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ORIGINAL PAPER

A rigidity theorem of ξ-submanifolds in $\mathbb{C}^2$

Xingxiao Li[1] · **Xiufen Chang**[1]

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Abstract In this paper, we first introduce the concept of ξ-submanifold which is a natural generalization of self-shrinkers for the mean curvature flow and also an extension of λ-hypersurfaces to the higher codimension. Then, as the main result, we prove a rigidity theorem for Lagrangian ξ-submanifold in the complex 2-plane $\mathbb{C}^2$.

Keywords ξ-Submanifold · The second fundamental form · Mean curvature vector · Torus

Mathematics Subject Classification (2000) Primary 53A30 · Secondary 53B25

1 Introduction

Let $x : M^n \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Then x is called a *self-shrinker* (to the mean curvature flow) in $\mathbb{R}^{n+p}$ if its mean curvature vector field H satisfies

$$H + x^{\perp} = 0, \tag{1.1}$$

where $x^{\perp}$ is the orthogonal projection of the position vector x to the normal space $T^{\perp}M^n$ of x.

It is well known that the self-shrinker plays an important role in the study of the mean curvature flow. Not only self-shrinkers correspond to self-shrinking solutions to the mean curvature flow, but also they describe all possible Type I blow ups at a given singularity of the

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✉ Xingxiao Li
xxl@henannu.edu.cn

Xiufen Chang
changxfff@163.com

1 School of Mathematics and Information Sciences, Henan Normal University, Xinxiang 453007, Henan, People's Republic of China

flow. Up to now, there have been a plenty of research papers on self-shrinkers among which are many that provide various results of classification or rigidity theorems. In particular, there are also interesting results about the Lagrangian self-shrinkers in the complex Euclidean n-space $\mathbb{C}^n$. For example, in [1], Anciaux gives new examples of self-shrinking and self-expanding Lagrangian solutions to the mean curvature flow. In [3], the authors classify all Hamiltonian stationary Lagrangian surfaces in the complex plane $\mathbb{C}^2$, which are self-similar solutions of the mean curvature flow and, in [4], several rigidity results for Lagrangian mean curvature flow are obtained. As we know, a canonical example of the compact Lagrangian self-shrinker in $\mathbb{C}^2$ is the Clifford torus $\mathbb{S}^1(1) \times \mathbb{S}^1(1)$.

Recently in [13], Li and Wang prove a rigidity theorem which improves a previous theorem by Castro and Lerma [4].

Theorem 1.1 (cf. [4,13]). *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian self-shrinker with h its second fundamental form. If $|h|^2 \leq 2$, then $|h|^2 = 2$ and $x(M^2)$ is the Clifford torus $\mathbb{S}^1(1) \times \mathbb{S}^1(1)$, up to a holomorphic isometry on $\mathbb{C}^2$.*

Remark 1.1 Castro and Lerma also proved Theorem 1.1 in [4] under the additional condition that the Gauss curvature K of M^2 is either non-negative or non-positive.

To make an extension of hypersurface self-shrinkers, Cheng and Wei recently introduce in [7] the definition of λ-hypersurface of weighted volume-preserving mean curvature flow in Euclidean space, and classify complete λ-hypersurfaces with polynomial area growth and $H - \lambda \geq 0$, which are generalizations of the results due to Huisken [12] and Colding-Minicozzi [9]. According to [7], a hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is called a λ-hypersurface if its mean curvature H_0 satisfies

$$H_0 + \langle x, N \rangle = \lambda \tag{1.2}$$

for some constant λ, where N is the unit normal vector of x. Some rigidity or classification results for λ-hypersurfaces are obtained, for example, in [6,8,11]; for the rigidity theorems for space-like λ-hypersurfaces see [15].

As a natural generalization of both self-shrinkers and λ-hypersurfaces, we introduce the concept of ξ-submanifolds. Precisely, an immersed submanifold $x : M^n \to \mathbb{R}^{n+p}$ is called a ξ-submanifold if there is a parallel normal vector field ξ such that the mean curvature vector field H satisfies

$$H + x^{\perp} = \xi. \tag{1.3}$$

Obviously, the Clifford tori $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ with positive numbers a and b are examples of Lagrangian ξ-submanifold in $\mathbb{C}^2$. Similar examples in higher dimensions can be listed as those in [5] for self-shrinkers. In this paper, we focus on the rigidity of compact Lagrangian ξ-submanifolds in $\mathbb{C}^2$, and our main theorem is as follows:

Theorem 1.2 *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian ξ-submanifold with the second fundamental form h and mean curvature vector H. Assume that*

$$|h|^2 + |H - \xi|^2 \leq |\xi|^2 + 4.$$

Then $|h|^2 + |H - \xi|^2 \equiv |\xi|^2 + 4$ and $x(M^2) = T^2$ is a topological torus.

Furthermore, if $\langle H, \xi \rangle$ is constant and one of the following four conditions holds:

$$(1)\ |h|^2 \geq 2, \quad (2)\ |H|^2 \geq 2, \quad (3)\ |h|^2 \geq \langle H, H - \xi \rangle, \quad (4)\ \langle H, \xi \rangle \geq 0, \tag{1.4}$$

then, up to a holomorphic isometry on $\mathbb{C}^2$, $x(M^2) = \mathbb{S}^1(a) \times \mathbb{S}^1(b)$ *is a standard torus, where* a *and* b *are positive numbers satisfying* $a^2 + b^2 \geq 2a^2b^2$.

Corollary 1.3 *Let* $x : M^2 \to \mathbb{C}^2$ *be a compact oriented Lagrangian self-shrinker. If*

$$|h|^2 + |H|^2 \leq 4,$$

then $|h|^2 + |H|^2 \equiv 4$ *and* $x(M^2) = \mathbb{S}^1(1) \times \mathbb{S}^1(1)$ *up to a holomorphic isometry on* $\mathbb{C}^2$.

Clearly, Corollary 1.3 can be viewed as a different new version of Theorem 1.1.

Remark 1.2 We believe that the last condition (1.4) in Theorem 1.2 can be removed. On the other hand, the condition that $\langle H, \xi \rangle$ is constant may also be removed. In fact, as suggested by the referee, we can use (3.11) and the compactness of M^2 to show that $|x|^2$ is constant when either $\langle H, H - \xi \rangle \leq 2$ or $\langle H, H - \xi \rangle \geq 2$. Then by the argument at the end of the paper, we can simplify Theorem 1.2 as follows:

Theorem 1.4 *Let* $x : M^2 \to \mathbb{C}^2$ *be a compact oriented Lagrangian* ξ*-submanifold with the second fundamental form* h *and mean curvature vector* H. *Assume that*

$$|h|^2 + |H - \xi|^2 \leq |\xi|^2 + 4.$$

Then $|h|^2 + |H - \xi|^2 \equiv |\xi|^2 + 4$ *and* $x(M^2) = T^2$ *is a topological torus.*

Furthermore, if either $\langle H, H - \xi \rangle \leq 2$ *or* $\langle H, H - \xi \rangle \geq 2$, *then, up to a holomorphic isometry on* $\mathbb{C}^2$, $x(M^2) = \mathbb{S}^1(a) \times \mathbb{S}^1(b)$ *is a standard torus for some* $a, b > 0$.

