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A note about connectedness theorems à la Barth

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Abstract We prove Barth-type connectedness results for low-codimension smooth subvarieties with good numerical properties inside certain "easy" ambient spaces (such as homogeneous varieties, or spherical varieties). The argument employs some basics from the theory of cones of cycle classes, in particular the notion of bigness of a cycle class.

Keywords Barth theorem · Connectedness · Positivity · Cones of cycle classes · Homogeneous varieties · Spherical varieties

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1 Introduction

The Mother Of All Connectedness Theorems is Barth's theorem. In its original version, Barth's theorem is about the cohomology of low-codimensional smooth subvarieties of projective space:

Theorem 1 (Barth [2]) *Let $X \subset \mathbb{P}^{n+r}(\mathbb{C})$ be a smooth subvariety of dimension n. Then restriction induces isomorphisms*

$$H^j(\mathbb{P}^{n+r}(\mathbb{C}), \mathbb{Q}) \xrightarrow{\cong} H^j(X, \mathbb{Q}) \quad \textit{for all } j \leq n-r.$$

Hartshorne [16] found a nice proof of Theorem 1 based on the hard Lefschetz theorem. Subsequent extensions of Barth's theorem also establish connectedness results for homotopy groups, as well as for low-codimensional subvarieties of other ambient spaces, such as Grassmannians, rational homogeneous varieties or abelian varieties (cf. [13,14], [19, Chapter 3] for

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comprehensive overviews). As is made clear by results of Debarre, in certain ambient spaces P a connectedness result holds for any subvariety X with an appropriate intersection-theoretic behaviour in P:

Theorem 2 (Debarre [8]) *Let P be a product of projective spaces or a Grassmannian, with* $\dim P = n + r$. *Let $X \subset P$ be a smooth subvariety of dimension $n \geq r + 1$ which is* bulky *(i.e., X meets all r-dimensional subvarieties of P). Then X is simply connected.*

Results similar in spirit have been obtained by Arrondo–Caravantes [1], and by Perrin [23,24]:

Theorem 3 (Arrondo–Caravantes [1]) *Let P be the Grassmannian of lines in a projective space, with* $\dim P = n+r$. *Let $X \subset P$ be a smooth bulky subvariety of dimension $n \geq r+2$. Then*

$$\mathrm{Pic}(X) = \mathbb{Z}.$$

Theorem 4 (Perrin [24]) *Let P be a rational homogeneous variety with Picard number* 1. *Let $X \subset P$ be a smooth bulky subvariety of codimension r, and assume $2r \leq \mathrm{coeff}(P) - 2$ (here, $\mathrm{coeff}(P)$ is a number in between 0 and $\dim P$, defined in* [24, Definition 0.9]*). Then the Néron–Severi group $NS(X)$ of X has rank 1:*

$$\mathrm{NS}(X) = \mathbb{Z}.$$

In this note, we aim for similar connectedness results for subvarieties that have certain intersection-theoretic properties (such as bulkiness). Our main result is a cohomological version of Theorem 2. This result applies to any ambient space P for which the cone $\mathrm{Eff}^n(P)$ of effective codimension n algebraic cycles modulo numerical equivalence is a closed cone (in particular, this applies when P is a spherical variety, cf. Corollary 20).

Theorem (=Theorem 17) *Let n, r be positive integers with $n \geq r + 1$. Let P be a smooth projective variety of dimension $n + r$, and assume there is equality*

$$\mathrm{Eff}^n(P) = \mathrm{Psef}^n(P)$$

(i.e., the cone $\mathrm{Eff}^n(P)$ is a closed cone).

Let $X \subset P$ be a smooth closed subvariety of dimension n, and assume X is strictly nef. Then the push-forward map

$$H^1(X, \mathbb{Q}) \ \to \ H^{2r+1}(P, \mathbb{Q})$$

is injective.

For the definition of "strictly nef", cf. Definition 10; on a homogeneous variety P, strict nefness is equivalent to bulkiness (Remark 13), which connects Theorem 17 to Theorem 2. The proof of Theorem 17 is a very straightforward adaptation of Hartshorne's proof [16] of Barth's theorem using the hard Lefschetz theorem. The ampleness in Hartshorne's proof is replaced by "bigness" (in the sense of: being in the interior of the pseudo-effective cone of codimension r cycles). Indeed, thanks to work of Fu [10], bigness of the class $[X]$ in the space $N^r(P)$ (of codimension r cycles modulo numerical equivalence) is (under certain conditions) sufficient to obtain a connectedness result.

