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ORIGINAL PAPER

The group of all finite-state automorphisms of a regular rooted tree has a minimal generating set

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Abstract We find some sufficient conditions under which the permutational wreath product of two groups has a minimal (irredundant) generating set. In particular we prove that for a regular rooted tree the group of all automorphisms and the group of all finite-state automorphisms of such a tree satisfy these conditions. Thereby we solve the problem that was stated by B. Csákány and F. Gécseg in 1965.

Keywords Minimal generating set · Permutational wreath product · Automorphisms of rooted tree · Finite-state automorphisms

Mathematics Subject Classification (2000) 20F05 · 20E22 · 20E08

1 Introduction

We consider the following problem

Problem 1 Do the group of all automorphisms and the group of all finite-state automorphisms of a regular rooted tree have any minimal generating set?

This problem was stated originally by Csákány and Gécseg [6] in terms of automata in 1965. They asked whether the semigroup of all automaton transformations, the semigroup of all finite automaton transformations, the group of all bijective automaton transformations, and the group of all finite bijective automaton transformations over a fixed finite alphabet with at least two elements have a minimal generating set?

The answer for semigroups is negative. This result was obtained independently by Aleshin [1] in 1970 and Dömösi [7] in 1972. The question about groups (i.e. Problem 1) was also

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formulated in papers of Dömösi. In particular, it appeared in [8, Problem 2.1] and [9, Problem 2.31]. Moreover this problem is mentioned in the papers [2,16].

Among works related to this problem we mention the result of Andriy Oliynyk from [17]. Namely, it was proven that finite-state wreath product of transformation semigroups is not finitely generated and in some cases doesn't have a minimal generating set. We also mention papers devoted to the study of generating sets in projective limits of wreath products of groups [3,4,14,18].

We find some sufficient conditions under which the permutational wreath product of a finite group and an infinite group has a minimal generating set (Theorem 2). We also give a several examples of groups and classes of groups satisfying such conditions. In particular we prove that for a regular rooted tree the group of all automorphisms and the group of all finite-state automorphisms of such a tree satisfy these conditions (Theorems 7 and 9). Therefore we obtain the main theorem of the paper.

Theorem 1 *The group of all automorphisms and the group of all finite-state automorphisms of a regular rooted tree have minimal generating sets.*

Thus Problem 1 is solved positively.

Most results of this paper were announced without proofs in [12,13].

2 Minimal generating sets in permutational wreath products

We first recall the notion of the permutational wreath product.

Let (A, X) be a permutation group and let H be a group. Then *the permutational wreath product* $(A, X) \wr H$ is the semi-direct product $(A, X) \leftthreetimes H^X$, where (A, X) acts on the direct power H^X by the respective permutations of the direct factors.

We will say that a permutation group (A, X) satisfies the condition **PS** if:

1. The group (A, X) is finite and transitive.
2. There are subsets X_1, X_2 of X and subgroups A_1, A_2 of A with the following properties:
   - (A_1, X) and (A_2, X) act transitively on X_1 and X_2 respectively and act trivially on $X \setminus X_1$ and $X \setminus X_2$ respectively.
   - X_1 and X_2 do not intersect.
   - $|X_1| \geq 2$, $|X_2| \geq 3$.
   - If $|X_1| = 2$ then there is $a \in A$ satisfying $a(X_1) \cap X_2 \neq \emptyset$ and $a(X_1) \nsubseteq X_2$.

We say that a group G satisfies *the L-condition*, if G is decomposed into permutational wreath product $G = (A, X) \wr H$ and there are a normal subgroup H_0 of H and an integer $k > 1$ with the following properties:

1. The permutation group (A, X) satisfies the condition **PS**,
2. The quotient H/H_0 has infinite minimal generating set,
3. $|H/H_0| \geq |H_0|$,
4. Either $H_0 < H'$ or $H' \lneq H_0 < H^k H'$, where H' is the commutator subgroup of H and $H^k = \langle \{h^k, h \in H\} \rangle$,
5. If $H' \lneq H_0 < H^k H'$ then there is a subset C of some minimal generating set of H/H_0 such that
   (a) $|C| = |H/H_0|$,
   (b) For every coset $c \in C$ there is h in the coset c such that $h^k = e$.