Remark 1.3 Cheng and Wei have introduced in [7] a weighted area functional $\mathcal{A}$ and derived a related variation formula. Besides the relation between λ-hypersurfaces and the weighted volume preserving mean curvature flow, they also prove that λ-hypersurfaces are the critical points of the weighted area functional. Based on this, we believe that similar conclusions will be valid for the ξ-submanifolds defined above. Furthermore, We reasonably believe that, if self-shrinkers and λ-hypersurfaces take the places of minimal submanifolds and constant mean curvature hypersurfaces, respectively, then ξ-submanifolds must take the place of submanifolds of parallel mean curvature vector.

2 Lagrangian submanifolds in $\mathbb{C}^n$ and their Maslov class

Let $\mathbb{C}^n$ be the complex Euclidean n-space with the canonical complex structure J. Through out this paper, $x : M^n \to \mathbb{C}^n$ always denotes an n-dimensional Lagrangian submanifold, and ∇, D, $\nabla^\perp$ denote, respectively, the Levi-Civita connections on M^n, $\mathbb{C}^n$, and the normal connection on the normal boundle $T^\perp M^n$. The formulas of Gauss and Weingarten are given by

$$D_X Y = \nabla_X Y + h(X, Y), \quad D_X \eta = -A_\eta X + \nabla^\perp_X \eta,$$

where X, Y are tangent vector fields on M^n and η is a normal vector field of x. The Lagrangian condition implies that

$$\nabla^\perp_X JY = J\nabla_X Y, \quad A_{JX} Y = -Jh(X, Y) = A_{JY} X,$$

where h and A are the second fundamental form and the shape operator of x, respectively. In particular, $\langle h(X, Y), JZ \rangle$ is totally symmetric as a 3-form, namely

$$\langle h(X, Y), JZ \rangle = \langle h(X, Z), JY \rangle = \langle h(Y, Z), JX \rangle. \tag{2.1}$$

From now on, we agree with the following convention on the ranges of indices:

$$1 \leq i, j, \cdots \leq n, \quad n+1 \leq \alpha, \beta, \cdots \leq 2n, \quad 1 \leq A, B, \cdots \leq 2n, \quad i^* = n+i.$$

For a Lagrangian submanifold $x : M^n \to \mathbb{C}^n$, there are orthonormal frame fields of the form $\{e_i, e_{i^*}\}$ for $\mathbb{C}^n$ along x, where $e_i \in TM^n$ and $e_{i^*} = Je_i$. Such a frame is called an *adapted Lagrangian frame field* in the literature. The dual frame field is always denoted by $\{\theta_i, \theta_{i^*}\}$, where $\theta_{i^*} = -J\theta_i$. Write

$$h = \sum h_{ij}^{k^*} \theta_i \theta_j e_{k^*}, \quad \text{where } h_{ij}^{k^*} = \langle h(e_i, e_j), e_{k^*} \rangle,$$

or equivalently,

$$h(e_i, e_j) = \sum_k h_{ij}^{k^*} e_{k^*}, \quad \text{for all } e_i, e_j.$$

Then (2.1) is equivalent to

$$h_{ij}^{k^*} = h_{kj}^{i^*} = h_{ik}^{j^*}, \quad 1 \leq i, j, k \leq n. \tag{2.2}$$

If θ_{ij} and $\theta_{i^*j^*}$ denote the connection forms of ∇ and $\nabla^\perp$, respectively, then the components $h_{ij,l}^{k^*}$, $h_{ij,lp}^{k^*}$ of the covariant derivatives of h are given respectively by

$$\sum_l h_{ij,l}^{k^*} \theta_l = dh_{ij}^{k^*} + \sum_l h_{lj}^{k^*} \theta_{li} + \sum_l h_{il}^{k^*} \theta_{lj} + \sum_m h_{ij}^{m^*} \theta_{m^*k^*}; \tag{2.3}$$

$$\sum_p h_{ij,lp}^{k^*} \theta_p = dh_{ij,l}^{k^*} + \sum_p h_{pj,l}^{k^*} \theta_{pi} + \sum_p h_{ip,l}^{k^*} \theta_{pj} + \sum_p h_{ij,p}^{k^*} \theta_{pl} + \sum_p h_{ij,l}^{p^*} \theta_{p^*k^*}. \tag{2.4}$$

Moreover, the equations of motion are as follows:

$$dx = \sum_i \theta_i e_i, \quad de_i = \sum_j \theta_{ij} e_j + \sum_{k,j} h_{ij}^{k^*} \theta_j e_{k^*}, \tag{2.5}$$

$$de_{k^*} = -\sum_{i,j} h_{ij}^{k^*} \theta_j e_i + \sum_l \theta_{k^*l^*} e_{l^*}. \tag{2.6}$$

Let R_{ijkl} and $R_{i^*j^*kl}$ denote the components of curvature operators of ∇ and $\nabla^\perp$, respectively. Then the equations of Gauss, Codazzi and Ricci are as follows:

$$R_{mijk} = \sum_l (h_{mk}^{l^*} h_{ij}^{l^*} - h_{mj}^{l^*} h_{ik}^{l^*}), \quad 1 \leq m, i, j, k \leq n, \tag{2.7}$$

$$h_{ij,l}^{k^*} = h_{il,j}^{k^*}, \quad 1 \leq i, j, k, l \leq n, \tag{2.8}$$

$$R_{i^*j^*kl} = \sum_m (h_{ml}^{i^*} h_{mk}^{j^*} - h_{mk}^{i^*} h_{ml}^{j^*}), \quad 1 \leq i, j, k, l \leq n. \tag{2.9}$$

The scalar curvature of ∇ is

$$R = |H|^2 - |h|^2 \quad \text{with } |H|^2 = \sum_k \left(\sum_i h_{ii}^{k^*} \right)^2, \quad |h|^2 = \sum_{i,j,k} (h_{ij}^{k^*})^2, \tag{2.10}$$

where the mean curvature vector field H is defined by

$$H = \sum_k H^{k^*} e_{k^*} = \sum_{i,k} h_{ii}^{k^*} e_{k^*}.$$

Combining (2.2) and (2.8), we know that $h^{k^*}_{ij,l}$ is totally symmetric, namely

$$h^{k^*}_{ij,l} = h^{i^*}_{jl,k} = h^{j^*}_{lk,i} = h^{l^*}_{ki,j}, \quad 1 \le i, j, k, l \le n, \tag{2.11}$$

and the Ricci identities are as follows:

$$h^{k^*}_{ij,lp} - h^{k^*}_{ij,pl} = \sum_m h^{k^*}_{mj} R_{imlp} + \sum_m h^{k^*}_{im} R_{jmlp} + \sum_m h^{m^*}_{ij} R_{k^*m^*lp}. \tag{2.12}$$