We establish some variants of Theorem 17 that similarly exploit this notion of bigness: in one variant (Theorem 22), there is no assumption on the ambient space P but the assumptions on X are stronger. As an application of Theorem 22, we obtain in particular the following improvement on the above-cited result of Perrin:

Corollary (=Corollary 23) *Let X and P be as in Theorem 4. Then*

$$\operatorname{Pic}(X) = \mathbb{Z}.$$

In another variant result (Proposition 26), we show that when P is a spherical variety, there is still a certain connectedness even for subvarieties X that may fail to be bulky.

Finally, we include a conditional result (Theorem 30) that proves connectedness for cohomology of degree >1. This result is conditional, because (apart from the codimension 2 case) we need to assume the standard Lefschetz conjecture $B(X)$ for the subvariety X. Theorem 30 implies in particular a conditional improvement on the above-cited result of Arrondo–Caravantes:

Corollary (=Corollary 33) *Let X and P be as in Theorem 3, and suppose either $r = 2$ or $B(X)$ holds. Then*

$$H^2(X, \mathbb{Z}) = \mathbb{Z}.$$

We present two more applications of a similar ilk (Corollaries 34 and 36). Just like Corollary 33, these applications prove a certain connectedness result for bulky subvarieties of codimension 2 and for bulky subvarieties verifying the standard Lefschetz conjecture.

Conventions *All varieties will be irreducible projective varieties over $\mathbb{C}$. A subvariety will always be a* closed *subvariety.*

2 Cones of cycle classes

Definition 5 Let M be a smooth projective variety of dimension m. Let $N^j(M)$ denote the $\mathbb{R}$-vector space of codimension j algebraic cycles on M (with $\mathbb{R}$-coefficients) modulo numerical equivalence. Let

$$\operatorname{Eff}^j(M) \subset N^j(M)$$

be the cone generated by effective algebraic cycles. Let

$$\operatorname{Psef}^j(M) := \overline{\operatorname{Eff}^j(M)} \subset N^j(M)$$

be the closure of the cone generated by effective algebraic cycles. A class $\gamma \in N^j(M)$ is called *big* if γ is in the relative interior of $\operatorname{Psef}^j(M)$.

The intersection product defines a perfect pairing

$$N^j(M) \times N^{m-j}(M) \ \to \ N^m(M) \cong \mathbb{R}.$$

Let

$$\operatorname{Nef}^j(M) \subset N^j(M)$$

be the cone dual to $\operatorname{Psef}^{m-j}(M)$ under this pairing.

The pseudo-effective cone $\operatorname{Psef}^j(M)$ is studied for instance in [9,11,12,20,21]. There is another notion of bigness, which is a priori more stringent:

Definition 6 Let M be a smooth projective variety. Let N^* denote the coniveau filtration on cohomology [3]. Let

$$\mathrm{Hpsef}^j(M) \subset N^j H^{2j}(M, \mathbb{R})$$

be the closure of the cone generated by effective algebraic cycles. A class $\gamma \in N^j H^{2j}(M, \mathbb{R})$ is called *homologically big* if γ is in the relative interior of $\mathrm{Hpsef}^j(M)$.

Remark 7 The "homologically pseudo-effective cone" $\mathrm{Hpsef}^j(M)$ is considered for instance in [10,27]. If Grothendieck's standard conjecture $D(M)$ is true (i.e., homological and numerical equivalence coincide on M), then there is a natural isomorphism

$$N^j H^{2j}(M, \mathbb{R}) \cong N^j(M) ,$$

and so the two notions of bigness coincide. In particular, since we know the standard conjecture D is true in codimension 1 and 2 [22, Corollary 1] and for curves ([22, Corollary 1], or alternatively [7, Proposition 1.1]), the two notions of bigness coincide for $j = 1$, for $j = 2$ and for $j = n - 1$. In general, in the absence of $D(M)$, we only know that a homologically big class in $N^j H^{2j}(M, \mathbb{R})$ projects to a big class in $N^j(M)$. For more on the standard conjectures, cf. [17,18].