Note that condition 2 of the definition of the L-condition imply that H is infinite.
In this section we prove the following theorem.

Theorem 2 *A group with the L-condition has a minimal generating set.*

Proof Proof of Theorem 2

At first we fix some notation.

Let $G = (A, X) \wr H$ and let H_0 be a normal subgroup of H. We assume that G, (A, X), H, and H_0 satisfy the conditions of Theorem 2. Let $X = \{0, 1, \dots, n\}$, $X_1 = \{0, 1, \dots, l_1\}$, and $X_2 = \{l_2, l_2 + 1, \dots, n\}$.

The symbol for the identity element is e and the symbol for the trivial group is E.

The group G is a semidirect product of its subgroups A and K, where K is the direct product of $n+1$ copies of H, i.e., $K = \underbrace{H \times \cdots \times H}_{n+1}$. We will also write whole subgroup K as $(\underbrace{H, \dots, H}_{n+1})$. The conjugation of $g = (g_0, \dots, g_n)$ by an element of (A, X) is a corresponding permutation of coordinates of the tuple.

Without loss of generality, we will make the following assumptions:

If there exists $a \in A$ such that $a(X_1) \cap X_2 \neq \emptyset$ and $a(X_1) \nsubseteq X_2$ then let $d_1 \in A$ be such that $d_1(0) \notin X_2$ and $d_1(1) = n$.

Otherwise, if $a(X_1) \cap X_2 \neq \emptyset$ implies that $a(X_1) \subseteq X_2$ for all $a \in A$ then $|X_1| \geq 3$ by the condition **PS**. In this case let $d_2 \in A$ be such that $d_2(0) = n$, $d_2(1) = n - 1$, and $d_2(2) = n - 2$.

By the L-condition there is a minimal generating set $\bar{F} = \{\bar{f_i} \mid i \in \mathcal{I}\}$ of H/H_0, where $\mathcal{I}$ denotes a set of indices. Let $\psi : H \to H/H_0$ be the canonical epimorphism. For every $i \in \mathcal{I}$ we fix some element $f_i \in H$ such that $\psi(f_i) = \bar{f_i}$. Denote

$$F = \{f_i \mid i \in \mathcal{I}\}.$$

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be subsets of $\mathcal{I}$ with the following properties:

- $|\mathcal{I}_2| = |\mathcal{I}|$.
- $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_2$.

Denote

$$F^{\mathcal{I}_j} = \{f_i \mid i \in \mathcal{I}_j\} \text{ for } j = 1, 2.$$

In the case of $H' \lneqq H_0 < H^k H'$ due to condition 5 of the L-condition we can assume that for every $i \in \mathcal{I}_2$ the following equality holds: $f_i^k = e$. In the case of $H_0 < H'$ we can assume that $\mathcal{I}_2 = \mathcal{I}$.

Since $\bar{F}$ is an infinite set the set of the finite words over $\bar{F}$ has the same cardinality as $\bar{F}$. By the L-condition $|H/H_0| \geq |H_0|$. It follows that

$$|\mathcal{I}_2| = |\mathcal{I}| = |H/H_0| \geq |H_0|.$$

Therefore we can fix a surjection $\phi : \mathcal{I}_2 \to H_0$. We also define the set of elements of G:

$$S_K = \{q_i = (f_i, e, \dots, e, \phi(i)) \mid i \in \mathcal{I}_2\} \cup \{q_i = (f_i, e, \dots, e) \mid i \in \mathcal{I}_1\}.$$

Let us fix a minimal generating set of A: $S_A = \{s_1, s_2, \dots, s_r\}$. Let $S = S_K \cup S_A$ and $N = \langle S \rangle$. Note that $A = \langle S_A \rangle$ is contained in N.

Lemma 1 *The set S is a minimal generating set of the group N.*

Proof Since G is the semidirect product $A \leftthreetimes K$ and $S_K \subset K$, any element s of S_A cannot be written as an product of elements of $S \setminus \{s\}$ and their inverses.