Note that, with respect to the adapted Lagrangian frame $\{e_i, e_{i^*}\}$, the connection forms $\theta_{i^*j^*} = \theta_{ij}$. It follows that

$$R_{m^*i^*jk} = R_{mijk}, \quad \forall m, i, j, k. \tag{2.13}$$

Furthermore, the first and second derivatives $H^{k^*}_{,i}$, $H^{k^*}_{,ij}$ of the mean curvature vector field H are given as

$$H^{k^*}_{,i} = \sum_j h^{k^*}_{jj,i}, \quad H^{k^*}_{,ij} = \sum_l h^{k^*}_{ll,ij}. \tag{2.14}$$

For any smooth function f on M^n, the covariant derivatives $f_{,i}$, $f_{,ij}$ of f, the Laplacian of f are respectively defined as follows:

$$df = \sum_i f_{,i}\theta_i, \quad \sum_j f_{,ij}\theta_j = df_{,i} - \sum_j f_{,j}\theta_{ij}, \quad \triangle f = \sum_i f_{,ii}. \tag{2.15}$$

Finally, we also need to introduce the Lagrangian angles, Maslov form and Maslov class of a Lagrangian submanifold in $\mathbb{C}^n$ which we shall make use of later.

Let $(z^1, \dots, z^n)$ be the standard complex coordinates on $\mathbb{C}^n$. Then $\Omega = dz^1 \wedge \cdots \wedge dz^n$ is a globally defined *holomorphic volume form* which is clearly parallel. For a Lagrangian submanifold $x : M^n \to \mathbb{C}^n$, *the Lagrangian angle* of x is by definition a multi-valued function $\beta : M^n \to \mathbb{R}/2\pi\mathbb{Z}$ given by

$$\Omega_M := x^*\Omega = e^{\sqrt{-1}\beta} dV_M.$$

As one knows, although the Lagrangian angle β can not be determined globally in general, its gradient $\nabla\beta$ is clearly a well-defined vector field on M^n, or the same, $\alpha := d\beta$ is a globally defined 1-form which is called the *Maslov form* of x. Clearly, α is closed and thus represents a cohomology class $[\alpha] \in H^1(M^n)$ called the *Maslov class*.

In [16], the author proved an important formula by which the mean curvature and the Lagrangian angle of a Lagrangian submanifold are linked to each other; A. Arsie has extended this result in [2] to Lagrangian submanifolds in a general Calabi-Yau manifold.

Theorem 2.1 ([16]) *Let $x : M^n \to \mathbb{C}^n$ be a Lagrangian submanifold and J be the canonical complex structure of $\mathbb{C}^n$. Then the mean curvature vector H and the Lagrangian angle β meet the following formula:*

$$x_*(\nabla\beta) = -JH. \tag{2.16}$$

Corollary 2.2 ([4,17]) *Let $x : M^n \to \mathbb{C}^n$ be a compact and oriented Lagrangian self-shrinkers. Then the Maslov class $[\alpha]$ can not be trivial. In particular, there does not exist any Lagrangian self-shrinker in $\mathbb{C}^n$ with the topology of a sphere.*

Remark 2.1 For our use in this paper, it is necessary to show that Corollary 2.2 is still true if we replace the self-shrinker by a ξ-submanifold. Precisely, we need

Proposition 2.3 *Let $x : M^n \to \mathbb{C}^n$ be a Lagrangian ξ-submanifold. If M is compact and orientable, then $[\alpha] \neq 0$; Consequently, there does not exist any Lagrangian ξ-submanifold in $\mathbb{C}^n$ with the topology of a sphere.*

Proof By the definition of a ξ-submanifold, we have $x = x^\top + \xi - H$. By Gauss and Weingarten formulas it follows that, for any $v \in TM^n$,

$$\begin{aligned} A_H v &= -D_v H + \nabla_v^\perp H = -D_v(\xi - x^\perp) + \nabla_v^\perp H \\ &= D_v x^\perp - D_v \xi + \nabla_v^\perp H = D_v x - D_v x^\top - D_v \xi + \nabla_v^\perp H \\ &= v - \nabla_v x^\top + A_\xi(v) - h(v, x^\top) + \nabla_v^\perp H, \end{aligned}$$

where A_H and A_ξ are Weingarten transformations with respect to H and ξ, respectively. Thus

$$A_H v = v - \nabla_v x^\top + A_\xi(v), \quad \nabla_v^\perp H = h(v, x^\top).$$

So that

$$\begin{aligned} \operatorname{div} JH &= \sum_i \langle \nabla_{e_i} JH, e_i\rangle = \sum_i \langle J\nabla_{e_i} JH, Je_i\rangle = \sum_i \langle -\nabla_{e_i}^\perp H, Je_i\rangle \\ &= \sum_i \langle -h(e_i, x^\top), Je_i\rangle = \sum_i -\langle h(e_i, e_i), Jx^\top\rangle \\ &= \sum_i \langle Jh(e_i, e_i), x^\top\rangle = \langle JH, x^\top\rangle, \end{aligned} \tag{2.17}$$

where div is the divergence operator. By (2.16) and (2.17) we obtain

$$\Delta\beta = \langle \nabla\beta, x^\top\rangle = \frac{1}{2}\langle \nabla\beta, \nabla|x|^2\rangle. \tag{2.18}$$

If $[\alpha] = 0$, then there exists a globally defined Lagrangian angle β such that $\alpha = -d\beta$, implying (2.18) holds globally on M^n. Then the compactness assumption and the maximum principle for a second linear elliptic partial equation (see [10], for example) assure that β must be constant. Hence $H = x_*(J\nabla\beta) \equiv 0$, contradicting to the fact that there are no compact minimal submanifolds in Euclidean space. This contradiction proves that $[\alpha] \neq 0$.

Since the first homology of a sphere S^n vanishes for $n > 1$, there can not be any Lagrangian ξ-submanifolds with the topology of a sphere. □

3 Proof of the main theorem

Let $x : M^n \to \mathbb{C}^n$ be a Lagrangian ξ-submanifold without boundary. Then, with respect to an orthonormal frame field $\{e_i\}$, the defining equation (1.3) is equivalent to

$$H^{k^*} = -\langle x, e_{k^*}\rangle + \xi^{k^*}, \quad 1 \le k \le n. \tag{3.1}$$

where $\xi = \sum \xi^{k^*} e_{k^*}$ is a given parallel normal vector field. From now on, we always assume that $n = 2$ if no other specification is given.