Thanks to work of Lehmann, there exists a nice volume-type function for cycle classes. This volume-type function acts as a bigness detector:

Theorem 8 (Lehmann [20]) *Let X be a smooth projective variety of dimension n. Consider the homogeneous function defined as*

$$\begin{aligned} \widehat{\mathrm{vol}}\colon\ N^j(X) &\rightarrow \mathbb{R}_{\geq 0} , \\ \widehat{\mathrm{vol}}(\alpha) &:= \sup_{\phi, A}\{A^n\} , \end{aligned}$$

where $\phi\colon Y \rightarrow X$ varies over all birational models of X, and A varies over all big and nef $\mathbb{R}$-Cartier divisors on Y such that $\phi_*(A^j) - \alpha \in \mathrm{Psef}^j(X)$. This function has the property that $\widehat{\mathrm{vol}}(\alpha) > 0$ if and only if α is big.

Proof This is [20, Section 7]. □

3 Strictly nef subvarieties

In this section, we prove the main result of this note (Theorem 17), which is about degree 1 cohomology of smooth strictly nef subvarieties.

Definition 9 Let P be a smooth projective variety, and let $X \subset P$ be a closed irreducible subvariety of codimension r. We say that X is *bulky* if X meets every dimension r subvariety of P, i.e. for every closed r-dimensional subvariety $a \subset P$, we have

$$X \cap a \neq \emptyset$$

(here $\cap$ indicates set-theoretic intersection).

Definition 10 Let P be a smooth projective variety, and let $X \subset P$ be a closed irreducible subvariety of codimension r. We say that X is *strictly nef* if for every non-zero $a \in \mathrm{Eff}_r(P)$ we have

$$[X] \cdot a > 0 \quad \text{in } H_0(P, \mathbb{R}) \cong \mathbb{R}.$$

Remark 11 The definition of bulkiness seems to originate with [8] (where it is called "une sous-variété encombrante"). In [24], the adjective "cumbersome" is used instead of bulky.

Remark 12 Strictly nef divisors are studied in [6].

Remark 13 Any strictly nef subvariety is bulky. On a homogeneous variety P, the converse is true (indeed, any non-zero effective class on P is represented by an effective cycle in general position with respect to X). On arbitrary varieties P, the converse is *not* true. (Here is an example that was kindly pointed out by the referee: Let $P_1, \ldots, P_{10}$ be 10 very general points on an elliptic curve $E \subset \mathbb{P}^2$. Let $S \to \mathbb{P}^2$ denote the blow-up with center the 10 points P_i, and let $\bar{E} \subset S$ be the strict transform of E. One can check that $\bar{E} \subset S$ is bulky. On the other hand, the self-intersection $\bar{E}^2$ is negative, so $\bar{E}$ is not nef.)

To recap, one could say that the notion of strict nefness (which is equivalent to bulkiness on homogeneous varieties) is the more natural notion for arbitrary varieties.

Example 14 Let P be a homogeneous variety, and $X \subset P$ a smooth subvariety with ample normal bundle. Then X is bulky [19, Example 8.4.6]. In particular, if P is a simple abelian variety, every smooth subvariety $X \subset P$ is bulky [19, Corollary 6.3.11].

Definition 15 Let P be a smooth projective variety, and let $X \subset P$ be a closed irreducible subvariety of codimension r. We will write

$$H^j(X)_{\text{van}} := \ker\big(H^j(X, \mathbb{C}) \ \to \ H^{j+2r}(P, \mathbb{C})\big).$$

Remark 16 It follows from mixed Hodge theory that the kernel

$$\ker\big(H^j(X, \mathbb{Q}) \ \to \ H^{j+2r}(P, \mathbb{Q})\big)$$

is a Hodge sub-structure [26]. Thus, it makes sense to write $\mathrm{Gr}^i_F H^j(X)_{\text{van}}$ (where F^* denotes the Hodge filtration).

Theorem 17 *Let n and r be positive integers with $n \geq r + 1$. Let P be a smooth projective variety of dimension $n + r$, and assume there is equality*

$$\mathrm{Eff}^n(P) = \mathrm{Psef}^n(P) \subset N^n(P)$$

(i.e., the cone $\mathrm{Eff}^n(P)$ is a closed cone).

Let $X \subset P$ be a smooth subvariety of dimension n which is strictly nef. Then

$$H^1(X)_{van} = 0.$$

Proof Suppose $n > r + 1$. There is a fibre diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & P' \\ \downarrow & & \downarrow f \\ X & \xrightarrow{\tau} & P\,, \end{array}$$

where $P' \subset P$ is a smooth complete intersection of dimension $n' + r$, and $X' \subset X$ is smooth of dimension n', and we have equality $n' = r + 1$.