Further, suppose, contrary to our claim, that the element q_i for some $i \in \mathcal{I}$ is a product of elements of $S \setminus \{q_i\}$ and their inverses. It is easy to check that this decomposition of q_i can be transformed to the product of the form $q_i = (q_{i_1}^{\epsilon_1})^{a_1} \dots (q_{i_m}^{\epsilon_m})^{a_m}$, where $i_1, \dots, i_m \in \mathcal{I} \setminus \{i\}$, $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$ and $a_1, \dots, a_m \in A$. Consider the 0-th coordinate of q_i. We have that f_i is a product of elements of $F \setminus \{f_i\}$, their inverses and elements of H_0. Applying ψ we conclude that $\bar{f}_i$ is a product of elements of $\bar{F} \setminus \{\bar{f}_i\}$ and their inverses. This contradicts the fact that $\bar{F}$ is a minimal generating set of the quotient H/H_0. □

We next show that the set S is a generating set of G, i.e., we next show that $N = G$.

Lemma 2 *For every $g \in \langle F \rangle$ the elements $u_{n-2} = (e, \dots, e, g, e, g^{-1})$ and $u_{n-1} = (e, \dots, e, g, g^{-1})$ are contained in N.*

Proof The element g can be decomposed into the product of elements of F and their inverses: $g = f_{i_1}^{\epsilon_1} \dots f_{i_m}^{\epsilon_m}$, where $\epsilon_1, \dots, \epsilon_m \in \{-1, 1\}$. For every $j \in X_1 \setminus \{0\}$ choose $b_j \in A_1$ such that $b_j(0) = j$. Then

$$t_j = b_j^{-1} q_{i_1}^{\epsilon_1} \dots q_{i_m}^{\epsilon_m} b_j (q_{i_1}^{\epsilon_1} \dots q_{i_m}^{\epsilon_m})^{-1} = (g^{-1}, e, \dots, e, g, e, \dots, e) \in N,$$

where g is located on the j-th coordinate of the tuple.

We consider all possible cases depending on the group A. We will use here the elements d_1 and d_2 which were defined at the beginning of the proof of the theorem.

1. There is $a \in A$ such that $a(X_1) \cap X_2 \neq \emptyset$ and $a(X_1) \nsubseteq X_2$. For every $m \in \{n-2, n-1\}$ choose $c_m \in A_2$ such that $c_m(n) = m$. Then $t_1^{d_1}(t_1^{d_1 c_m})^{-1} = u_m \in N$ for $m = n-2, n-1$.
2. For all $a \in A$, the inequality $a(X_1) \cap X_2 \neq \emptyset$ implies that $a(X_1) \subseteq X_2$. Then $|X_1| \geq 3$ and elements $u_{n-1} = t_1^{d_2}$ and $u_{n-2} = t_2^{d_2}$ are contained in N.

Lemma 3 *The subgroup $(E, \dots, E, H')$ of K is contained in N.*

Proof Let $h_1, h_2 \in H$. Then there exist $g_j \in \langle F \rangle$, $i_j \in \mathcal{I}_2$ for $j = 1, 2$ such that $h_j = g_j \phi(i_j)$. By the construction, the set S contains elements $q_{i_j} = (f_{i_j}, e, \dots, e, \phi(i_j))$ for $j = 1, 2$. Let $a \in A_1$ be such that $a(0) = 1$. Then $q_{i_2}^a = (e, f_{i_2}, e, \dots, e, \phi(i_2)) \in N$. By Lemma 2 elements $t_1 = (e, \dots, e, g_1^{-1}, e, g_1)$ and $t_2 = (e, \dots, e, g_2^{-1}, g_2)$ are contained in N. Therefore $h_1' = t_1 q_{i_1} = (f_{i_1}, e, \dots, e, g_1^{-1}, e, g_1\phi(i_1))$ and $h_2' = t_2 q_{i_2}^a = (e, f_{i_2}, e, \dots, e, g_2^{-1}, g_2\phi(i_2))$ are contained in N. Hence $h_1' h_2' {h_1'}^{-1} {h_2'}^{-1} = (e, \dots, e, h_1 h_2 {h_1}^{-1} {h_2}^{-1}) \in N$. Thus $(E, \dots, E, H') < N$. □

Lemma 4 *If $H' \lneq H_0 < H^k H'$ then $(E, \dots, E, H^k) < N$.*

Proof If $H' \lneq H_0 < H^k H'$ then for every $i \in \mathcal{I}_2$ the following equality holds: $f_i^k = e$.