We start with a well-known operator $\mathcal{L}$ acting on smooth functions defined by

$$\mathcal{L} = \triangle - \langle x, \nabla\cdot\rangle = e^{\frac{|x|^2}{2}} \operatorname{div}(e^{-\frac{|x|^2}{2}}\nabla\cdot), \tag{3.2}$$

which was first introduced by Colding and Minicozzi [9] to the study of self-shrinkers. Since then, the operator $\mathcal{L}$ has been one of the most effect tools adapted by many authors. In particular, the following is a fundamental lemma related to $\mathcal{L}$:

Lemma 3.1 ([14]) *Let $x : M^n \to \mathbb{R}^{n+p}$ be a complete immersed submanifold. If u and v are C^2-smooth functions with*

$$\int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|x|^2}{2}} dV_M < \infty,$$

then it holds that

$$\int_M u\mathcal{L}v e^{-\frac{|x|^2}{2}} dV_M = -\int_M \langle \nabla u, \nabla v\rangle e^{-\frac{|x|^2}{2}} dV_M.$$

Now, to make the whole argument more readable, we divide our proof into the following lemmas and propositions:

Lemma 3.2 (cf. [13]) *Let $x : M^2 \to \mathbb{C}^2$ be a Lagrangian ξ-submanifold. Then*

$$H^{k^*}_{,i} = \sum_j h^{k^*}_{ij}\langle x, e_j\rangle, \quad 1 \le i, k \le 2, \tag{3.3}$$

$$H^{k^*}_{,ij} = \sum_m h^{k^*}_{im,j}\langle x, e_m\rangle + h^{k^*}_{ij} - \sum_{m,p}(H-\xi)^{p^*} h^{k^*}_{im} h^{p^*}_{mj}, \quad 1 \le i, j, k \le 2. \tag{3.4}$$

Lemma 3.3 *It holds that*

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|h|^2 + |H-\xi|^2) = {} & |\nabla h|^2 + |\nabla^\perp H|^2 + |h|^2 \\ & - \frac{1}{2}(|h|^2 - |H|^2)(3|h|^2 - 2|H|^2 + \langle H, H-\xi\rangle) \\ & + \langle H, H-\xi\rangle - \sum_{i,j,k,l} h^{k^*}_{ij} h^{l^*}_{ij}(H-\xi)^{k^*}(H-\xi)^{l^*} \\ & - \sum_{i,j,k,l} h^{k^*}_{ij} h^{l^*}_{ij} H^{k^*}(H-\xi)^{l^*}.\end{aligned} \tag{3.5}$$

Proof By a direct computation using Lemma 3.2 we find (cf. [13])

$$\begin{aligned}\frac{1}{2}\mathcal{L}|h|^2 = {} & |\nabla h|^2 + |h|^2 - \frac{3}{2}|h|^4 + \frac{5}{2}|H|^2|h|^2 - |H|^4 \\ & + \frac{1}{2}\langle H, H-\xi\rangle(|H|^2 - |h|^2) - \sum_{i,j,k,l} H^{k^*} h^{k^*}_{ij} h^{l^*}_{ij}(H-\xi)^{l^*};\end{aligned} \tag{3.6}$$

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|H-\xi|^2) &= \frac{1}{2}\triangle(|H-\xi|^2) - \frac{1}{2}\langle x, \nabla|H-\xi|^2\rangle \\ &= \sum_{i,k}(H-\xi)^{k^*} H^{k^*}_{,ii} + |\nabla^\perp H|^2 - \sum_{i.k}(H-\xi)^{k^*} H^{k^*}_{,i}\langle x, e_i\rangle \\ &= \langle H-\xi, H\rangle + |\nabla^\perp H|^2 - \sum_{i,j,k,l}(H-\xi)^{k^*} h^{k^*}_{ij} h^{l^*}_{ij}(H-\xi)^{l^*}.\end{aligned} \tag{3.7}$$

By taking the sum we obtain (3.5). □

Lemma 3.4 *It holds that*

$$\frac{1}{2}\triangle(|x^\top|^2) = \sum_{i,j,k} h_{ij}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle(\xi - H)^{k^*} - \sum_{i,j,k,l} h_{il}^{k^*} h_{lj}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle + 2 - 2\langle H, H - \xi\rangle + \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}(H - \xi)^{k^*}(H - \xi)^{l^*}. \quad (3.8)$$

Proof We find

$$\begin{aligned}\frac{1}{2}\triangle(|x^\top|^2) &= \frac{1}{2}\sum_{i,j}\langle x, e_j\rangle^2_{,ii} = \sum_{i,j}(\langle x, e_j\rangle\langle x, e_j\rangle_i)_{,i}\\ &= \sum_{i,j}\big(\langle x, e_j\rangle\langle x_i, e_j\rangle\big)_{,i} + \sum_{i,j,k}\Big(\langle x, e_j\rangle\langle x, h_{ji}^{k^*} e_{k^*}\rangle\Big)_{,i}\\ &= 2 + 2\sum_{i,k} h_{ii}^{k^*}\langle x, e_{k^*}\rangle + \sum_{j,k} H_{,j}^{k^*}\langle x, e_j\rangle\langle x, e_{k^*}\rangle\\ &\quad + \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}\langle x, e_{l^*}\rangle\langle x, e_{k^*}\rangle - \sum_{i,j,k,l} h_{ij}^{k^*} h_{il}^{k^*}\langle x, e_j\rangle\langle x, e_l\rangle\\ &= 2 - 2\langle H, H - \xi\rangle + \sum_{i,j,k} h_{ij}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle(\xi - H)^{k^*}\\ &\quad + \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}(H - \xi)^{l^*}(H - \xi)^{k^*} - \sum_{i,j,k,l} h_{il}^{k^*} h_{lj}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle,\end{aligned}$$

and the lemma is proved. □

Lemma 3.5 *It holds that*

$$\triangle(\langle H, \xi\rangle) = \sum_{i,j,k} h_{ij}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle\xi^{k^*} + \langle H, \xi\rangle - \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}\xi^{k^*}(H - \xi)^{l^*}, \quad (3.9)$$

$$\mathcal{L}(\langle H, \xi\rangle) = \langle H, \xi\rangle - \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}\xi^{k^*}(H - \xi)^{l^*}. \quad (3.10)$$