Lemma 18 *The class*

$$(\tau')^*[X'] \in N^r(X')$$

is homologically big.

Proof First, since $r = n' - 1$ (i.e., $(\tau')^*[X']$ is a curve class), the notions of bigness and homological bigness are the same (Remark 7). We are thus reduced to proving bigness, i.e. we need to prove $(\tau')^*[X']$ is in the relative interior of $\mathrm{Eff}^r(X')$. Let $A \subset P$ be a codimension r intersection of ample divisors. Then

$$A' := (\tau')f^*(A) \in N^r(X')$$

is the class of a codimension r intersection of ample divisors; as such, A' is in the relative interior of $\mathrm{Eff}^r(X')$ ([14, Lemma 2.11], or alternatively Theorem 8). Hence, to prove bigness of $(\tau')^*[X']$, it suffices to prove that

$$(\tau')^*[X'] - \epsilon A' \in \mathrm{Psef}^r(X') \,, \tag{1}$$

for some $\epsilon > 0$ sufficiently small.

Now let $D \in \mathrm{Nef}^1(X')$. Then we have

$$\begin{aligned}\big((\tau')^*[X'] - \epsilon A'\big) \cdot D &= \Big((\tau')^* f^*([X] - \epsilon A)\Big) \cdot D \\ &= \big([X] - \epsilon A\big) \cdot f_*(\tau')_*(D) \\ &\geq 0\,,\end{aligned}$$

for some $\epsilon > 0$ sufficiently small. Here, the first equality is just the fact that $f^*[X] = [X']$, and the second equality is the projection formula. As for the last line, note that $X \subset P$ is strictly nef, which combined with the assumption that $\mathrm{Eff}^n(P)$ is a closed cone implies that $[X]$ is strictly positive on $\mathrm{Psef}^n(P)\backslash\{0\}$, i.e. $[X]$ is in the relative interior of $\mathrm{Nef}^r(P)$. On the other hand, $\mathrm{Nef}^1(X') \subset \mathrm{Psef}^1(X')$, and so the push-forward $f_*(\tau')_*(D)$ is pseudo-effective, hence (by assumption) effective. This means that there exists $\epsilon > 0$ such that $\big([X] - \epsilon A\big) \cdot f_*(\tau')_*(D) \geq 0$. This proves the inclusion (1), and hence the lemma.

Homological bigness is relevant to us, because of the following hard Lefschetz type result:

Lemma 19 (Fu [10]) *Let M be a smooth projective variety of dimension n, and let $\gamma \in N^r H^{2r}(M, \mathbb{Q})$ be homologically big. Then the homomorphism "cup product with γ" induces an injection*

$$\cup\gamma\colon\quad \mathrm{Gr}^0_F\, H^{n-r}(M, \mathbb{C}) \;\to\; \mathrm{Gr}^r_F\, H^{n+r}(M, \mathbb{C})$$

(here F^* denotes the Hodge filtration).

Proof This is [10, Lemma 3.3]. The proof exploits the second Hodge–Riemann bilinear relation, and is inspired by ideas of Voisin [27]. □

Applying Lemma 19 to the homologically big class $(\tau')^*[X'] \in N^r H^{2r}(X', \mathbb{Q})$, we find that

$$\cup(\tau')^*[X']\colon\;\; \mathrm{Gr}^0_F\, H^1(X', \mathbb{C}) \;\to\; \mathrm{Gr}^{n'-1}_F\, H^{2n'-1}(X', \mathbb{C})$$

is injective (and hence, for dimension reasons, an isomorphism). Using the fact that $\mathrm{Gr}^1_F\, H^1$ is the complex conjugate of $\mathrm{Gr}^0_F\, H^1$, we find that

$$\cup(\tau')^*[X']\colon\;\; H^1(X', \mathbb{C}) \;\to\; H^{2n'-1}(X', \mathbb{C})$$

is also injective. On the other hand, it follows from the normal bundle formula that there is a factorization

$$\cup(\tau')^*[X']\colon\; H^1(X', \mathbb{C}) \xrightarrow{(\tau')_*} H^{2r+1}(P', \mathbb{C}) \xrightarrow{(\tau')^*} H^{2n'-1}(X', \mathbb{C}).$$