Let $h \in H$. Then $h = g h_0$ for some $g \in \langle F \rangle$ and $h_0 \in H_0$. Since $H_0 > H'$ and $F^{\mathcal{I}_1} \cup F^{\mathcal{I}_2} = F$ there exist $g_1 \in \langle F^{\mathcal{I}_1} \rangle$, $g_2 \in \langle F^{\mathcal{I}_2} \rangle$, and $h_1 \in H'$ such that $g = g_1 g_2 h_1$. Since $H_0 > H'$ there exists $i \in \mathcal{I}_2$ satisfying $\phi(i) = h_1 h_0$. Thus we have $h = g_1 g_2 \phi(i)$. By the construction, the set S contains the element $q_i = (f_i, e, \dots, e, \phi(i))$. By Lemma 2 element $t_2 = (e, \dots, e, g_2^{-1}, g_2)$ is contained in N. Let $a \in A$ be such that $a(0) = n$. Note that element $t_1 = a^{-1}(g_1, e, \dots, e)a = (e, \dots, e, g_1)$ is contained in N. Therefore $h' = t_1 t_2 q_i = (f_i, e, \dots, e, g_2^{-1}, g_1 g_2 \phi(i))$ is contained in N. By the condition of the lemma there is $h_2 \in H'$ such that $g_2^p = h_2$. Let $a_1 \in A_2$ be such that $a_1(n-1) = n$. Then $(e, \dots, e, h_2, e) = a_1^{-1}(e, \dots, e, h_2) a_1 \in N$ by Lemma 3. Therefore ${h'}^k (e, \dots, e, h_2, e) = (e, \dots, e, e, h^k) \in N$. Thus $(E, \dots, E, H^k) < N$. □

Lemma 5 $N = G$.

Proof If $H_0 < H'$ then $(E, \ldots, E, H_0) < N$ by Lemma 3. If $H_0 > H'$ then the conditions of Lemma 4 hold by condition 5 of the L-condition. Thus in this case $(E, \ldots, E, H_0) < N$ too. By construction $(F, E, \ldots, E)$ is contained in $\langle S_K, (E, \ldots, E, H_0)\rangle$. Therefore the set $(F, E, \ldots, E)$ is contained in N. Let $a \in A$ be such that $a(0) = n$. Then $a^{-1}(F, E, \ldots, E)a = (E, \ldots, E, F) \subset N$. Since $\langle F\rangle H_0 = H$ we have $(E, \ldots, E, H) < N$. Also by transitivity of (A, X) we obtain $(H, \ldots, H) < N$. It follows that $N = G$. □

Now the assertion of Theorem 2 follows immediately from Lemma 1 and Lemma 5.

3 Applications and examples

We first give natural constructions of groups with property **PS**.

Proposition 3 *The following groups satisfy* **PS***:*

1. *The symmetric group of degree* $m \geq 5$.
2. *The permutational wreath product* $(B_1, Y_1) \wr (B_2, Y_2)$*, where* (B_1, Y_1) *and* (B_2, Y_2) *are finite transitive permutation groups and* $|Y_1| \geq 2$*,* $|Y_2| \geq 3$.
3. *The permutational wreath product* $(B_1, Y_1) \wr (B_2, Y_2) \wr (B_3, Y_3)$*, where* (B_i, Y_i) *is a finite transitive permutation group and* $|Y_i| \geq 2$ *for* $i = 1, 2, 3$.

Now we formulate two corollaries from Theorem 2 which are more applicable.

Proposition 4 *Let* $G = (A, X) \wr H$ *and there is an integer* $k > 1$ *with the following properties:*

1. (A, X) *satisfies* **PS**.
2. H *is an infinite group.*
3. $H' > H^k$.
4. $|H'| \leq |H/H'|$.

