 , and since it does not diverge to infinity, it must contain exactly one other point with a vertical tangent plane, say p_β , such that the geodesic β from $q_0 = \beta(0)$ to $q_\beta = \beta(\bar{s}) = \pi(p_\beta)$ is the intersection of β with $\pi(\Sigma^k)$. Let \tilde{B} be the set of all these points p_β for all such geodesics β plus the point p_0 . Then \tilde{B} is the set of points in Σ^k with vertical tangent planes and we see that the frontier of $\pi(\Sigma^k)$ is contained in $B = \pi(\tilde{B}) \subset \pi(\Sigma^k)$, so $\pi(\Sigma^k)$ is a closed set in \mathbb{H}^k with boundary B .

Next we claim that \tilde{B} is a smooth hypersurface in Σ^k . In a neighborhood $V \subset \Sigma^k$ of the point p_β let N be the normal vector field to V . Let $\pi' : \mathbb{H}^k \times \mathbb{R} \rightarrow \mathbb{R}$ be the projection on the second factor, and let $d\pi'_p : T\Sigma^k_p \rightarrow T\mathbb{R}_{f(p)} \cong \mathbb{R}$ be its differential at $p \in V$. In view of the strictly positive curvature of Σ^k , $d\pi'_p$ is surjective for every $p \in \tilde{B} \cap V$, so the implicit function theorem shows that near p , $\tilde{B} \cap V$ is a smooth hypersurface in V . Furthermore, $\pi|_{\tilde{B}} : \tilde{B} \rightarrow B$ is a diffeomorphism and B is a smooth hypersurface in \mathbb{H}^k , as claimed.

It remains to show that B has strictly positive curvature in \mathbb{H}^k . Let N be the external unit normal field to B . Since Σ^k has a vertical tangent plane at every point of \tilde{B} , the external unit normal field \tilde{N} to Σ^k is horizontal along \tilde{B} and coincides with N , i.e., at $p \in \tilde{B}$, $d\pi_p \tilde{N}(p) = N(\pi(p))$. For a tangent vector $\tilde{X} \in T_p \tilde{B}$ with $d\pi_p \tilde{X} = X$ we have

$$A(X) = -\nabla_X = -\tilde{\nabla}_{\tilde{X}} = \tilde{A}(\tilde{X})$$

where ∇ and $\tilde{\nabla}$ are the Riemannian connections on \mathbb{H}^k and $\mathbb{H}^k \times \mathbb{R}$ and A and \tilde{A} are the shape operators associated to the second fundamental forms of B and Σ^k . Thus the principal curvatures of B at the point $\pi(p)$, which are the eigenvalues of A , coincide with the eigenvalues of the restriction of \tilde{A} to \tilde{B} , the principal curvatures of Σ^k along \tilde{B} at p , and these are all positive since Σ^k has strictly positive curvature. Thus B has strictly positive curvature. \square

Proof of Proposition 4.3 By the lemma, the frontier B of $\pi(\Sigma^k)$ is a smooth hypersurface in \mathbb{H}^k contained in $\pi(\Sigma^k)$ and it has strictly positive curvature. By hypothesis, near θ the image $\pi(\Sigma^k)$ and its frontier B must be disjoint from the totally geodesic hyperplane Q . For any totally geodesic 2-plane η in \mathbb{H}^k that is orthogonal to Q and has θ as a limit point, the intersection $\eta \cap Q$ is a geodesic in \mathbb{H}^k , while $\eta \cap B$ is a strictly convex curve (unless it is empty), so the distance of points on this curve from $\eta \cap Q$ increases with the distance from θ . Outside a tiny neighborhood of the sphere at infinity, the curvature of the boundary B of $\pi(\Sigma^k)$ is greater than some positive number ϵ . This forces points on $\pi(\Sigma^k)$ to become more distant from Q uniformly as their distance from θ increases. Hence there cannot be any other accumulation point of $\pi(\Sigma^k)$ in addition to θ . Since $\pi(\Sigma^k)$ is connected, it must lie entirely on one side of Q . \square