We can thus conclude that

$$(\tau')_*: \quad H^1(X',\mathbb{C}) \ \to\ H^{2r+1}(P',\mathbb{C})$$

is injective. We have a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathbb{C}) & \xrightarrow{\tau_*} & H^{2r+1}(P,\mathbb{C}) \\ \downarrow & & \downarrow \\ H^1(X',\mathbb{C}) & \xrightarrow{(\tau')_*} & H^{2r+1}(P',\mathbb{C}) \end{array}$$

where vertical arrows are injective (weak Lefschetz, note that $\dim P' = 2r+1$). It follows that

$$\tau_*: \quad H^1(X,\mathbb{C}) \ \to\ H^{2r+1}(P,\mathbb{C})$$

is injective. □

As a corollary, we obtain the following:

Corollary 20 *Let P be a smooth projective variety of dimension $n+r$, and suppose a connected solvable linear algebraic group acts on P with finitely many orbits. Let $X \subset P$ be a smooth subvariety of dimension $n \geq r+1$ which is strictly nef. Then*

$$H^1(X,\mathbb{Q}) = 0.$$

Proof For P as in Corollary 20, it is known that all cones $\mathrm{Eff}^r(P)$ are closed rational polyhedral cones, generated by the orbit closures [15, Corollary to Theorem 1]. Theorem 17 thus applies; this gives

$$H^1(X)_{\mathrm{van}} = 0.$$

But P has no odd cohomology since the cycle class map is an isomorphism [15, Corollary to Theorem 2], and so $H^1(X,\mathbb{Q}) = 0$. □

Remark 21 Suppose P is a Grassmannian or a product of projective spaces (of dimension $n+r$), and $X \subset P$ smooth and bulky (of dimension $n \geq r+1$) as in Corollary 20. Then, as noted in the introduction, Debarre has proven that X is simply connected [8]. Can one also prove simple-connectedness in the more general set-up of Corollary 20?

Here is a variant of Theorem 17 where we make no assumption on the ambient space P.

Theorem 22 *Let n, r be positive integers with $n \geq r+1$. Let P be a smooth projective variety of dimension $n+r$. Let $X \subset P$ be a smooth subvariety of dimension n that is strictly nef. Assume that* $\dim N^1(X) = 1$. *Then*

$$H^1(X)_{van} = 0.$$

Proof This is similar to Theorem 17. Again, in case $n > r+1$, we consider a fibre diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & P' \\ \downarrow & & \downarrow f \\ X & \xrightarrow{\tau} & P\ , \end{array}$$

where $P' \subset P$ is a generic smooth complete intersection of dimension $n'+r$, and $X' \subset X$ is smooth of dimension n', and we have equality $n' = r+1$. Taking P' sufficiently generic, we will have $\dim N^1(X') = 1$ (this follows from weak Lefschetz in case $n' \geq 3$, and from Noether–Lefschetz in case $n' = 2$). Hence, to test the bigness of the curve class $(\tau')^*[X']$, it suffices to intersect with one ample divisor $D \in \mathrm{Nef}^1(X')$. But any ample divisor is effective, and so the push-forward $f_*(\tau')_*(D)$ is effective. It follows that the intersection is positive, by strict nefness of X:

$$\begin{aligned}(\tau')^*[X'] \cdot D &= (\tau')^* f^*[X] \cdot D \\ &= [X] \cdot f_*(\tau')_*(D) \\ &> 0.\end{aligned}$$

We conclude that $(\tau')^*[X']$ is big. The rest of the argument is the same as Theorem 17. □

Thanks to Theorem 22, we can "complete" certain results of Perrin:

Corollary 23 *Let P be a rational homogeneous variety with Picard number* 1*, and* $\dim P = n+r$*. Let $X \subset P$ be a smooth bulky subvariety of dimension n, and assume* $2r \leq \mathrm{coeff}(P)-2$ *(here,* $\mathrm{coeff}(P)$ *is a number in between* 0 *and* $\dim P$*, defined in* [24, Definition 0.9]*). Then*

$$\mathrm{Pic}(X) = \mathbb{Z}.$$

Proof Note that bulkiness and strict nefness coincide on P (Remark 13). Perrin has proven [24, Theorem 0.10] that the Néron–Severi group $\mathrm{NS}(X)$ is $\mathbb{Z}$, so that $N^1(X) = \mathbb{R}$. The result now follows from Theorem 22, in view of the exact sequence (coming from the exponential sequence)

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0.$$

□

4 Not so bulky subvarieties

In this section, we consider a refinement of Theorem 17 for certain special ambient spaces P. The connectedness result of this section (Proposition 26) improves on Theorem 17 because it applies to subvarieties X that may fail to be bulky (cf. Remark 29).