Then the group G satisfies the L-condition.

Proof Due to Theorem 2 we only need to show that H/H' has infinite minimal generating set. By the conditions of the theorem H/H' has exponent k. We conclude from result of [11, Proposition3.7] that an abelian group of a bounded exponent has a minimal generating set, and the proposition follows. □

Proposition 5 *Let* $G = (A, X) \wr H$ *and there is an infinite subgroup* M *of* H *with the following properties:*

1. (A, X) *satisfies* **PS**.
2. M *has exponent 2.*
3. $|M| = |H|$.
4. $M \cap H^2 = E$.

Then the group G satisfies the L-condition.

Proof Set $H_0 = H^2$. Then H/H_0 and M have minimal generating sets (Hamel basis) as vector spaces over the field with two elements.

Let I be an index set, and let $B = \{b_i \mid i \in I\}$ be a minimal generating set of M. Let also $\psi : H \to H/H^2$ be the canonical epimorphism. Since $M \cap H^2 = E$ the restriction ψ onto M is a bijection and the set $\psi(B)$ is a minimal generating set of $\langle \psi(B) \rangle$. Since B is infinite we have $|\psi(B)| = |B| = |M| = |H|$. Therefore we have $|H| = |H/H_0|$ and $\psi(B)$ can be complemented to Hamel basis F of the space H/H_0. It is also evident that $b_i^2 = e$ for every $b_i \in B$. Note that inclusion $H' < H^2$ is always true. Thus the group G satisfies the L-condition. □

Note that we use existence of Hamel basis of a vector space over a field in the proof of Proposition 5. Hence this proof uses the axiom of choise in some cases.

In the next section we will apply Proposition 5 to some groups of automorphisms of rooted trees, and particularly give positive answer to Problem 1.

3.1 Automorphism groups of rooted trees

We first recall necessary definitions related to rooted trees and groups acting on rooted trees. All notions which will be defined in this section are well-known, see for instance [10,19,20] for more details.

Let us fix our notation. Let $\mathsf{X} = (\mathsf{X}_1, \mathsf{X}_2, \ldots)$ be a sequence of finite sets $\mathsf{X}_i = \{0, 1, \ldots, n_i\}$ (we assume $n_i \geq 1$ for all i). Let X^n denote the set of all words of the form $x_1 x_2 \ldots x_n$, where $x_i \in \mathsf{X}_i$ for $i = 1, \ldots, n$. Let X^* denote the set which consist of the empty word $\emptyset$ and all words of the form $x_1 x_2 \ldots x_n$, where $n \in \mathbb{N}$ and $x_i \in \mathsf{X}_i$ for $i = 1, \ldots, n$. Let X^ω denote the set of all infinite words of the form $x_1 x_2 \ldots$, where $x_i \in \mathsf{X}_i$. We denote by $\mathsf{X}^{(k)}$ the infinite sequence $(\mathsf{X}_k, \mathsf{X}_{k+1}, \ldots)$.

We can consider the set of words X^* as rooted tree T_X which can be defined as follows: a vertex $x_1 x_2 \ldots x_n$ is adjacent to $x_1 x_2 \ldots x_{n-1}$, $\emptyset$ is the root. For the rooted tree T_X we also define *the vertex subtree* T_v ($v \in \mathsf{X}^*$) whose vertices are the words of the form $v\mathsf{X}^*$. We call the set of vertices X^n the *n-th level* of T_X.

Let $\operatorname{Aut} T_\mathsf{X}$ be the group of all automorphisms of the tree T_X. Let $G < \operatorname{Aut} T_\mathsf{X}$. We recall the definitions of some standard subgroups of G:

- The subgroup of all elements of G fixing every vertex of n-th level, denoted by $\operatorname{Stab}_G(n)$, is called *the stabilizer of the n-th level*.
- For every $v \in \mathsf{X}^*$ the subgroup of all elements of G fixing every vertex outside the subtree T_v, denoted by $\mathsf{rist}_G(v)$, is called *the rigid stabilizer of the vertex v*.
- The group generated by the set $\bigcup_{v \in \mathsf{X}^n} \mathsf{rist}\, v$, denoted by $\mathsf{Rist}_G(n)$, is called *the rigid stabilizer of the nth level*.