5 Examples of simple ends

We use the upper half-space model of hyperbolic n -space \mathbb{H}^n ,

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with the Riemannian metric given explicitly as $ds^2 = x_n^{-2} \sum_{i=1}^n dx_i^2$. Then $\mathbb{H}^n = \mathbb{R}_+^n$ with this metric has constant curvature -1 . We shall give examples of complete embedded hypersurfaces with strictly positive curvature and with a simple end by explicit computations. The hypersurfaces which we consider are invariant under the group of parabolic isometries that leave invariant each horizontal horosphere in \mathbb{R}_+^n and fix ∞ , the point at infinity. Explicitly, for $a = (a_1, \dots, a_{n-1}) \in \mathbb{R}^{n-1}$, consider the parabolic isometry $f_a : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$, defined by setting $f_a(x) = x + (a, 0)$, where $(a, 0) = (a_1, \dots, a_{n-1}, 0)$. Then f_a preserves the horizontal horospheres $\{x_n = b\}$, and

$$F_a : \mathbb{H}^n \times \mathbb{R} \rightarrow \mathbb{H}^n \times \mathbb{R}, F_a(x, t) = (f_a(x), t)$$

is an isometry for every $a \in \mathbb{R}^{n-1}$.

Consider an interval $\mathcal{I} = (t_1, t_2)$, $-\infty < t_1 < t_2 < +\infty$, and a positive smooth function $u : \mathcal{I} \rightarrow \mathbb{R}$,

$$u(t) = c_1 \ln(t - t_1) + c_2 \ln(t_2 - t) \tag{1}$$

where c_1 and c_2 are negative constants and

$$t_2 - t_1 \leq e^{-1}. \tag{2}$$

It is clear that $u(t)$ is positive and $\lim_{t \rightarrow t_1} u(t) = \lim_{t \rightarrow t_2} u(t) = +\infty$. Let

$$\alpha = \{(0, 0, \dots, 0, u(t), t) \mid t \in \mathbb{R}\} \subset \mathbb{H}^n \times \mathbb{R}$$

be the graph of u as a curve in the $x_n t$ -plane and consider the set

$$\Sigma^n = \cup_{a \in \mathbb{R}^{n-1}} F_a(\alpha) \subset \mathbb{H}^n \times \mathbb{R},$$

parametrized by

$$\varphi(x_1, x_2, \dots, x_{n-1}, t) = (x_1, x_2, \dots, x_{n-1}, u(t), t). \tag{3}$$

Clearly Σ^n is a complete smooth embedded hypersurface. We shall show that (2) implies that it has strictly positive curvature and has a simple end.

The hypersurface Σ^n is preserved by $A \times \text{id}_{\mathbb{R}} : \mathbb{H}^n \times \mathbb{R} \rightarrow \mathbb{H}^n \times \mathbb{R}$ for every Euclidean isometry $A : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ that preserves x_n ; this includes not only horizontal translations, but also rotations of \mathbb{R}_+^n and compositions, and these all produce isometries of $\mathbb{H}^n \times \mathbb{R}$.

Let $\{\partial_1 = \partial/\partial x_1, \dots, \partial_n = \partial/\partial x_n, \partial_t = \partial/\partial t\}$ be the usual frame in $\mathbb{H}^n \times \mathbb{R}$ and let $\lambda = x_n^{-1}$ be the conformal factor of the metric. Then the Riemannian connection $\bar{\nabla}$ in $\mathbb{H}^n \times \mathbb{R}$ is given by

$$\begin{aligned} \bar{\nabla}_{\partial_i} \partial_j &= \delta_{ij} \lambda \partial_n \quad \text{for } i, j < n \\ \bar{\nabla}_{\partial_i} \partial_n &= \bar{\nabla}_{\partial_n} \partial_i = -\lambda \partial_i \quad \text{for } i < n \\ \bar{\nabla}_{\partial_n} \partial_n &= -\lambda \partial_n \\ \bar{\nabla}_{\partial_t} \partial_i &= \bar{\nabla}_{\partial_i} \partial_t = \bar{\nabla}_{\partial_t} \partial_t = 0 \quad \text{for } i \leq n, \end{aligned}$$

where δ_{ij} is the Kronecker delta.