Definition 24 Let G be a connected reductive algebraic group. A *spherical variety* is a normal G-variety for which there is a Borel subgroup $B \subset G$ with a dense orbit.

Remark 25 More on spherical varieties can be found in [4,5,25] and the references given there.

Proposition 26 *Let P be a smooth projective spherical variety of dimension $n + r$. Let $X \subset P$ be a smooth subvariety of dimension $n \geq r + 1$, verifying the following:*

(i) X is in general position with respect to the n-dimensional orbit closures on P;
(ii) $X \subset P$ is big.

Then

$$H^1(X, \mathbb{Q}) = 0.$$

Proof As before, in case $n > r + 1$, we consider a fibre diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & P' \\ \downarrow g & & \downarrow \\ X & \xrightarrow{\tau} & P \,, \end{array}$$

where $P' \subset P$ is a smooth complete intersection of dimension $n' + r$, and $X' \subset X$ is smooth of dimension n', and we have equality $n' = r + 1$.

Lemma 27 *The class $\tau^*[X] \in N^r(X)$ is big.*

Proof As the cone $\mathrm{Eff}^r(P)$ is generated by the n-dimensional orbit closures [15], assumption (i) implies that

$$\tau^*\big(\mathrm{Eff}^r(P)\big) \subset \mathrm{Eff}^r(X).$$

Dually, this amounts to an inclusion

$$\tau_*\big(\mathrm{Nef}^{n-r}(X)\big) \subset \mathrm{Nef}^n(P).$$

Let $A \in N^1(P)$ denote the class of an ample divisor. The class $\tau^*(A^r)$ lies in the relative interior of $\mathrm{Eff}^r(X)$. Hence, proving Lemma 27 is equivalent to showing

$$\tau^*[X] - \epsilon\tau^*(A^r) \in \mathrm{Eff}^r(X) \tag{2}$$

for some $\epsilon > 0$ sufficiently small.

Let $D \in \mathrm{Nef}^{n-r}(X)$. As we have seen, $\tau_*(D) \in \mathrm{Nef}^n(P)$. It follows that

$$\big(\tau^*[X] - \epsilon\tau^*(A^r)\big) \cdot D = \big([X] - \epsilon A^r\big) \cdot \tau_*(D) \geq 0 \,,$$

for some $\epsilon > 0$ sufficiently small. This proves inclusion (2), and hence Lemma 27. □

Lemma 28 *The class $(\tau')^*[X'] \in N^r(X')$ is homologically big.*

Proof Since $\tau^*[X]$ is big (Lemma 27), we can write

$$\tau^*[X] = A^r + e \quad \text{in } N^r(X) \,,$$

where A is an ample divisor on X, and e is an effective class (here, we have again used the fact that complete intersection classes A^r are big; this is [11, Lemma 2.11], or, alternatively, can be seen using the volume-type function of Theorem 8). For a generic choice of X', the restriction $e' = g^*(e)$ is still effective, and (obviously) $A' = g^*(A)$ is still ample. It follows that

$$(\tau')^*[X'] = (A')^r + e' \quad \text{in } N^r(X')$$

is big.

Because $r = n' - 1$ (i.e., we look at a curve class on X') the class $(\tau')^*[X']$ is also homologically big (Remark 7). □

The rest of the argument is identical to that of Theorem 17: Applying Lemma 19 to the homologically big class $(\tau')^*[X']$, we find that

$$(\tau')_*\colon \; H^1(X', \mathbb{Q}) \;\to\; H^{2r+1}(P', \mathbb{Q})$$

is injective. The commutative diagram

$$\begin{array}{ccc} H^1(X,\mathbb{C}) & \xrightarrow{\tau_*} & H^{2r+1}(P,\mathbb{C}) \\ \downarrow & & \downarrow \\ H^1(X',\mathbb{C}) & \xrightarrow{(\tau')_*} & H^{2r+1}(P',\mathbb{C}) \end{array}$$

(where vertical arrows are injective by weak Lefschetz) then proves the proposition. □

Remark 29 Let X be a smooth projective spherical variety. It is known [21, Theorem 1.1] that there are inclusions of cones

$$\mathrm{Nef}^j(P) \subset \mathrm{Eff}^j(P) \quad \text{for all } j.$$

That is, any bulky subvariety $X \subset P$ verifies hypothesis (ii) of Proposition 26.