Let $\operatorname{Aut}_k T_\mathsf{X}$ be the subgroup of $\operatorname{Aut} T_\mathsf{X}$ such that an automorphism g of T_X is in $\operatorname{Aut}_k T_\mathsf{X}$ if and only if the equality $g(uv) = g(u)v$ is valid for any $u \in \mathsf{X}^k$ and any $v \in \mathsf{X}^*$. Note that $\operatorname{Aut}_k T_\mathsf{X}$ acts by permutations faithfully on the set X^k. Note also that $\operatorname{Stab}_{\operatorname{Aut} T_\mathsf{X}}(k) = \mathsf{Rist}_{\operatorname{Aut} T_\mathsf{X}}(k)$. Therefore the group $\operatorname{Aut} T_\mathsf{X}$ can be decomposed into semidirect product of its subgroups $\operatorname{Aut}_k T_\mathsf{X} \leftthreetimes \mathsf{Rist}_{\operatorname{Aut} T_\mathsf{X}}(k)$. It follows that $\operatorname{Aut} T_\mathsf{X}$ is isomorphic to the permutational wreath product $(\operatorname{Aut}_k T_\mathsf{X}, \mathsf{X}^k) \wr \mathsf{rist}_{\operatorname{Aut} T_\mathsf{X}}(v)$, where $v \in \mathsf{X}^k$ and $\mathsf{Rist}_{\operatorname{Aut} T_\mathsf{X}}(k)$ is the base subgroup of this wreath product.

We define subgroup $M_0 < \operatorname{Aut} T_\mathsf{X}$ as infinite direct product: $M_0 = \prod_{i \geq 0} C_2^{(i)}$, where each $C_2^{(i)}$ is isomorphic to the group of order 2. The action of elements of M_0 on the tree T_X can be defined in the following way. A nontrivial element of $C_2^{(i)}$ acts as follows $\underbrace{00\ldots0}_{i}10v \to$

$\underbrace{00\ldots0}_{i}11v$, $\underbrace{00\ldots0}_{i}11v \to \underbrace{00\ldots0}_{i}10v$ for every $v \in \mathsf{X}^{(i+1)}$, and $w \to w$ for the other words of X^*.

Lemma 6 *The intersection $M_0 \cap (\mathrm{Aut}\, T_{\mathsf{X}})^2$ is trivial.*

Proof For every $g \in \mathrm{Aut}\, T_{\mathsf{X}}$ and $n \geq 0$ we can write $g = g_n(g_{v_1}, \ldots, g_{v_k})$, where $g_n \in \mathrm{Aut}_n\, T_{\mathsf{X}}$, $(g_{v_1}, \ldots, g_{v_k}) \in \mathsf{Rist}_{\mathrm{Aut}\, T_{\mathsf{X}}}(n)$, and $\{v_1, \ldots, v_k\} = \mathsf{X}^n$. Write $\Pi_n g = \prod_{v \in \mathsf{X}^n} g_v$ for every $n \geq 0$.

It is evident that $\Pi_n g^2$ is an even permutation of X_{n+1} for every $g \in \mathrm{Aut}\, T_{\mathsf{X}}$ and $n \geq 0$. Therefore $\Pi_n h$ is an even permutation of X_{n+1} for every $h \in (\mathrm{Aut}\, T_{\mathsf{X}})^2$ and $n \geq 0$. But for every nontrivial element $g \in M_0$ there is $m \geq 0$ such that $\Pi_m g$ is an odd permutation of X_{m+1}. Thus the intersection $M_0 \cap (\mathrm{Aut}\, T_{\mathsf{X}})^2$ is trivial. □

Proposition 6 *Let G be an infinite automorphism group of T_{X} and there is a positive integer k with the following properties:*

1. *The group G can be decomposed into a semidirect product of its subgroups $(G \cap \mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k) \leftthreetimes \mathsf{Rist}_G(k)$ provided the group $(G \cap \mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k)$ satisfies* **PS**.
2. $|M_0 \cap G| = |G|$.