Proof By (3.3) and (3.4),

$$\begin{aligned}\triangle(\langle H, \xi\rangle) &= \sum_{i,k}(H^{k^*}\xi^{k^*})_{,ii} = \sum_{i,k} H_{,ii}^{k^*}\xi^{k^*}\\ &= \sum_{i,k,l,m}(h_{im,i}^{k^*}\langle x, e_m\rangle + h_{ii}^{k^*} - (H - \xi)^{l^*} h_{im}^{k^*} h_{mi}^{l^*})\xi^{k^*}\\ &= \sum_{i,k} H_{,i}^{k^*}\langle x, e_i\rangle\xi^{k^*} + \langle H, \xi\rangle - \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}\xi^{k^*}(H - \xi)^{l^*}\\ &= \sum_{i,j,k} h_{ij}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle\xi^{k^*} + \langle H, \xi\rangle - \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*}\xi^{k^*}(H - \xi)^{l^*};\\ \langle x, \nabla\langle H, \xi\rangle\rangle &= \sum_{i}\langle H, \xi\rangle_{,i}\langle x, e_i\rangle = \sum_{i,j,k} h_{ij}^{k^*}\langle x, e_i\rangle\langle x, e_j\rangle\xi^{k^*}.\end{aligned}$$

Thus, by adding them up, we get (3.10). □

Lemma 3.6 (cf. [5,9]; also [13]) *It holds that*

$$\frac{1}{2}\triangle(|x|^2) = 2 - \langle H, H - \xi\rangle, \tag{3.11}$$

$$\frac{1}{2}\mathcal{L}(|x|^2) = |\xi|^2 + 2 - (|x|^2 + \langle H, \xi\rangle). \tag{3.12}$$

Proof From (3.1), we find

$$\frac{1}{2}\triangle(|x|^2) = 2 + \langle x, \triangle x\rangle = 2 + \sum_k H^{k^*}\langle x, e_{k^*}\rangle = 2 - \langle H, H - \xi\rangle,$$

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|x|^2) &= \frac{1}{2}\triangle(|x|^2) - \frac{1}{2}\langle x, \nabla |x|^2\rangle = 2 - |H|^2 + \langle H, \xi\rangle - |x^\top|^2 \\ &= 2 + |\xi|^2 - (|x|^2 + \langle H, \xi\rangle).\end{aligned}$$

□

Proposition 3.7 *Let M^2 be oriented and compact. If*

$$|h|^2 + |H - \xi|^2 \le |\xi|^2 + 4,$$

then

$$|h|^2 + |H - \xi|^2 \equiv |\xi|^2 + 4 \tag{3.13}$$

and $x(M^2)$ is a topological torus.

Proof By Lemma 3.6,

$$\begin{aligned}\int_M |H - \xi|^2 dV_M &= \int_M (|\xi|^2 + 2(|H|^2 - \langle H, \xi\rangle) - |H|^2) dV_M \\ &= \int_M (|\xi|^2 + 4 - |H|^2) dV_M.\end{aligned} \tag{3.14}$$

Let K be the Gauss curvature of M^2. Then the Gauss equation gives that

$$2K = |H|^2 - |h|^2.$$

Denote by $\text{gen}(M^2)$ the genus of M^2. Then from the Gauss-Bonnet theorem and (3.14) it follows that

$$\begin{aligned}8\pi(1 - \text{gen}(M^2)) &= 2\int_M K dV_M = \int_M (|H|^2 - |h|^2) dV_M \\ &= \int_M (|\xi|^2 + 4 - (|h|^2 + |H - \xi|^2)) dV_M \ge 0,\end{aligned} \tag{3.15}$$

implying that $\text{gen}(M^2) \le 1$. So M^2 is topologically either a 2-sphere or a torus. But Proposition 2.3 excludes the first possibility. So $\text{gen}(M^2) = 1$ and (3.13) is proved. □

Lemma 3.8 *Let $p_0 \in M^2$ be a point where $|x|^2$ attains its minimum on M^2. If M^2 is orientable, compact and*

$$|h|^2 + |H - \xi|^2 = \text{const},$$

then

$$\nabla^\perp H(p_0) = 0, \quad (\nabla h)(p_0) = 0. \tag{3.16}$$

Proof Since $(|x|^2)_{,j} = 0, 1 \le j \le 2$ at p_0, it holds that $\langle x, e_j\rangle(p_0) = 0, 1 \le j \le 2$. So by (3.3) we have

$$H^{k*}_{,i} = 0,\ 1 \le i, k \le 2, \quad |H - \xi|^2_{,i} = 2\sum_k (H - \xi)^{k*} H^{k*}_{,i} = 0, \quad 1 \le i \le 2 \ \text{ at } \ p_0 \tag{3.17}$$

where the first set of equalities are exactly $\nabla^{\perp} H(p_0) = 0$, which give

$$h^{1*}_{11,1} + h^{1*}_{22,1} = 0, \quad h^{1*}_{11,2} + h^{1*}_{22,2} = 0, \quad h^{2*}_{11,1} + h^{2*}_{22,1} = 0, \quad h^{2*}_{11,2} + h^{2*}_{22,2} = 0. \tag{3.18}$$

On the other hand, from

$$|h|^2 + |H - \xi|^2 = \text{const}, \tag{3.19}$$

we obtain

$$|h|^2_{,k} + |H - \xi|^2_{,k} \equiv 0, \quad 1 \le k \le 2, \tag{3.20}$$

which with (3.17) implies that

$$(|h|^2)_{,k} = 0, \quad 1 \le k \le 2 \ \text{ at } p_0.$$

Since

$$|h|^2 = (h^{1*}_{11})^2 + 2(h^{1*}_{12})^2 + (h^{1*}_{22})^2 + (h^{2*}_{11})^2 + 2(h^{2*}_{12})^2 + (h^{2*}_{22})^2,$$

we find that

$$h^{1*}_{11}h^{1*}_{11,1} + 2h^{1*}_{12}h^{1*}_{12,1} + h^{1*}_{22}h^{1*}_{22,1} + h^{2*}_{11}h^{2*}_{11,1} + 2h^{2*}_{12}h^{2*}_{12,1} + h^{2*}_{22}h^{2*}_{22,1} = 0, \tag{3.21}$$
$$h^{1*}_{11}h^{1*}_{11,2} + 2h^{1*}_{12}h^{1*}_{12,2} + h^{1*}_{22}h^{1*}_{22,2} + h^{2*}_{11}h^{2*}_{11,2} + 2h^{2*}_{12}h^{2*}_{12,2} + h^{2*}_{22}h^{2*}_{22,2} = 0 \tag{3.22}$$

hold at p_0. From (2.11) and (3.18) we get

$$h^{1*}_{22,1} = -h^{1*}_{11,1}, \quad h^{1*}_{22,2} = -h^{1*}_{11,2}, \quad h^{2*}_{22,2} = h^{1*}_{11,1} \ \text{ at } p_0. \tag{3.23}$$