We can say more: as shown in [21], there are "many" spherical varieties P for which there are *strict* inclusions

$$\mathrm{Nef}^j(P) \subsetneq \mathrm{Eff}^j(P) \quad \text{for all } j.$$

(More precisely: let P be either a toric variety different from a product of projective spaces, or a toroidal spherical variety different from a rational homogeneous space. Then these inclusions are strict for all j [21, Theorem 1.2].) The conclusion is that in these cases Proposition 26 gives a connectedness result even for subvarieties X that fail to be bulky; it suffices that X be only "slightly bulky", in the sense of hypothesis (ii).

5 A conditional result

In this final section, we prove a conditional connectedness result for cohomology groups in degree > 1. This result is conditional to one of the standard conjectures. The reason we need to assume a standard conjecture is that there might a priori be a difference between the two notions of bigness defined in Sect. 2 (cf. Remark 7).

Theorem 30 *Let P be a smooth projective variety of dimension $n+r$, and $\tau\colon X \subset P$ a smooth subvariety of dimension n. Assume the following:*

(i) There is an inclusion of cones

$$\mathrm{Nef}^n(P) \subset \mathrm{Eff}^n(P)\ ;$$

(ii) $X \subset P$ is strictly nef;

(iii) There is an inclusion

$$\tau^*\big(\mathrm{Psef}^r(P)\big) \subset \mathrm{Psef}^r(X)\ ;$$

(iv) Either $r = 2$, or the standard Lefschetz conjecture $B(X)$ holds.

Then

$$\mathrm{Gr}^0_F H^j(X)_{van} = 0 \quad \textit{for all } j \le n - r.$$

Proof First, in case $j < n - r$ we take generic hyperplane sections. That is, we consider (as before) a fibre diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tau'} & P' \\ \downarrow^{g} & & \downarrow \\ X & \xrightarrow{\tau} & P\ , \end{array}$$

where $P' \subset P$ is a smooth complete intersection of dimension $n'+r$, and $X' \subset X$ is smooth of dimension n', and we have equality $j = n' - r$.

Lemma 31 *The class $\tau^*[X] \in N^r(X)$ is (homologically) big.*

Proof Let $A \in N^1(P)$ be an ample divisor class. To prove bigness of $\tau^*[X]$, it suffices to prove

$$\tau^*[X] - \epsilon\tau^*(A^r) \in \mathrm{Psef}^r(X) \tag{3}$$

for some $\epsilon > 0$.

Let $a \in \mathrm{Nef}^{n-r}(X)$. It follows from assumption (iii) (by duality) that

$$\tau_*(a) \in \mathrm{Nef}^n(P).$$

It follows from assumption (i) that $\tau_*(a)$ is effective. Also, assumptions (i) and (ii) combined imply that $[X] \in N^r(P)$ is big. Now, using the projection formula we find that

$$\big(\tau^*[X] - \epsilon\tau^*(A^r)\big) \cdot a = \big([X] - \epsilon A^r\big) \cdot \tau_*(a) \geq 0\,,$$

for some sufficiently small $\epsilon > 0$. This proves inclusion (3) and hence the bigness of $\tau^*[X]$. Since we have assumed that either $r = 2$ or $B(X)$ holds, the two notions of bigness coincide (Remark 7), and so $\tau^*[X]$ is homologically big. □

Lemma 32 *The class $(\tau')^*[X'] \in N^r(X') = N^r H^{2r}(X', \mathbb{R})$ is homologically big.*

Proof The fact that $(\tau')^*[X']$ is big can be deduced from Lemma 31 along the lines of the proof of Lemma 28.

In case $r = 2$, the two notions of bigness coincide (Remark 7). Otherwise, since property $B(X)$ implies $B(X')$ [18], the two notions of bigness also coincide on X'; this proves the lemma. □

Applying Lemma 19 to the homologically big class $(\tau')^*[X'] \in N^r(X') = N^r H^{2r}(X', \mathbb{R})$, we find that

$$\cup(\tau')^*[X']\colon\ \ \mathrm{Gr}^0_F\, H^{n'-r}(X', \mathbb{C}) \ \to\ \mathrm{Gr}^r_F\, H^{n'+r}(X', \mathbb{C})$$

is injective (and hence, for dimension reasons, an isomorphism). On the other hand, it follows from the normal bundle formula that there is a factorization

$$\cup(\tau')^*[X']\colon\ \ \mathrm{Gr}^0_F\, H^{n'-r}(X', \mathbb{C}) \xrightarrow{(\tau')_*} \mathrm{Gr}^r_F\, H^{n'+r}(P', \mathbb{C}) \xrightarrow{(\tau')^*} \mathrm{Gr}^r_F\, H^{n'+r}(X', \mathbb{C}).$$

We can thus conclude that

$$(\tau')_*\colon\ \ \mathrm{Gr}^0_F\, H^j(X', \mathbb{C}) \ \to\ \mathrm{Gr}^r_F\, H^{j+2r}(P', \mathbb{C})$$

is injective.