Using the parametrization (3) and setting $u_t = \partial u/\partial t$ as usual, we see that the vector fields

$$\begin{aligned} \varphi_i &= \varphi_{x_i} = \partial_i, \quad \text{for } i = 1, \dots, n - 1, \text{ and} \\ \varphi_n &= \varphi_t = u_t \partial_n + \partial_t \end{aligned}$$

form a basis of the tangent space to Σ^n . We can calculate the coefficients of the first fundamental form $g_{ij} = \langle \varphi_i, \varphi_j \rangle$ of Σ^n to be

$$g_{ij} = \begin{cases} \lambda^2 & \text{if } i = j \text{ with } i = 1, \dots, n - 1 \\ \lambda^2 u_t^2 + 1 & \text{for } i = j = n \\ 0 & \text{for } i \neq j \text{ with } i, j = 1, \dots, n. \end{cases}$$

The unit normal vector field to Σ^n is

$$N = (\lambda^{-1} \partial_n - \lambda u_t \partial_t)/m$$

where $m = \sqrt{1 + \lambda^2 u_t^2}$. Then we compute the coefficients $b_{ij} = \langle \bar{\nabla}_{\varphi_j} \varphi_i, N \rangle$ of the second fundamental form to be

$$b_{ij} = \begin{cases} \lambda^2/m & \text{if } i = j \text{ with } i = 1, \dots, n - 1 \\ \lambda(-\lambda u_t^2 + u_{tt})/m & \text{for } i = j = n \\ 0 & \text{for } i \neq j \text{ with } i, j = 1, \dots, n. \end{cases}$$

Therefore the matrix $(b_{ij})_{n \times n}$ of coefficients of the second fundamental form is a diagonal matrix with the eigenvalues

$$\begin{aligned} \mu_i &= \lambda^2/m \quad \text{for } i = 1, \dots, n - 1 \text{ and} \\ \mu_n &= \lambda(-\lambda u_t^2 + u_{tt})/m \end{aligned}$$

along the diagonal. Since $m = \sqrt{1 + \lambda^2 u_t^2} > 0$, it is clear that $\mu_i > 0$ for $i = 1, \dots, n - 1$.

Lemma 5.1 *Condition (2) implies that $-u_t^2 + uu_{tt} > 0$.*

Since $\lambda = 1/u > 0$ this lemma implies that μ_n is also positive. Thus Σ^n , as defined by the explicit formulas given above, is a complete embedded hypersurface in $\mathbb{H}^n \times \mathbb{R}$ with a simple end and strictly positive curvature.

Proof of Lemma 5.1 Calculate the derivatives u_t and u_{tt} from (1), the definition of $u(t)$. Then a short calculation gives

$$\begin{aligned} -\lambda u_t^2 + u_{tt} &= -\frac{c_2^2}{(t - t_1)^2} [1 + \ln(t - t_1)] - \frac{c_1^2}{(t_2 - t)^2} [1 + \ln(t_2 - t)] \\ &\quad + \frac{c_1 c_2}{(t - t_1)(t_2 - t)} \left[2 - \frac{(t - t_1)}{(t_2 - t)} \ln(t - t_1) - \frac{(t_2 - t)}{(t - t_1)} \ln(t_2 - t) \right]. \end{aligned}$$

Since $t_2 - t_1 \leq e^{-1}$, $t_1 < t < t_2$, and the parameters c_1 and c_2 are negative, it is easy to check that each of the terms on the right hand side is positive. □

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