Since, by (2.2) and (2.11), both h^{k*}_{ij} and $h^{k*}_{ij,l}$ are totally symmetric, we obtain by (3.23), (3.21) and (3.22) that

$$(h^{1*}_{11} - 3h^{1*}_{22})h^{1*}_{11,1} - (h^{2*}_{22} - 3h^{2*}_{11})h^{1*}_{11,2} = 0, \tag{3.24}$$
$$(h^{2*}_{22} - 3h^{2*}_{11})h^{1*}_{11,1} + (h^{1*}_{11} - 3h^{1*}_{22})h^{1*}_{11,2} = 0 \ \text{ at } p_0. \tag{3.25}$$

We claim that

$$(\nabla h)(p_0) = 0. \tag{3.26}$$

Otherwise, we should have $(h^{1*}_{11,1})^2 + (h^{1*}_{11,2})^2 \ne 0$ at p_0. Then from (3.24) and (3.25) it follows that

$$(h^{1*}_{11} - 3h^{1*}_{22})^2 + (h^{2*}_{22} - 3h^{2*}_{11})^2 = 0 \ \text{ at } p_0.$$

Thus

$$|h|^2(p_0) = \frac{4}{3}((h^{1*}_{11})^2 + (h^{2*}_{22})^2), \quad |H|^2(p_0) = \frac{16}{9}((h^{1*}_{11})^2 + (h^{2*}_{22})^2). \tag{3.27}$$

Now by the definition of p_0 and Lemma 3.6,

$$0 \le \frac{1}{2}\triangle|x|^2(p_0) = 2 - \langle H, H-\xi\rangle(p_0).$$

It follows that

$$\begin{aligned}|h|^2 + |H-\xi|^2 &= (|h|^2 + |H-\xi|^2)(p_0)\\ &= \frac{3}{4}|H|^2(p_0) + 2\langle H, H-\xi\rangle(p_0) - |H|^2(p_0) + |\xi|^2\\ &= |\xi|^2 + 2\langle H, H-\xi\rangle(p_0) - \frac{1}{4}|H|^2(p_0) \qquad (3.28)\\ &\le |\xi|^2 + 4. \qquad (3.29)\end{aligned}$$

Therefore, by Proposition 3.7, $|h|^2 + |H-\xi|^2 = |\xi|^2 + 4$. But it is easy to see that the equality in (3.29) holds if and only if $|H|^2(p_0) = 0$ and $\langle H, H-\xi\rangle(p_0) = 2$, which is of course not possible! This contradiction proves the above claim and completes the proof of Lemma 3.8. □

Remark 3.1 Our main observation here is that, if $p_0 \in M^2$ is a minimum point of $|x|^2$ then

$$x^\top(p_0) = \sum_i \langle x, e_i\rangle e_i(p_0) = 0,$$

implying

$$\nabla^\perp H(p_0) = \nabla^\perp(H-\xi)(p_0) = 0.$$

In particular, p_0 is also a minimum point of $|x^\top|^2$.

Proposition 3.9 *Let $x : M^2 \to \mathbb{C}^2$ be a compact and oriented Lagrangian ξ-submanifold. Suppose that*

$$|h|^2 + |H-\xi|^2 = |\xi|^2 + 4$$

and $\langle H, \xi\rangle$ is constant. If one of the followings holds,

$$(1)\ |h|^2 \ge 2, \quad (2)\ |H|^2 \ge 2, \quad (3)\ |h|^2 \ge \langle H, H-\xi\rangle, \quad (4)\ \langle H, \xi\rangle \ge 0, \qquad (3.30)$$

then $|x|^2$ is a constant.

Proof As above, let p_0 be a minimum point of $|x|^2$. Then, by Lemma 3.3 and Lemma 3.8, it holds at p_0 that

$$\begin{aligned}0 &= \frac{1}{2}\mathcal{L}(|h|^2 + |H-\xi|^2)\\ &= |h|^2 - \frac{1}{2}(|h|^2 - |H|^2)(3|h|^2 - 2|H|^2 + \langle H, H-\xi\rangle) + \langle H, H-\xi\rangle\\ &\quad - \sum_{i,j,k,l} h^{k^*}_{ij} h^{l^*}_{ij}(H-\xi)^{k^*}(H-\xi)^{l^*} - \sum_{i,j,k,l} h^{k^*}_{ij} h^{l^*}_{ij} H^{k^*}(H-\xi)^{l^*}. \qquad (3.31)\end{aligned}$$

Furthermore, form Lemma 3.4 and Lemma 3.5 it follows that, at p_0

$$0 \le \frac{1}{2}\triangle(|x^\top|^2) = 2 - 2\langle H, H-\xi\rangle + \sum_{i,j,k,l} h^{k^*}_{ij} h^{l^*}_{ij}(H-\xi)^{k^*}(H-\xi)^{l^*}, \qquad (3.32)$$

$$0 = \frac{1}{2}\mathcal{L}(\langle H, \xi\rangle) = \frac{1}{2}(\langle H, \xi\rangle - \sum_{i,j,k,l} h^{k^*}_{ij} h^{l^*}_{ij}\xi^{k^*}(H-\xi)^{l^*}), \qquad (3.33)$$

implying

$$-\sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*} (H-\xi)^{k^*} (H-\xi)^{l^*} \le 2 - 2\langle H, H-\xi \rangle, \tag{3.34}$$

and

$$\begin{aligned}
&-\sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*} (H-\xi)^{l^*} H^{k^*} \\
&= -\sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*} (H-\xi)^{k^*} (H-\xi)^{l^*} - \sum_{i,j,k,l} h_{ij}^{k^*} h_{ij}^{l^*} \xi^{k^*} (H-\xi)^{l^*} \\
&\le 2 - 2\langle H, H-\xi \rangle - \langle H, \xi \rangle = 2 - \langle H, H-\xi \rangle - |H|^2.
\end{aligned}$$

Consequently, we have at p_0

$$\begin{aligned}
0 &= \frac{1}{2}\mathcal{L}(|h|^2 + |H-\xi|^2) \\
&\le -\frac{1}{2}(|h|^2 - |H|^2)(3|h|^2 - 2|H|^2 + \langle H, H-\xi \rangle) \\
&\quad + |h|^2 - |H|^2 + 2(2 - \langle H, H-\xi \rangle).
\end{aligned}$$

On the other hand, from

$$|h|^2 + |H-\xi|^2 = |\xi|^2 + 4,$$

we know that

$$|h|^2 - |H|^2 = 2(2 - \langle H, H-\xi \rangle) \ge 0 \quad \text{at } p_0. \tag{3.35}$$