To return to X, we consider a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}^0_F\, H^j(X, \mathbb{C}) & \xrightarrow{\tau_*} & \mathrm{Gr}^r_F\, H^{j+2r}(P, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{Gr}^0_F\, H^j(X', \mathbb{C}) & \xrightarrow{(\tau')_*} & \mathrm{Gr}^r_F\, H^{j+2r}(P', \mathbb{C}) \end{array}$$

where vertical arrows are injective (this is an application of weak Lefschetz; note that $\dim X' = n' > j$ and $\dim P' = j + 2r$). It follows from this commutative diagram that

$$\tau_*\colon\ \ \mathrm{Gr}^0_F\, H^j(X, \mathbb{C}) \ \to\ \mathrm{Gr}^r_F\, H^{j+2r}(P, \mathbb{C})$$

is injective. □

Corollary 33 *Let n, r be positive integers with $n \geq r + 2$. Let P be a Grassmannian of lines in a projective space, and* $\dim P = n + r$. *Let $X \subset P$ be a smooth bulky subvariety of dimension n. Assume either $r = 2$ or $B(X)$ holds. Then*

$$H^2(X, \mathbb{Z}) = \mathbb{Z}.$$

Proof As mentioned in the introduction, Arrondo and Caravantes have proven [1] that $\mathrm{Pic}(X) = \mathbb{Z}$.

We now check that all assumptions of Theorem 30 are satisfied. Any Grassmannian P has $\mathrm{Nef}^j(P) = \mathrm{Eff}^j(P)$ for all j so assumption (i) is OK. Assumption (ii) is OK by Remark 13. Assumption (iii) of Theorem 30 is satisfied, because (by homogeneity) any subvariety $a \subset P$ is homologically equivalent to a subvariety in general position with respect to X. Applying Theorem 30, we find that $H^2(X, \mathcal{O}_X) = 0$. The result now follows from the exponential sequence.

Corollary 34 *Let P be a product $\mathbb{P}^m \times \mathbb{P}^m$, and let $X \subset P$ be a smooth subvariety of codimension r and dimension $n \geq r + 2$. Assume the two projection maps $X \to \mathbb{P}^m$ are surjective. Assume also that either $r = 2$ or $B(X)$ holds. Then*

$$H^2(X, \mathbb{Z}) = \mathbb{Z}^2.$$

Proof Arrondo and Caravantes have proven that $\mathrm{Pic}(X) = \mathbb{Z}^2$ [1, Theorem 3.1]. The assumption about the projection maps ensures that X is bulky [8, Proposition 2.6], hence (by homogeneity of P) strictly nef. Applying Theorem 30, we find that $H^2(X, \mathcal{O}_X) = 0$.

Definition 35 (*Perrin* [24]) Let $\mathbb{G}_Q(p, m)$ and $\mathbb{G}_\omega(p, 2m)$ be the Grassmannians of isotropic subspaces of dimension p in a vector space of dimension m (resp. $2m$) endowed with a non-degenerate quadratic form Q (resp. symplectic form ω).

Corollary 36 *Let n, r be positive integers with $n \geq r+3$. Let P be $\mathbb{G}_Q(2, 2m+1)$, $\mathbb{G}_\omega(2, 2m)$ or $\mathbb{G}_Q(2, 4m)$. Let $X \subset P$ be a smooth bulky subvariety of dimension n and codimension r. Assume either $r = 2$, or $B(X)$ holds. Then*

$$H^2(X, \mathbb{Z}) = \mathbb{Z}.$$

Proof Perrin has proven that $\mathrm{Pic}(X) = \mathbb{Z}$ [24, Corollary 0.11]. Since P is homogeneous, the conditions of Theorem 30 are again fulfilled, so we also have $H^2(X, \mathcal{O}_X) = 0$.

Remark 37 It would be interesting if one could prove Theorem 30 (or even the Corollaries 33 and 34 and 36) for $r > 2$ without assuming some standard conjecture for the subvariety X. I have not been able to do so.

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