Then the group G satisfies the L-condition.

Proof Let G be a group and $k \in \mathbb{N}$ be such that all conditions of the statement are satisfied. Let $v = 0\ldots0 \in \mathsf{X}^k$. Then G is isomorphic to the permutational wreath product $(G \cap \mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k) \wr \mathsf{rist}_G(v)$.

Consider the subgroup $M = M_0 \cap \mathsf{rist}_G(v)$ of the group G. It is obvious that M has exponent 2. Since $M_0 \cap G = (M_0 \cap G \cap \mathrm{Aut}_k\, T_{\mathsf{X}}) \times M$ we have $|M| = |M_0 \cap G|$. Combining it with the second condition of the statement we obtain $|M| = |G|$. It follows that $|M| = |G| = |\,\mathsf{rist}_G(v)|$. By Lemma 6 we have $M \cap (\mathsf{rist}_G(v))^2 < M_0 \cap (\mathrm{Aut}\, T_{\mathsf{X}})^2 = E$. Thus the group G satisfies the L-condition by Proposition 5, and the statement follows. □

3.1.1 Examples of uncountable groups of automorphisms with the L-condition

We recall the definitions of some classes of automorphisms of T_{X}.

- An automorphism g is called *finitary* if there is a positive integer n such that the equality $g(uv) = g(u)v$ is valid for every $u \in \mathsf{X}^n$ and every $v \in \mathsf{X}^*$.
- An automorphism g is called *weakly finitary* if for every $w \in \mathsf{X}^\omega$ there are $n \in \mathbb{N}$, $u \in \mathsf{X}^n$, and $v \in \mathsf{X}^\omega$ such that $w = uv$ and $g(uv) = g(u)v$.
- Two words $w_1, w_2 \in \mathsf{X}^\omega$ are called *cofinal* if there are $n \in \mathbb{N}$, $u_1, u_2 \in \mathsf{X}^n$, $v \in \mathsf{X}^\omega$ satisfying $w_1 = u_1 v$ and $w_2 = u_2 v$.
  An automorphism g is called *cofinal* if it maps cofinal words to cofinal words.
- An automorphism g is called *bicofinal* if both g and g^{-1} are cofinal.

Denote by $\mathrm{Aut}_f\, T_{\mathsf{X}}$, $\mathrm{Aut}_{wf}\, T_{\mathsf{X}}$, $\mathrm{Aut}_b\, T_{\mathsf{X}}$ the sets of all finitary, weakly finitary and bicofinal automorphisms of T_{X} respectively. All of these sets are groups.

Note that, by definitions, we have the following inclusions:

$$\mathrm{Aut}_f\, T_{\mathsf{X}} < \mathrm{Aut}_{wf}\, T_{\mathsf{X}} < \mathrm{Aut}_b\, T_{\mathsf{X}}.$$

For more details on these groups we refer the reader to [15].

Theorem 7 *The groups* Aut T_{X}, $\mathrm{Aut}_{wf}\, T_{\mathsf{X}}$ *and* $\mathrm{Aut}_b\, T_{\mathsf{X}}$ *satisfy the L-condition and so have minimal generating sets.*

Proof Let G be one of the above groups. Then G can be decomposed into semidirect product of its subgroups $(\mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k) \leftthreetimes \mathsf{Rist}_G(k)$ for a positive integer k. The permutation group $(\mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k)$ satisfies the condition **PS** for $k \geq 3$ by Proposition 3. It is clear that $M_0 < \mathrm{Aut}_{wf}\, T_{\mathsf{X}} < \mathrm{Aut}_b\, T_{\mathsf{X}} < \mathrm{Aut}\, T_{\mathsf{X}}$. It follows that G satisfies all conditions of Proposition 6, and the statement follows. □

3.1.2 Examples of countable groups of automorphisms with L-condition

Proposition 8 *Let G be a countable automorphism group of the rooted tree T_{X} with the following properties:*

- $\mathrm{Aut}_f\, T_{\mathsf{X}} < G$.
- $\mathsf{Rist}_G(k) = \mathrm{Stab}_G(k)$ *for some integer* $k \geq 3$.