Thus, if one of (3.30) holds, then at p_0

$$\begin{aligned}
0 &= \frac{1}{2}\mathcal{L}(|h|^2 + |H-\xi|^2) \\
&\le -\frac{1}{2}(|h|^2 - |H|^2)(2|h|^2 - |H|^2 + 4 - \langle H, H-\xi \rangle) + 2(|h|^2 - |H|^2) \\
&= -\frac{1}{2}(|h|^2 - |H|^2)(2|h|^2 - |H|^2 - \langle H, H-\xi \rangle) \\
&= -\frac{1}{2}(|h|^2 - |H|^2)(|h|^2 - |H|^2 + |h|^2 - \langle H, H-\xi \rangle) \\
&= -\frac{1}{2}(|h|^2 - |H|^2)(|h|^2 - |H|^2 + |h|^2 - 2 + 2 - \langle H, H-\xi \rangle) \\
&= -\frac{1}{2}(|h|^2 - |H|^2)(2(|h|^2 - |H|^2) + \langle H, \xi \rangle) \\
&= -\frac{1}{2}(|h|^2 - |H|^2)(2(|h|^2 - |H|^2) + |H|^2 - 2 + 2 - \langle H, H-\xi \rangle) \le 0.
\end{aligned}$$

Consequently

$$|h|^2 - |H|^2 = 2 - \langle H, H-\xi \rangle = 0 \quad \text{at } p_0. \tag{3.36}$$

It follows that

$$\begin{aligned}|x|^2 + \langle H, \xi\rangle &\geq |x|^2(p_0) + \langle H, \xi\rangle(p_0)\\ &= |H-\xi|^2(p_0) + \langle H, \xi\rangle(p_0)\\ &= \langle H, H-\xi\rangle(p_0) + |\xi|^2\\ &= |\xi|^2 + 2.\end{aligned}$$

This together with Lemma 3.1 (for $u = 1$, $v = |x|^2$) and 3.6 gives that

$$0 = \int_M \frac{1}{2}\mathcal{L}(|x|^2)e^{-\frac{|x|^2}{2}}dV_M = \int_M (|\xi|^2 + 2 - (|x|^2 + \langle H, \xi\rangle))e^{-\frac{|x|^2}{2}}dV_M \leq 0$$

implying that $|x|^2 + \langle H, \xi\rangle = |\xi|^2 + 2$. In particular, $|x|^2 = \text{const}$. □

Proposition 3.10 *Let $x : M^n \to N^n$ be a Lagrangian submanifold in a Kähler manifold N^n. If both M^n and N^n are flat, then around each point $p \in M^n$, there exists some orthonormal frame field $\{e_i, e_{i^*}\}$ with $e_{i^*} = Je_i$ $(1 \leq i \leq n)$, such that*

$$h_{ij}^{k^*} := \langle h(e_i, e_j), e_{k^*}\rangle = \lambda_i^{k^*}\delta_{ij}, \quad 1 \leq i, j, k \leq n.$$

Proof For $p \in M^n$, we pick an orthonormal tangent frame $\{\bar{e}_i\}$ and an orthonormal normal frame $\{\bar{e}_\alpha\}_{n+1\leq\alpha\leq 2n}$. Define

$$\bar{h}_{ij}^\alpha = \langle h(\bar{e}_i, \bar{e}_j), \bar{e}_\alpha\rangle.$$

Since M^n is flat, x is Lagrangian and N^n is Kähler, $T^\perp M^n$ is also flat with respect to the normal connection. By the Ricci equation and the flatness of N^n,

$$0 = \langle R^\perp(e_i, e_j)e_\alpha, e_\beta\rangle = \sum_k \left(h_{ik}^\beta h_{jk}^\alpha - h_{ik}^\alpha h_{jk}^\beta\right).$$

Hence we can choose another orthonormal tangent frame $\{e_i\}$ such that

$$h_{ij}^\alpha := \langle h(e_i, e_j), e_\alpha\rangle = \mu_i^\alpha\delta_{ij}.$$

Write $e_{k^*} = \sum_\alpha a_{k^*}^\alpha e_\alpha$. Then

$$\begin{aligned}h_{ij}^{k^*} &= \langle h(e_i, e_j), e_{k^*}\rangle = \left\langle h(e_i, e_j), \sum_\alpha a_{k^*}^\alpha e_\alpha\right\rangle = \sum_\alpha a_{k^*}^\alpha\langle h(e_i, e_j), e_\alpha\rangle\\ &= \sum_\alpha a_{k^*}^\alpha h_{ij}^\alpha = \sum_\alpha a_{k^*}^\alpha\mu_i^\alpha\delta_{ij} = \lambda_i^{k^*}\delta_{ij} \text{ with } \lambda_i^{k^*} := \sum_\alpha a_{k^*}^\alpha\mu_i^\alpha.\end{aligned} \tag{3.37}$$

Thus Proposition 3.10 is proved. □

Proof of Theorem Since $|x|^2 = \text{const}$, $|H|^2 - |h|^2 \leq 0$ on M^2 by (3.35). Then it follows from (3.15) that $|H|^2 - |h|^2 \equiv 0$ which with the Gauss equation shows that M^2 is flat. Therefore, due to Proposition 3.10, we can choose $\{e_1, e_2\}$ such that

$$h_{12}^{1^*} = h_{12}^{2^*} = 0. \tag{3.38}$$

It follows that

$$h_{22}^{1^*} = h_{11}^{2^*} = 0. \tag{3.39}$$

On the other hand, since $\nabla h \equiv 0$, we have

$$0 = \sum h_{ijl}^{k^*}\theta_l = dh_{ij}^{k^*} - \sum h_{lj}^{k^*}\theta_{il} - \sum h_{il}^{k^*}\theta_{jl} + \sum h_{ij}^{p^*}\theta_{p^*k^*}.$$

It follows that

(1) $i = j = 1, k = 2$, we get $h_{11}^{1^*}\theta_{1^*2^*} = 0$,

(2) $i = j = 2, k = 1$, we get $h_{22}^{2^*}\theta_{2^*1^*} = 0$,

(3) $i = j = k = 1, 0 = dh_{11}^{1^*} + h_{11}^{1^*}\theta_{1^*1^*} = dh_{11}^{1^*}$, we get $h_{11}^{1^*} = \text{const}$,

(4) $i = j = k = 2, 0 = dh_{22}^{2^*} + h_{22}^{2^*}\theta_{2^*2^*} = dh_{22}^{2^*}$, we get $h_{22}^{2^*} = \text{const}$.