Then the group G satisfies the L-condition.

Proof By the condition of the proposition $M_0 \cap G > M_0 \cap \mathrm{Aut}_f\, T_{\mathsf{X}}$. Since the intersection $M_0 \cap \mathrm{Aut}_f\, T_{\mathsf{X}}$ is countable, $|M_0 \cap G| = |G|$. Since $\mathrm{Aut}_f\, T_{\mathsf{X}} < G$, $\mathrm{Aut}_k\, T_{\mathsf{X}} < G$. Therefore G is decomposed into semidirect product $(\mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k) \leftthreetimes \mathsf{Rist}_G(k)$ of its subgroups. The permutation group $(\mathrm{Aut}_k\, T_{\mathsf{X}}, \mathsf{X}^k)$ satisfies condition **PS** for $k \geq 3$ by Proposition 3. It follows that G satisfies all conditions of Proposition 6, and the statement follows. □

From now we assume that $X_1 = X_2 = \ldots$. In this case T_{X} is called *regular rooted tree*.

A vertex subtree T_v of T_{X} for every $v \in \mathsf{X}^n$ can be naturally identified with the whole tree T_{X}:

$$\pi_v : v x_{n+1} \ldots x_m \mapsto x_{n+1} x_{n+2} \ldots x_m.$$

Thus for every $g \in \mathrm{Aut}\, T_{\mathsf{X}}$ and $v \in \mathsf{X}^*$ we can define automorphism $g|_v \in \mathrm{Aut}\, T_{\mathsf{X}}$ in the following way: $g|_v(u) = w$ if and only if $g(vu) = g(v)w$ for every $u, w \in \mathsf{X}^*$. We call the automorphism $g|_v \in \mathrm{Aut}\, T_{\mathsf{X}}$ *the state of g in v.*

An automorphism $g \in \mathrm{Aut}\, T_{\mathsf{X}}$ is *a finite-state* automorphism if the set of its states is finite. All finite-state automorphisms of the tree T_{X} form the *group* FAut T_{X} *of finite-state automorphisms* of the tree T_{X}.

Let us define the number $\Theta_n(g) = \#\{v \in \mathsf{X}^n \mid g|_v \neq e\}$ for every $g \in \mathrm{Aut}\, T_{\mathsf{X}}$.

The set of all finite-state automorphisms $g \in \mathrm{FAut}\, T_{\mathsf{X}}$ such that the sequence $\Theta_n(g)$ is bounded by a polynomial in n of degree m forms the *group* Pol(m) *of polynomial automorphisms of degree m* of the tree T_{X}. The group Pol(0) is also called the *group of bounded automorphisms*. The group of *polynomial automorphisms*, denoted by Pol(∞), is defined to be the union of increasing chain of groups: $\mathrm{Pol}(\infty) = \bigcup_{m=0}^{\infty} \mathrm{Pol}(m)$.

A subgroup G of Aut T_{X} is *self-similar* provided $g|_v \in G$ for all $g \in G$ and $v \in \mathsf{X}^*$. *The group* RAut T_{X} *of functionally recursive automorphisms* of T_{X} can be defined as the union of all finitely generated self-similar subgroups of Aut T_{X}.

We refer the reader to [5,19,20] for details concerning groups defined above.

Theorem 9 *The groups* FAut T_{X}, Pol(m) *(*$m \geq 0$*),* Pol(∞)*, and* RAut T_{X} *satisfy the L-condition and so have minimal generating sets.*

Proof Let G be one of the groups above. It is well known that G is countable group. By definition, the group G contains the group $\mathrm{Aut}_f\, T_{\mathsf{X}}$. Furthermore, we have $\mathrm{Stab}_G(k) = \mathsf{Rist}_G(k)$ for every positive integer k. It follows that G satisfies the L-condition by Proposition 8, and the theorem follows. □

Finally we obtain the statement of Theorem 1 as an immediate corollary of Theorems 2, 7 and 9.

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