Since x can not be totally geodesic, $(h_{11}^{1^*})^2 + (h_{22}^{2^*})^2 = |h|^2 \neq 0$ by (3.38) and (3.39). Without loss of generality we can assume that $h_{11}^{1^*} \neq 0$. Thus, by (1), we have $\theta_{1^*2^*} = 0$ which with $\nabla J = 0$ shows that $\theta_{12} = 0$. Now let

$$\tilde{e}_1 = e_1 \cos\theta - e_2 \sin\theta, \quad \tilde{e}_2 = e_1 \sin\theta + e_2 \cos\theta$$

be another frame field such that

$$\tilde{h}_{ij}^{k^*} := \langle h(\tilde{e}_i, \tilde{e}_j), \tilde{e}_{k^*}\rangle = \tilde{\lambda}_i^{k^*}\delta_{ij}.$$

Then a direct computation shows that

$$\sin\theta\cos\theta(h_{11}^{1^*}\cos\theta + h_{22}^{2^*}\sin\theta) = \sin\theta\cos\theta(h_{11}^{1^*}\sin\theta - h_{22}^{2^*}\cos\theta) = 0.$$

Since $(h_{11}^{1^*})^2 + (h_{22}^{2^*})^2 \neq 0$, we have $\sin 2\theta = 0$, that is $\theta = 0$, or $\frac{\pi}{2}$ or π. Clearly, by choosing $\theta = \frac{\pi}{2}$, we can change the sign of $h_{11}^{1^*}h_{22}^{2^*}$; while by choosing $\theta = \pi$, we can change the sign of both $h_{11}^{1^*}$ and $h_{22}^{2^*}$. Thus we can always assume that $h_{11}^{1^*} > 0$ and $h_{22}^{2^*} \geq 0$. It then follows that $\{e_1, e_2\}$ can be uniquely determined and, in particular, is globally defined.

Now we claim that $h_{22}^{2^*} > 0$. In fact, if $h_{22}^{2^*} = 0$, then $\theta_{22^*} = 0$. This with $\theta_{12} = \theta_{21^*} = 0$ shows that e_2 is constant in $\mathbb{C}^2$ along M which means that M contains a family of parallel straight lines, contradicting the assumption that M is compact.

Define

$$V_1 = \text{Span}_{\mathbb{R}}\{e_1, e_{1^*}\} = \text{Span}_{\mathbb{C}}\{e_1\}, \quad V_2 = \text{Span}_{\mathbb{R}}\{e_2, e_{2^*}\} = \text{Span}_{\mathbb{C}}\{e_2\}. \quad (3.40)$$

Since

$$de_1 = \nabla e_1 + \sum h_{1j}^{k^*}\theta_j e_{k^*} = h_{11}^{1^*}\theta_1 e_{1^*} \in V_1,$$
$$de_{1^*} = J de_1 = h_{11}^{1^*}\theta_1 J e_{1^*} = -h_{11}^{1^*}\theta_1 e_1 \in V_1,$$

we know that V_1 is a 1-dimensional constant complex subspace of $\mathbb{C}^2$.

Similarly, V_2 is also a 1-dimensional constant complex subspace of $\mathbb{C}^2$. Furthermore, V_1 and V_2 are clearly orthogonal. So, up to a holomorphic isometry on $\mathbb{C}^2$, we can assume that $V_1 = \mathbb{C}^1$, $V_2 = \mathbb{C}^1$ so that $\mathbb{C}^2 = V_1 \times V_2$. Write

$$x = (x^1, x^2) \in V_1 \times V_2 \equiv \mathbb{C}^2.$$

Then

$$0 = e_i(|x|^2) = e_i(|x^1|^2) + e_i(|x^2|^2), \quad i = 1, 2,$$

which with the definitions (3.40) of V_1 and V_2 shows that

$$e_1(|x^1|^2) = e_2(|x^1|^2) = 0, \quad e_1(|x^2|^2) = e_2(|x^2|^2) = 0,$$

that is,

$$|x^1|^2 = \text{const}, \quad |x^2|^2 = const.$$

It is easily seen that both $|x^1|^2$ and $|x^2|^2$ are positive since x is non-degenerate. Thus we can write $|x^1|^2 = a^2 > 0$, $|x^2|^2 = b^2 > 0$. It then follows that $M^2 = \mathbb{S}^1(a) \times \mathbb{S}^1(b)$.

Finally, by the assumption (1.4), $|h|^2 \geq 2$, it should holds that $a^2 + b^2 \geq 2a^2b^2$. □

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References

1. Anciaux, H.: Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$. Geom. Dedic. **120**, 37–48 (2006)
2. Arsie, A.: Maslov class and minimality in Calabi–Yau manifolds. J. Geom. Phys. **35**, 145–156 (2000)
3. Castro, I., Lerma, A.M.: Hamiltonian stationary self-similar solutions for Lagrangian mean curvature flow in the complex Euclidean plane. Proc. Am. Math. Soc. **138**, 1821–1832 (2010)
4. Castro, I., Lerma, A.M.: The Clifford torus as a self-shrinker for the Lagrangian mean curvature flow. Int. Math. Res. Not. **6**, 1515–1527 (2014)
5. Cao, H.-D., Li, H.Z.: A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension. Calc. Var. Partial Diff. Equ. **46**, 879–889 (2013)
6. Cheng, Q.-M., Ogata, S., Wei, G.: Rigidity theorems of λ-hypersurfaces. arXiv arXiv:1403.4123v2 [math. DG], 1 Apr (2014)
7. Cheng, Q.-M., Wei, G.: Complete λ-hypersurfaces of weighted volume-preserving mean curvature flow. arXiv:1403.3177v3 [math. DG], 10 Jun (2014)
8. Cheng, Q.-M., Wei, G.: The Gauss image of λ-hypersurfaces and a Bernstein type problem. arXiv:1410.5302v1 [math.DG], 20 Oct (2014)
9. Colding, T.H., Minicozzi II, W.P.: Generic mean curvature flow I; generic singularities. Ann. Math. **175**, 755–833 (2012)
10. Gilbarg, D., Trudinger, N.S.: Elleptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
11. Guang, Q.: Gap and rigidity theorems of λ-hypersurfaces. arXiv:1405.4871v1 [math. DG], 19 May (2014)
12. Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**, 285–299 (1990)
13. Li, H. Z., Wang, X.F.: A new characterization of the clifford torus as a Lagrangian self-shrinker. arXiv:1505.05593v1 [math.DG], 21 May (2015)
14. Li, H.Z., Wei, Y.: Classification and rigidity of self-shrinkers in the mean curvature flow. J. Math. Soc. Jpn. **66**, 709–734 (2014)
15. Li, X.X., Chang, X.F.: Rigidity theorems on the space-like λ-hypersurfaces in the Lorentzian space $\mathbb{R}^{n+1}_1$, preprint
16. Morvan, J.-M.: Classe de Maslov d'une immersion lagrangienne et minimalité. C. R. Acad. Sci. Paris I **292**, 633–636 (1981)
17. Smoczyk, K.: Self-shrinkers of the mean curvature flow in arbitrary codimension. Int. Math. Res. Not. **48**, 2983–3004 (2005)